8 DIFFERENTIAL EQUATIONS

8.1 THE CONCEPT OF DIFFERENTIAL EQUATIONS

Definition: Differential equation

A *differential equation* is an equation that involves both an unknown function and its derivatives or differentials.

There are ordinary and partial differential equations.

Definition: Ordinary differential equation

A differential equation for a one variable function is called an ordinary differential equation.

The general form of an ordinary differential equation can be written as

$$F(x, y, y', y'', y''', \dots, y^{(n)}) = 0$$

or

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

where $y(x)$ is an unknown function and $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}, \dots,$ $y^{(n)} = \frac{d^ny}{dx^n}$ are derivatives of the function *y(x).*

Example 8.1

The following two equations,

$$y' + xy = x^3$$

$$y'' - 5y' + 6y = 13\sin(3x)$$

are ordinary differential equations for an unknown one-variable function y=y(x).

These equations can be also written as:

$$\frac{dy}{dx} + xy = x^3$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13\sin(3x)$$

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Definition: Partial differential equation

A differential equation for a function of several variables is called a ***partial differential equation*** (PDE). PDE contains partial derivatives.

Example 8.2

The equation of a string vibration

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2}$$

is a partial differential equation for the function of two variables *U=U(x,t).*

In this topic **only ordinary differential equations** of the first and second order will be considered.

Definition: Order of a differential equation

An order of a differential equation is the order of the highest derivative it contains.

Example 8.3

The first order ODE:	$y' + xy = x^3$
The second order ODE*:*	$y" - 5y' + 6y = 13sin\,(3x)$
The third order ODE:	$y''' - x\,ln(x) = 0$

Definition: Solution of a differential equation

The *solution* of a differential equation is any function that satisfies given equation identically.

It means that the given equation becomes identical after substituting its solution into the differential equation.

Definition: General and particular solutions of a differential equation

A solution of an ordinary differential equation of order n, which involves exactly n (maximum number) of essential arbitrary constants is called a ***general solution***.

A solution of a differential equation obtained by substituting the defined numerical values instead of arbitrary constants in the general solution of a differential equation is called a ***particular solution***.

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Definition: Singular solution of a differential equation

A solution of an ordinary differential equation that does not contain arbitrary constants and cannot be obtained from the general solution is called a singular solution of a differential equation.

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8.2 FIRST-ORDER DIFFERENTIAL EQUATIONS

The main concept for first-order differential equations will be given in this chapter. The separable variable equations and first-order linear differential equations with their solving methods will be considered in detail. Two solving methods of first-order linear differential equations will be presented, i.e. the method of variation of constants and the Bernoulli method (solving by using substitution).

8.2.1 *Main concept of first-order differential equations*

An ordinary differential equation of the first order can be given in the following standard forms:

a) in implicit form

$$F(x, y, y') = 0,$$

b) in explicit form

$$y' = f(x, y)$$

c) in the differential form, since $y' = dy/dx$

$$P(x, y)dx + Q(x, y)dy = 0$$

where $f(x, y)$, $P(x, y)$, $Q(x, y)$ are functions of x and y, in general.

We also point out that $f(x, y) = -P(x, y)/Q(x, y)$.

Example 8.4

Let us consider a differential equation that is given in the implicit form:

$$(x^2 + 1)y' - 2xy - 3x = 0$$

This equation can be written in explicit form:

$$y' = \frac{2xy + 3x}{x^2 + 1}$$

We can also obtain this equation in a differential form, if we substitute $\mathrm{dy/dx}$ instead of $\mathrm{y'}$ into the first equation and multiply both its sides by *dx*:

$$(2xy + 3x)dx - (x^2 + 1)dy = 0$$

The general solution of a first-order differential equation involves one arbitrary constant C and it can be written in the explicit form $y' = \varphi(x, C)$ or in the implicit form $\Phi(x, y, C) = 0$.

Definition: General integral

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The general solution of a differential equation in the implicit form $\Phi(x, y, C) = 0$ is called a *general integral*.

We get a particular solution of a differential equation if we substitute a defined number instead of a constant *C* into the general solution.

Example 8.5

Let us consider one of the easiest first-order differential equations:

$$y' = 2x.$$

To find the unknown function *y(x)* we use integration with respect to x, taking into account that $y = \int y' dx$. As a result, we have

$$y = \int 2x dx = x^2 + C$$

where C is an integration constant.

The function $y = x^2 + C$ is the **general solution** of the given differential equation, since it satisfies the given equation, i.e. $y' = (x^2 + C)' = 2x,$ and contains one essential arbitrary constant C.

Assigning particular values to the arbitrary constant C in the general solution, we get so-called particular solutions. For instance,

$y = x^2 - 2$ is the particular solution which corresponds to $C = -2$,

$y = x^2 - 1$ is the particular solution which corresponds to $C = -1$,

$y = x^2$ is the particular solution which corresponds to $C = 0$,

$y = x^2 + 1$ is the particular solution which corresponds to $C = 1$,

$y = x^2 + 2$ is the particular solution which corresponds to $C = 2$.

Visualization of the obtained solutions, presents a set of parabolas, where each parabola corresponds to a particular value of the constant C (see Fig.1).

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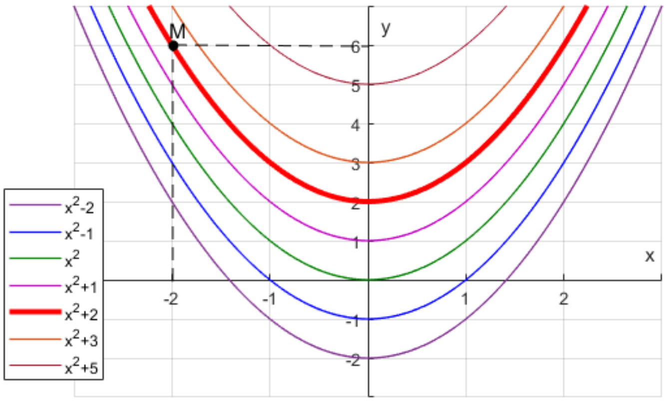

Figure 8.1

Due to the arbitrary constant *C*, the general solution is not just one function, but a set of functions. To each specific numerical value of the constant C in the general solution there corresponds one particular solution.

It is often necessary to find only one particular solution from a set of solutions. In this case, a differential equation is given together with an additional condition $y(x_0) = y_0$, which means that a value of the unknown function is equal to y_0 at some particular value x_0 of the argument. This additional condition is called an ***initial condition***.

Definition: Initial value problem

The ***initial value problem*** (Cauchy problem) is a problem which involves an ordinary differential problem $\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{y}') = 0$ together with an initial condition $\mathrm{y}(\mathrm{x}_0) = \mathrm{y}_0$ which specifies the value of an unknown function at a given point x_0. That is a problem of finding a particular solution of the differential equation that satisfies the initial condition $\mathrm{y}(\mathrm{x}_0) = \mathrm{y}_0$.

Thus, the solutions to a differential equation can be viewed as a family of solution curves in the x y-plane. Besides, from a geometric point of view, the initial condition $y(x_0) = y_0$ is the same as a point (x_0, y_0) that the solution curve must pass through.

An initial value problem is solved in the following way

1. Find a general solution to the given differential equation that involves an arbitrary constant C.

2. Substitute x_0 and y_0 from the initial condition into the general solution instead of x and y.

3. Solve the obtained equation with respect to C.

4. Substitute the result back into the general solution.

Example 8.6

Let us take the differential equation from Example 8.5 and consider it as an initial value problem

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$$y' = 2x \text{ and } \quad y(-2) = 6.$$

Thus, the task is to find a particular solution of the given differential equation that satisfies the initial condition $y(-2) = 6$, which means that at x=-2, the value of the solution function is y=6.

The general solution of the differential equation from Example 8.5 is

$$y = x^2 + C$$

In order to find the corresponding C value, we substitute $x = -2$ and $y = 6$ (i.e. the initial condition) into the general solution:

$$6 = (-2)^2 + C$$

$$6 = 4 + C$$

$$C = 2$$

The solution of the initial value problem is found by substituting the obtained C value into the general solution. It means that the particular solution which satisfies the given initial condition is:

$$y = x^2 + 2$$

The graphical interpretation of the result (see Fig.1) means that from the set of curves, only the curve $y = x^2 + 2$ which goes through the point M(-2,6), must be selected.

There are different types of the first order differential equations which are solved by different methods. Separable variable equations and linear equations and their respective solutions are discussed below.

8.2.2 *Separable variable equations*

Consider a differential equation given in the form $y' = f(x, y)$.

Definition:

A first-order differential equation $y' = f(x, y)$ is called a separable variable equation if the function $f(x, y)$ can be factored into the product of two functions of x and y (i.e., it can be divided into multipliers so that each multiplier depends on only one variable):

$$y' = f_1(x) \cdot f_2(y)$$

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where $f_1(x)$ and $f_2(y)$ are continuous functions.

Solution method:

1) Replace y' with $\frac{dy}{dx}$

$$\frac{dy}{dx} = f_1(x) \cdot f_2(y)$$

2) Separate the variables, i.e. move all the y terms (including *dy*) to one side of the equation and all the x terms (including *dx*) to the other side.

For this purpose, multiply both sides of the equation by *dx* and divide by $f_2(y)$:

$$\frac{dy}{f_2(y)} = f_1(x)dx$$

Here we suppose that $f_2(y) \neq 0$.

3) Integrate directly both sides of the equation with respect to their variables.

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx$$

4) Solve the obtained equation if possible.

If *y(x)* cannot be expressed in an explicit form, the expression on the right-hand side of the equation is shifted to the left side. In this case, the general integral of the differential equation is obtained.

5) Dividing by $f_2(y)$ we assume that $f_2(y) \neq 0$. It may cause loss of the solution $f_2(y) = 0$. If there are y values for which $f_2(y) = 0$ and these values satisfy the given differential equation, then these values will also be solutions of the differential equation. Therefore, we should proof if $f_2(y) = 0$ is a solution of the differential equation, and if it is a solution, then check if it is a singular solution.

Let us consider a differential equation given in the form

$$P(x, y)dx + Q(x, y)dy = 0$$

Definition:

Differential equation $P(x, y)dx + Q(x, y)dy = 0$ is called a separable variable equation if each function $P(x, y)$ and $Q(x, y)dy$ can be factored into a product of two functions so that each multiplier depends on only one variable:

$$P_1(x) \cdot P_2(y)dx + Q_1(x) \cdot Q_2(y)dy = 0$$

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Solution method:

1) Separate the variables:

$$Q_1(x) \cdot Q_2(y)dy = -P_1(x) \cdot P_2(y)dx$$

$$\frac{Q_2(y)dy}{P_2(y)} = -\frac{P_1(x)dx}{Q_1(x)}$$

2) Integrate both sides of the obtained expression:

$$\int \frac{Q_2(y)dy}{P_2(y)} = -\int \frac{P_1(x)dx}{Q_1(x)}$$

3) Obtain a solution in an explicit or implicit form.

4) Check out if $P_2(y) = 0$ and $Q_1(y) = 0$ are the singular solutions of the differential equation.

Example 8.7

Solve the differential equation $y' - (x+2) \cdot e^{-y} = 0$.

This equation can be rewritten as

$$y' = (x+2) \cdot e^{-y}$$

This equation is a separable variable equation because the function on the right-hand side of the equation is a product of two functions. One function depends on *x* only and other one depends on *y* only.

We solve the equation by the following steps:

1) Replace y' with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = (x+2) \cdot e^{-y}$$

2) Separate variables, by multiplying both sides of the equation by *dx* and dividing by e^{-y}.

$$\frac{dy}{e^{-y}} = (x+2)dx$$

Since $e^{-y} \neq 0$, we do not miss any solutions dividing by e^{-y} .

3) Now we have an expression that contains only *y* terms on the left-hand side and only *x* terms on the right-hand side. It means that we can integrate both sides of the expression:

$$\int \frac{dy}{e^{-y}} = \int (x+2)dx$$

Let us find separately both integrals:

$$\int \frac{dy}{e^{-y}} = \int e^y dy = e^y + C_1$$

$$\int (x+2)dx = \int (x+2)d(x+2) = \frac{(x+2)^2}{2} + C_2$$

4) As a result, we have

$$e^y + C_1 = \frac{(x+2)^2}{2} + C_2$$

$$e^y = \frac{(x+2)^2}{2} + C_2 - C_1$$

Since C_1 and C_2 are constants, $C_2 - C_1 = C$ is also a constant. Therefore, after integration of both sides of the expression, only one constant C of integration is usually placed on only one side of the expression.

$$e^y = \frac{(x+2)^2}{2} + C$$

The solution in the explicit form is

$$y = \ln\left(\frac{(x+2)^2}{2} + C\right)$$

The last expression is the general solution of the given differential equation.

Example 8.8

Solve the differential equation $(y + xy)dx + (x - xy)dy = 0$.

The given equation can be rewritten in the form

$$y \cdot (1+x)dx + x \cdot (1-y)dy = 0$$

This equation is a separable variable equation because the functions before *dx* and before *dy* have the form of a product of only one variable functions.

We solve the equation by the following steps:

1) Switch the places of the terms

$$x \cdot (1-y)dy = -y \cdot (1+x)dx$$

2) Separate variables by dividing both sides of the equation by y and x ($x \neq 0$ and $y \neq 0$).

$$\frac{x \cdot (1-y)dy}{x \cdot y} = -\frac{y \cdot (1+x)}{x \cdot y}dx$$

$$\frac{(1-y)dy}{y} = -\frac{(1+x)}{x}dx$$

3) Integrate each side of the obtained equation:

$$\int \frac{(1-y)dy}{y} = -\int \frac{(1+x)}{x}dx$$

$$\int \left(\frac{1}{y} - 1\right) dy = -\int \left(\frac{1}{x} + 1\right) dx$$

$$\ln|y| - y = -\ln|x| - x + C$$

$$\ln|y| + \ln|x| - y + x = C$$

As a result, we have obtained the general solution of the equation in the form of a general integral:

$$\ln|y \cdot x| + x - y = C$$

where $x \neq 0$ and $y \neq 0$, to satisfy the condition on an argument of logarithmic function.

4) We check for possibly missed solutions because of dividing by x and y:

Both $x = 0$ and $y = 0$ satisfies the given differential equations, but they cannot be obtained from the general solution, so that $x = 0$ and $y = 0$ are singular solutions of the differential equation.

In general, ***solution method for the separable variable equations is:***

1) Separate variables, i.e., rewrite the equation thus that the terms depending on x and terms depending on y appear on opposite sides, so that there is only one variable on each side of the equation.

2) Integrate one side of obtained expression with respect to y and the other side with respect to x.

3) Simplify.

4) Check for possibly missed solutions, i.e. check for existence of singular solutions of the differential equation.

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8.2.3 *First-order linear differential equations*

Definition: Linear differential equation

A first-order differential equation is called a ***linear differential equation*** if it can be written in the form

$$y' + p(x)y = f(x)$$

where $p(x)$ and $f(x)$ are continuous functions.

Example 8.9

1) If the function $f(x)$ on the right-hand side of the equation is equal to zero, then the differential equation is called a ***homogeneous linear equation***:

$$y' + p(x)y = 0, \quad (f(x) = 0)$$

2) If the function $f(x)$ in on right-hand side of the equation is not equal to zero, then the differential equation is called a ***nonhomogeneous linear equation***:

$$y' + p(x)y = f(x), \quad (f(x) \neq 0)$$

Solution method:

There are two methods to solve a linear differential equation. These are the method of Variation of a Constant and Bernoulli method. Both methods will be considered here.

1. Method of Variation of a Constant

The method consists of the following steps

1) Find a general solution to the corresponding homogeneous equation

$$y' + p(x)y = 0.$$

The general solution of the homogeneous equation contains a constant of integration C.

2) Replace the constant C with a certain (but still unknown) function C(x).

3) Determine the unknown function C(x) by substituting this general solution of the homogeneous equation into the given nonhomogeneous differential equation.

Example 8.10

Solve the differential equation $xy' + y = sinx$ by the *Method of Variation of a Constant.*

This equation is a first-order linear differential equation and can be rewritten in the form

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$$y' + \frac{y}{x} = \frac{sinx}{x}$$

1) We solve the corresponding homogeneous linear equation:

$$y' + \frac{y}{x} = 0$$

This is a separable variable equation. Therefore, we replace y' with $\frac{dy}{dx}$, and move the second term from the left-hand side of the equation to the right-hand side:

$$\frac{dy}{dx} = -\frac{y}{x}$$

Then we separate variables, provided that $y \neq 0$

$$\frac{dy}{y} = -\frac{dx}{x}$$

and integrate both sides of the equation:

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln|y| = -\ln|x| + C$$

Since C is an arbitrary constant, it may be written in the form $\ln|C|$:

$$\ln|y| = -\ln|x| + \ln|C|$$

Then, using properties of a logarithm, we have:

$$\ln|y| = \ln\left|\frac{C}{x}\right|$$

As a result, we obtain the general solution of the homogeneous equation in the form

$$y_0 = \frac{C}{x}$$

2) In order to find a general solution of the nonhomogeneous equation, we replace the constant C with an unknown function C(x):

$$y = \frac{C(x)}{x}$$

4) The unknown function C(x) is found by substituting $y = \frac{C(x)}{x}$ into the given nonhomogeneous differential equation together with its derivative y':

$$y' = \left(\frac{C(x)}{x}\right)' = \frac{x \cdot C'(x) - x' \cdot C(x)}{x^2} = \frac{x \cdot C'(x) - C(x)}{x^2} = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$$

After substituting y and y' into the given equation we have

$$\left(\frac{C'(x)}{x} - \frac{C(x)}{x^2}\right) + \frac{C(x)}{x^2} = \frac{sinx}{x}$$

This equation can be simplified as

$$C'(x) = sinx$$

To find the unknown function C(x), we integrate the obtained expression with respect to x:

$$C(x) = \int C'(x)dx = \int sinx\, dx = -cosx + C$$

Substituting the obtained function *C(x)* into the expression for y, we have the general solution of the given nonhomogeneous linear differential equation:

$$y = \frac{-\cos(x) + C}{x}$$

2. Bernoulli method (Solution by using substitution $y = U \cdot V$)

The main idea is that the solution y of a linear differential equation $y' + p(x)y = f(x)$ is sought as a product of two functions $\underline{y = U \cdot V}$, where $U = U(x)$ un $V = V(x)$ are unknown functions. One of these functions can be chosen arbitrarily, but the other function must be chosen the way that the multiplication $U(x) \cdot V(x)$ satisfies the differential equation.

Steps of solving:

1) Substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given linear differential equation:

$$y' + p(x)y = f(x)$$

The equation takes the form

$$U'V + UV' + p(x)UV = f(x)$$

or

$$U'V + U \cdot (V' + p(x)V) = f(x)$$

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2) The function *V* is chosen to make the expression in the brackets equal to zero:

$$V' + p(x)V = 0$$

Then the last equation in the 1st step becomes:

$$U'V + U \cdot 0 = f(x) \quad \text{or} \quad U'V = f(x).$$

As a result, the following system is to be solved

$$\begin{cases} V' + p(x)V = 0 \\ U'V = f(x) \end{cases}$$

3) The equation $V' + p(x)V = 0$ is a separable variable differential equation so that it is solved for V by separating variables:

$$\frac{dV}{dx} = -p(x)V$$

$$\frac{dV}{V} = -p(x)dx$$

$$\int \frac{dV}{V} = -\int p(x)dx$$

$$\ln|V| = -\int p(x)dx$$

$$V = e^{-\int p(x)dx} + C_1$$

Here it is assumed that the constant $C_1 = 0$, because the equation suffices to have only one particular solution.

4) Substitute the obtained V back into the equation $U'V = f(x)$:

$$U' \cdot e^{-\int p(x)dx} = f(x)$$

6) Solve the last equation for U.

7) Finally, substitute the obtained U and V into *y = UV* and get a general solution.

Example 8.11

Solve the linear differential equation $xy' + y = sinx$ by *using substitution y=UV.*

Beforehand, we write the equation in the form

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$$y' + \frac{y}{x} = \frac{sinx}{x}$$

1) We substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given differential equation:

$$U'V + UV' + \frac{UV}{x} = \frac{sinx}{x}$$

or

$$U'V + U\left(V' + \frac{V}{x}\right) = \frac{sinx}{x}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' + \frac{V}{x} = 0$$

Then the equation can be written as a system of two equations:

$$\begin{cases} V' + \frac{V}{x} = 0 \\ U'V = \frac{sinx}{x} \end{cases}$$

3) We solve the first equation of the system for V. This is a separable variable equation:

$$\frac{\mathrm{dV}}{\mathrm{dx}} = -\frac{V}{x}$$

$$\frac{\mathrm{dV}}{\mathrm{V}} = -\frac{dx}{x}$$

$$\int \frac{\mathrm{dV}}{\mathrm{V}} = -\int \frac{\mathrm{dx}}{\mathrm{x}}$$

$$\ln|V| = -\ln|x| + C_1$$

where we assume $C_1 = 0$ so that:

$$\ln|V| = -\ln|x| \quad \Rightarrow \quad \ln|V| = \ln|x^{-1}|$$

$$V = \frac{1}{x}$$

4) We substitute the obtained V back into the second equation of the system:

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$$\mathrm{U'V} = \frac{\mathrm{sinx}}{\mathrm{x}}$$

$$U' \cdot \frac{1}{x} = \frac{sinx}{x}$$

We simplify and solve the obtained equation for U:

$$U' = sinx$$

$$\mathrm{U} = \int \mathrm{U'dx} = \int \mathrm{sin}x\, dx = -\mathrm{cos}x + C$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

$$\mathrm{y} = \mathrm{UV} = (-\mathrm{cos}x + C) \cdot \frac{1}{x}$$

$$\mathrm{y} = \frac{\mathrm{C} - \mathrm{cos}x}{\mathrm{x}}$$

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8.2.4 *Exercises*

Exercise 8.1.

Solve the initial value problem (Cauchy problem)

$$y' = y \cdot \cot x\,, \qquad y\left(\frac{\pi}{2}\right) = 3.$$

Solution:

This equation is a separable variable equation because the function on the right-hand side of the equation is a product of two functions. One function depends on x only and other one depends on y only.

We solve the equation in the following steps:

1) Replace y' with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y \cdot \cot x$$

2) We separate variables, multiplying both sides of the equation by *dx* and dividing by y:

$$\frac{dy}{y} = \cot x \cdot dx$$

Here we suppose $y \neq 0$.

3) We have the equation that contains only y terms on the left-hand side and only x terms on the right-hand side. It means that we can integrate both sides of the equation:

$$\int \frac{dy}{y} = \int \cot x dx$$

Let us find the integral on the right-hand side of the expression:

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$$

4) As a result, we have

$$\ln|y| = \ln|\sin x| + C$$

where C is a constant.

In order to simplify the obtained solution, we can write $\ln|C_1|$ on the right side of the expression instead of C, where C_1 is also a constant ($C_1 \neq 0$). It is allowed due to both C and $\ln|C_1|$ being arbitrary constants.

$$\ln|y| = \ln|\sin x| + \ln|C_1|$$

According to the properties of logarithmic functions, we have

$$\ln|y| = \ln|C_1 \cdot \sin x|$$

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As a result, we obtain the **general solution** of the given differential equation in the explicit form:

$$y = C_1 \cdot \sin x\,, \qquad (C_1 \neq 0\,).$$

4) Check for possibly missed solutions due to dividing by y:

The $y = 0$ satisfies the given differential equations, but it will be not the singular solution if we rewrite the obtained general solution of the differential equation in the form

$$y = C \cdot \sin x\,,$$

where C is an arbitrary constant (it can be equal by zero).

In this case the solution $y = 0$ can be obtained from the general solution at C=0, therefore it is the particular solution of the given equation.

As a result, the general solution of the given differential equation is

$$y = C \cdot \sin x$$

5) To solve the initial value problem, we should find only one particular solution of the differential equation that satisfies the initial condition $y\left(\frac{\pi}{2}\right) = 3$, i.e. the value of the function y(x) must be equal to 3 at $x = \frac{\pi}{2}$. In order to find this particular solution, we insert $y = 3$ and x= $\frac{\pi}{2}$ into the general solution.

$$3 = C \cdot \sin\frac{\pi}{2}$$

$$3 = C \cdot 1$$

$$C = 3\,.$$

We insert the obtained value of the constant C onto the general solution and get a **particular solution** of the given initial value problem.

$$y = 3 \cdot \sin x$$

Exercise 8.2.

Solve the initial value problem (Cauchy problem)

$$2(x^2 y + y)y' + \sqrt{1 + y^2} = 0\,, \qquad y(0) = 2.$$

Solution:

This equation can be rewritten as

$$2y(x^2 + 1)y' = -\sqrt{1 + y^2}$$

This equation is a separable variable equation because the function before y' on the left-hand side of the equation is a product of two functions. One function depends on x only and the other one depends on y only. The function on the right-hand side of the equation depends only on y.

We solve the equation by the following steps:

1) At first, we replace y' with $\frac{dy}{dx}$:

$$2y(x^2+1)\frac{dy}{dx} = -\sqrt{1+y^2}$$

2) We separate the variables, multiplying both sides of the equation by *dx* and dividing by (x^2+1) and by $\sqrt{1+y^2}$:

$$\frac{2ydy}{\sqrt{1+y^2}} = -\frac{dx}{(x^2+1)}$$

Since $\sqrt{1+y^2} \neq 0$ and $(x^2+1) \neq 0$, we do not miss any solution dividing by (x^2+1) and by $\sqrt{1+y^2}$.

3) Now we have an expression that contains only y terms on the left-hand side and only x terms on the right-hand side. This means that we can integrate both sides of the expression:

$$\int \frac{2ydy}{\sqrt{1+y^2}} = -\int \frac{dx}{(x^2+1)}$$

Let us find the integral in the left-hand side of the expression:

$$\int \frac{2ydy}{\sqrt{1+y^2}} = \int \frac{d(y^2)}{\sqrt{1+y^2}} = \int (1+y^2)^{-\frac{1}{2}} d(1+y^2) = 2(1+y^2)^{\frac{1}{2}} + C$$

4) As a result, we have

$$2\sqrt{1+y^2} = -\text{arctan}x + C$$

or

$$2\sqrt{1+y^2} + \text{arctan}x = C$$

where C is an arbitrary constant.

The last expression is the general solution of the given differential equation.

5) To solve the initial value problem, we should find only one particular solution of the differential equation that satisfies the initial condition $y(0) = 2$. In order to find this particular solution, we insert $y = 2$ and x$= 0$ into the general solution.

$$2\sqrt{1+2^2} + \text{arctan}0 = C$$

$$2\sqrt{5} + 0 = C \;\rightarrow\; C = 2\sqrt{5}$$

We insert the obtained value of the constant C onto the general solution and get the particular solution of the given initial value problem in implicit form

$$2\sqrt{1+y^2} + \text{arctan}x = 2\sqrt{5}$$

Exercise 8.3.

Solve the differential equation $(e^{2x}+3)dy + y \cdot e^{2x}dx = 0.$

Solution:

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This equation is a separable variable equation because the function before *dy* depends only on x and the function *dx* has the form of a product of only one variable functions.

We solve the equation by the following steps:

1) Switch the places of the terms

$$(e^{2x}+3)dy = -y \cdot e^{2x}dx$$

2) We separate variables by dividing both sides of the equation by y and by $e^{2x}+3$ ($y \neq 0$).

$$\frac{dy}{y} = -\frac{e^{2x}}{e^{2x}+3}dx$$

3) We integrate each side of the obtained equation:

$$\int \frac{dy}{y} = -\int \frac{e^{2x}}{e^{2x}+3}dx$$

$$\int \frac{dy}{y} = -\frac{1}{2}\int \frac{d(e^{2x}+3)}{e^{2x}+3}$$

$$\ln|y| = -\frac{1}{2}\ln|e^{2x}+3| + \ln|C|$$

We simplify the obtained solution by using the properties of a logarithmic function:

$$\ln|y| = \ln(e^{2x}+3)^{-\frac{1}{2}} + \ln|C|$$

$$\ln|y| = \ln\left|C \cdot (e^{2x}+3)^{-\frac{1}{2}}\right|$$

$$y = C \cdot (e^{2x}+3)^{-\frac{1}{2}}$$

As a result, we obtain the general solution of the given differential equation in the form:

$$y = \frac{C}{\sqrt{e^{2x}+3}}$$

4) We check for possibly missed solutions due to dividing by y:

$y = 0$ satisfies the given differential equations, and it can be obtained from the general solution at C=0. Therefore, it is not a singular solution.

Excercise 8.4.

Solve the initial value problem (Cauchy problem)

$$y' + y\tan x = \frac{1}{\cos x}, \qquad y(\pi) = 0.$$

Solution:

This is a first-order linear differential equation, which is to be solved using substitution y=UV.

1) We substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given differential equation:

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$$U'V + UV' + UV \cdot tanx = \frac{1}{cosx}$$

$$U'V + U(V' + V \cdot tanx) = \frac{1}{cosx}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' + V \cdot tanx = 0$$

Then the last equation in step 1 can be written as a system of two equations:

$$\begin{cases} V' + V \cdot tanx = 0 \\ U'V = \dfrac{1}{cosx} \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{dV}{dx} = -V \cdot tanx$$

$$\frac{dV}{V} = -tanx\, dx$$

$$\int \frac{dV}{V} = -\int tanx\, dx$$

The right-hand side is equal to

$$-\int tanx\, dx = -\int \frac{sinx}{cosx}\, dx = \int \frac{1}{cosx}\, d(cosx) = \ln|cosx| + C_1$$

Therefore,

$$\ln|V| = \ln|cosx| + C_1$$

Assuming $C_1 = 0$, we have

$$\ln|V| = \ln|\cos x|$$

$$V = \cos x$$

4) We substitute the obtained V back into the second equation of the system:

$$U'V = \frac{1}{cosx}$$

$$U' cosx = \frac{1}{cosx}$$

We simplify and solve the obtained above equation:

$$U' = \frac{1}{\cos^2 x}$$

$$U = \int U'dx = \int \frac{1}{\cos^2 x} dx = tanx + C$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

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$$y = UV = (\tan x + C) \cdot \cos x = \left(\frac{\sin x}{\cos x} + C\right) \cdot \cos x = \sin x + C \cdot \cos x$$

As a result, the general solution of the differential equation is

$$y = \sin x + C \cdot \cos x$$

6) We solve the initial value problem, it means that we should find only one particular solution of the differential equation that satisfies the initial condition $y(\pi) = 0$, i.e. the value of the function y(x) must to be equal to 0 at $x = \pi$. In order to find this particular solution, we substitute $y = 0$ and $x = \pi$ into the general solution.

$$0 = \sin\pi + C \cdot \cos\pi$$

$$0 = 0 + C \cdot (-1) \;\rightarrow\; -C = 0 \;\rightarrow\; C = 0.$$

We substitute the obtained value of the constant C onto the general solution and get the particular solution of the given initial value problem

$$y = \sin x$$

Exercise 8.5.

Solve the differential equation $y' - 3x^2y = x \cdot e^{x^3}$.

Solution:

This is a first-order linear differential equation. We will solve it by the substitution y=UV.

1) We substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given differential equation:

$$U'V + UV' - 3x^2 \cdot UV = x \cdot e^{x^3}$$

$$U'V + U(V' - V \cdot 3x^2) = x \cdot e^{x^3}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' - V \cdot 3x^2 = 0$$

The last equation in step 1 can be written as a system of two equations:

$$\begin{cases} V' - V \cdot 3x^2 = 0 \\ U'V = x \cdot e^{x^3} \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{dV}{dx} = V \cdot 3x^2$$

$$\frac{dV}{V} = 3x^2\, dx$$

$$\int \frac{dV}{V} = \int 3x^2\, dx$$

$$\ln|V| = x^3 + C_1$$

On assuming $C_1 = 0$, we have:

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$$\ln|V| = x^3 \quad \rightarrow \quad V = e^{x^3}$$

4) We substitute the obtained *V* back into the second equation of the system:

$$U'V = x \cdot e^{x^3} \quad \rightarrow \quad U'e^{x^3} = x \cdot e^{x^3}$$

We simplify and solve the equation for *U*:

$$U' = x$$

$$U = \int U'dx = \int x\,dx = \frac{x^2}{2} + C$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

$$y = UV = \left(\frac{x^2}{2} + C\right) \cdot e^{x^3}$$

Exercise 8.6.

Solve the initial value problem

$$(1 + x^2)y' = 2xy + (1 + x^2)^2\,, \qquad y(1) = 4.$$

Solution:

First, we rewrite the given equation in the form

$$(1 + x^2)y' - 2xy = (1 + x^2)^2$$

We divide both sides of the equation by $(1 + x^2)$:

$$y' - \frac{2xy}{(1 + x^2)} = (1 + x^2)$$

Now it is clear, that this differential equation is a first-order linear differential equation, which can be solved by substitution *y=UV* (Bernoulli method).

1) We substitute function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the differential equation:

$$U'V + UV' - \frac{2x \cdot UV}{(1 + x^2)} = (1 + x^2)$$

$$U'V + U\left(V' - \frac{2x \cdot V}{(1 + x^2)}\right) = (1 + x^2)$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' - \frac{2x \cdot V}{(1 + x^2)} = 0$$

The last equation in step 1 can be written as a system of two equations:

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$$\begin{cases} V' - \dfrac{2x \cdot V}{(1+x^2)} = 0 \\ \mathrm{U'V} = (1+x^2) \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{dV}{dx} = \frac{2x \cdot V}{(1+x^2)}$$

$$\int \frac{\mathrm{dV}}{\mathrm{V}} = \int \frac{2x}{(1+x^2)}\, dx$$

To evaluate integral on the right-hand side of the equation, we use $2xdx = d(x^2) = d(1+x^2)$,

$$\int \frac{2x}{(1+x^2)}\, dx = \int \frac{1}{(1+x^2)}\, d(x^2+1) = \ln|x^2+1| + C_1$$

Therefore,

$$\ln|\mathrm{V}| = \ln|x^2+1| + \mathrm{C}_1$$

Assuming $\mathrm{C}_1 = 0$, we have

$$\ln|\mathrm{V}| = \ln|x^2+1| \quad \rightarrow \quad \mathrm{V} = x^2+1$$

4) We substitute the obtained *V* back into the second equation of the system:

$$U'V = x^2+1$$

$$U'(x^2+1) = x^2+1$$

Simplify and solve the equation for *U*:

$$U' = 1$$

$$\mathrm{U} = \int \mathrm{U'dx} = \int 1\,\mathrm{dx} = x + \mathrm{C}$$

5) Substituting the obtained U and V into *y = UV* we get the general solution for the given linear differential equation:

$$\mathrm{y} = \mathrm{UV} = (\mathrm{x}+\mathrm{C}) \cdot (x^2+1)$$

As a result, the general solution of the differential equation is

$$\mathrm{y} = (\mathrm{x}+\mathrm{C}) \cdot (x^2+1)$$

In order to find a particular solution, we substitute $\mathrm{y} = 4$ and x$= 1$ into the general solution.

$$4 = (1+\mathrm{C}) \cdot (1^2+1)$$

$$4 = (1+C) \cdot 2 \quad \rightarrow \quad \mathrm{C}+1 = 2 \quad \rightarrow \quad \mathrm{C} = 1$$

Substitute the obtained value of the constant C onto the general solution to get the particular solution of the given initial value problem:

$$\mathrm{y} = (\mathrm{x}+1) \cdot (x^2+1) = x^3 + x^2 + x + 1$$

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8.3 SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

In this section we shortly consider basic concepts for second-order differential equations. The second-order linear differential equations with constant coefficients and two solving methods will be considered in detail, i.e. the method of variation of constants and the method of undetermined coefficients.

8.3.1 *Basic concepts for second-order differential equations. Second-order linear differential equations.*

A second-order differential equation can be written in the general (implicit) form

$$F(x, y, y', y'') = 0$$

or in the explicit form

$$y'' = f(x, y, y')$$

where *y=y(x)* is an unknown function.

The general solution of a second-order differential equation involves two arbitrary constants C_1 and C_2. It can be written in the explicit form $y = \varphi(x, C_1, C_2)$ or the implicit form $\Phi(x, y, C_1, C_2) = 0$.

Example 8.12

Consider the simplest second-order differential equation

$$y'' = 6x$$

The unknown function *y(x)* is found by integrating both sides of the equation two times with respect to x:

$$y' = \int 6x dx = 3x^2 + C_1$$

$$y = \int (3x^2 + C_1) dx = x^3 + C_1 x + C_2$$

where C_1, C_2 are arbitrary integration constants.

The general solution of the given equation is:

$$y = x^3 + C_1 x + C_2$$

In order to find only one particular solution of a second-order differential equation, two additional conditions are necessary. These additional conditions can be given as

1) *Initial value conditions*, when the functions' y(x) and y'(x) values are prescribed at defined x_0 value of x:

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$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

2) *Boundary conditions*, when the function's y(x) values are prescribed at different x_1 and x_2 values of x.

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2$$

Example 8.13

Let us consider the differential equation from Example 8.12 as an initial value problem

$$y'' = 6x \text{ and } y(0) = 1\,,\, y'(0) = 2$$

The general solution of the differential equation from Example 8.11 is

$$y = x^3 + C_1x + C_2.$$

In order to find the corresponding C_1 and C_2 values, we do the following:

1) Substitute $x = 0$ and $y = 1$ (i.e., the initial condition for *y*) into the general solution:

$$1 = 0^3 + C_1 0 + C_2$$

2) Find a *y'(x)* derivative of the general solution y (x) and substitute $x = 0$ and $y' = 2$ into obtained expression:

$$y' = 3x^2 + C_1$$

$$2 = 3 \cdot 0^2 + C_1$$

As a result, we find constants corresponding to the initial conditions:

$$C_1 = 2 \quad \text{and} \quad C_2 = 1$$

The solution of the initial value problem is found by substituting obtained C_1 and C_2 values into the general solution, and the particular solution that satisfies the given initial condition is:

$$y = x^3 + 2x + 1$$

In the following chapter we will consider second-order linear differential equations with constant coefficients.

Definition: Second-order linear differential equations

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A second-order differential equation is called a ***linear differential equation,*** if it can be written in the form

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x)$$

where $a_1(x)$, $a_2(x)$ and $a_3(x)$ are continuous functions and $a_1(x) \neq 0$.

Definition: Homogeneous and Nonhomogeneous linear differential equations

1) If the function $f(x)$ on the left-hand side of a linear equation is not equal to zero $(f(x) \neq 0)$, then the differential equation is called a ***nonhomogeneous linear equation***:

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x)$$

2) If the function $f(x)$ on the left-hand side of a linear equation is equal to zero $(f(x) = 0)$, then the differential equation is called a ***homogeneous linear equation:***

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = 0$$

Definition: Second-order linear differential equations with constant coefficients

A second-order linear differential equation is called *a **linear differential equation with constant coefficients*** if coefficients before y'', y' and y are constants

$$a_1y'' + a_2y' + a_3y = 0$$

where a_1 , a_2 and a_3 are constants and $a_1 \neq 0$.

8.3.2 *Second-order linear Homogeneous differential equations with constant coefficients*

Consider a second-order **linear homogeneous differential equation with constant coefficients**:

$$a_1y'' + a_2y' + a_3y = 0$$

where a_1 , a_2 and a_3 are some constant coefficients and $a_1 \neq 0$.

Solution method:

For each of the linear homogeneous differential equation with constant coefficients can be written the, so-called, *characteristic (*also called *auxiliary)* equation:

$$a_1k^2 + a_2k + a_3 = 0$$

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation. There exist three cases, as follows:

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1. The discriminant of the characteristic quadratic equation $\underline{D > 0}$.

   In this case, the roots of the characteristic equations ***are real and distinct*** $k_1 \neq k_2$, and the general solution of the homogeneous differential equation in this case has the form:

   $$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

   where C_1 and C_2 are arbitrary real numbers.

2. The discriminant of the characteristic equation $\underline{D = 0}$.

   In this case, the roots **are real and** equal $k_1 = k_2 = k$ (***repeated***), and the general solution of the differential equation has the form:

   $$y = C_1 e^{kx} + C_2 x e^{kx} \quad \text{or} \quad y = (C_1 + C_2 x) e^{kx}$$

3. The discriminant of the characteristic equation $\underline{D < 0}$.

   In this case, the roots **are complex** and conjugate, $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$ $(i = \sqrt{-1})$ and the general solution is written as

   $$y = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$$

Example 8.14

Let us consider the following linear differential equation with constant coefficients:

$$y'' + 3y' - 10y = 0$$

The corresponding characteristic (auxiliary) equation is

$$k^2 + 3k - 10 = 0$$

The discriminant of this equation $D = 49 > 0$; therefore, the roots are real and distinct:

$$k_1 = 2 \quad \text{and} \quad k_2 = -5$$

Then the general solution of the differential equation is

$$y = C_1 e^{2x} + C_2 e^{-5x}.$$

Example 8.15

Consider the equation:

$$y'' - 4y' + 4y = 0$$

Its characteristic (auxiliary) equation is

$$k^2 - 4k + 4 = 0$$

The discriminant of the quadratic equation $D = 0$, and the roots are real and repeated:

$$k_1 = k_2 = 2$$

The general solution of the differential equation is

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$$y = C_1 e^{2x} + C_2 x e^{2x}$$

Example 8.16

Consider the equation:

$$y'' + 2y' + 10y = 0$$

Its characteristic (auxiliary) equation is:

$$k^2 + 2k + 10 = 0$$

The discriminant of the quadratic equation $D = -36 < 0$, and the roots complex and conjugate:

$$k_1 = -1 + 3i \quad \text{and} \quad k_2 = -1 - 3i$$

The general solution of the differential equation is

$$y = C_1 e^{-1 \cdot x} \cos 3x + C_2 e^{-1 \cdot x} \sin 3x$$

8.3.3 *Second-order linear Nonhomogeneous differential equations with constant coefficients*

A **nonhomogeneous** linear differential equation with constant coefficient has the form

$$a_1 y'' + a_2 y' + a_3 y = f(x)$$

where a_1 , a_2 and a_3 are arbitrary constants and $a_1 \neq 0$.

For each nonhomogeneous linear differential equation its related homogeneous differential equation can be written as

$$a_1 y'' + a_2 y' + a_3 y = 0$$

1.1.1.1 Theorem.

A general solution of a nonhomogeneous equation is the sum of the general solution $y_c(x)$ of the related homogeneous equation and a *particular* solution $Y(x)$ of the nonhomogeneous equation:

$$y = y_c(x) + Y(x)$$

There exist two general approaches to find a particular solution $Y(x)$ of a nonhomogeneous differential equation. These are the Method of Undetermined Coefficients, and the Method of Variation of Constants.

1.1.1.2 Method of Variation of Constants

The Lagrangian constant variation method can be used for any type of function *f(x)* on the right-hand side of the nonhomogeneous linear differential equation.

Steps of solving:

1) First, solve an associated homogeneous equation

$$a_1 y'' + a_2 y' + a_3 y = 0$$

and find the general solution $y_c(x)$ of this equation. The general solution of the homogeneous equation contains two constants C_1 and C_2 and can be written in the form

$$y_c = C_1 \cdot y_1 + C_2 \cdot y_2$$

where C_1 , C_2 are constants and functions y_1, y_2 depend on the roots of the characteristic equation.

2) Replace the constants C_1 and C_2 with arbitrary (still unknown) functions $C_1(x)$ and $C_2(x)$ and find the ***general solution*** of the ***given nonhomogenous equation*** in the form

$$y = C_1(x) \cdot y_1 + C_2(x) \cdot y_2$$

3) Taking into account that $y = C_1(x)y_1 + C_2(x)y_2$ satisfies the given nonhomogeneous equation with the right-hand side f(x), it can be shown that the unknown functions $C_1(x)$ and $C_2(x)$ can be determined from the system of two equations:

$$\begin{cases} C'_1(x) \cdot y_1 + C'_2(x) \cdot y_2 = 0 \\ C'_1(x) \cdot y'_1 + C'_2(x) \cdot y'_2 = \dfrac{f(x)}{a_1} \end{cases}$$

4) Find $C_1{}'(x)$ and $C_2{}'(x)$ from the system.

5) By integration find $C_1(x) = \int C_1{}'(x)dx$ and $C_2(x) = \int C_2{}'(x)dx$

6) Substitute the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution.

Example 8.17

Solve the equation:

$$y'' + 9y = \frac{1}{\cos 3x}$$

We solve an associated homogeneous equation

$$y'' + 9y = 0$$

Its characteristic (auxiliary) equation is

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$$k^2 + 9 = 0 \quad => \quad k^2 = -9$$

The roots are complex and conjugate:

$$\mathrm{k}_1 = \sqrt{-9} = 3\mathrm{i} = 0 + 3\mathrm{i} \quad \text{and} \quad \mathrm{k}_2 = -\sqrt{-9} = -3\mathrm{i} = 0 - 3\mathrm{i}$$

The general solution of the associated *homogeneous differential* equation is

$$\mathrm{y_c} = \mathrm{C_1 e^{0\cdot x} cos3x} + \mathrm{C_2 e^{0\cdot x} sin3x}$$

or

$$\mathrm{y_c} = \mathrm{C_1 cos3x} + \mathrm{C_2 sin3x}$$

where $\mathrm{C_1}$ and $\mathrm{C_2}$ are arbitrary constants.

2) We replace the constants $\mathrm{C_1}$ and $\mathrm{C_2}$ with the arbitrary (still unknown) functions $\mathrm{C_1(x)}$ and $\mathrm{C_2(x)}$ and find the *general solution* of the given *nonhomogeneous* differential equation in the form:

$$\mathrm{y} = \mathrm{C_1(x) cos3x} + \mathrm{C_2(x) sin3x}$$

3) To determine the unknown functions $\mathrm{C_1(x)}$ and $\mathrm{C_2(x)}$, we write a system of equations for derivatives of the unknown functions

$$\begin{cases} \mathrm{C'_1(x)} \cdot \cos 3x + \mathrm{C'_2(x)} \cdot \sin 3x = 0 \\ \mathrm{C'_1(x)} \cdot (\cos 3x)' + \mathrm{C'_2(x)} \cdot (\sin 3x)' = \dfrac{1}{\cos 3x} \end{cases}$$

The system can be written in the form

$$\begin{cases} \mathrm{C'_1(x)} \cdot \cos 3x + \mathrm{C'_2(x)} \cdot \sin 3x = 0 \\ \mathrm{C'_1(x)} \cdot (-3\sin 3x) + \mathrm{C'_2(x)} \cdot 3\cos 3x = \dfrac{1}{\cos 3x} \end{cases}$$

4) We will solve the system by using Cramer's rule, so that we need to find a determinant of the coefficient matrix:

$$\text{D=} \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

and the determinants

$$D_1 = \begin{vmatrix} 0 & \sin 3x \\ \dfrac{1}{\cos 3x} & 3\cos 3x \end{vmatrix} = 0 - \frac{\sin 3x}{\cos 3x} = -\tan 3x$$

$$D_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \dfrac{1}{\cos 3x} \end{vmatrix} = \frac{\cos 3x}{\cos 3x} - 0 = 1$$

Then

$$C'_1(x) = \frac{D_1}{D} = -\frac{\tan 3x}{3} \qquad \text{and} \qquad C'_2(x) = \frac{D_2}{D} = \frac{1}{3}$$

5) We find unknown functions $C_1(x)$ and $C_2(x)$ by integrating:

$$C_1(x) = \int C'_1(x)\,dx \quad \text{and} \quad C_2(x) = \int C'_2(x)\,dx \;.$$

$$C_1(x) = \int \left(-\frac{\tan 3x}{3}\right) \mathrm{dx} = -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x}\,\mathrm{dx} = \frac{1}{9}\int \frac{\mathrm{d}(\cos 3x)}{\cos 3x} = \frac{1}{9}\ln|\cos 3x| + \tilde{C}_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int \frac{1}{3}\,\mathrm{dx} = \frac{1}{3}\int 1\,\mathrm{dx} = \frac{1}{3}\mathrm{x} + \tilde{C}_2$$

Thus,

$$C_1(x) = \frac{1}{9}\ln|\cos 3x| + \tilde{C}_1 \quad \text{and} \quad C_2(x) = \frac{1}{3}\mathrm{x} + \tilde{C}_2$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constants.

6) Substitute the obtained functions $C_1(x)$ and $C_2(x)$ into the form of general solution of the nonhomogeneous differential equation:

$$y = \left(\frac{1}{9}\ln|\cos 3x| + \tilde{C}_1\right)\cos 3x + \left(\frac{1}{3}\mathrm{x} + \tilde{C}_2\right)\sin 3x$$

As the result, the general solution of given nonhomogeneous differential equation is

$$y = \tilde{C}_1\cos 3x + \tilde{C}_2\sin 3x + \frac{1}{9}\ln|\cos 3x| \cdot \cos 3x + \frac{1}{3}\mathrm{x}\cdot\sin 3x$$

or

$$y = C_1\cos 3x + C_2\sin 3x + \frac{1}{9}\ln|\cos 3x| \cdot \cos 3x + \frac{1}{3}\mathrm{x}\cdot\sin 3x$$

where C_1 and C_2 are also arbitrary constants.

Note that the sum of the two first terms in the obtained solution is the general solution for associated homogenous differential equation, and the sum of the last two terms is the particular solution of the nonhomogeneous differential equation.

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8.3.4 *Method of Undetermined Coefficients*

Consider second-order nonhomogeneous differential equations with right-hand functions that has derivatives that vary little (in type of function) from their parent functions. These functions are: polynomial $P_n(x)$ functions, exponential functions $e^{\alpha x}$, trigonometric functions (sine and cosine ($\sin\beta x$, $\cos\beta x$)), as well as the sum, difference and multiplication of these functions. In this case we can predict the form of solution of this differential equation taking into account the form of its right-hand function.

The main idea of the Method of Undetermined Coefficients is to construct the form of a particular solution Y(x) of the given nonhomogenous equation corresponding to the form (based on the form) of a function f(x) on the right side of the equation. Y(x) is written down as a function with undefined coefficients, then is substituted into the equation and the coefficients are found.

As was mentioned before, this method works only for a restricted class of functions on the right-hand side of the equation, such as

$$f(x) = P_n(x)e^{\alpha x}$$

$$f(x) = (P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)) \cdot e^{\alpha x}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degrees n and m, respectively.

In both cases a choice for the particular solution should match the structure of the right-hand side function of the nonhomogeneous equation. It depends on the right side of the equation as well as on the roots of the characteristic equation.

Let us consider in detail how to construct the form of a particular solution Y(x) of a given nonhomogenous equation.

Consider three cases for a function on the right-hand side of the equation:

1) $f(x) = P_n(x)e^{\alpha x}$ $(\beta = 0)$

The particular solution has the same form as f(x), only instead of polynomial $P_n(x)$ we write polynomial with undefined coefficients. Furthermore, if the coefficient α in the *argument of the exponential* function *coincides with a root* of the auxiliary (characteristic) equation, the particular solution will contain the additional factor x^s, where s is the order of the root α in the characteristic equation.

This means that the particular solution *Y* is written down in the form

$$Y = \widetilde{P_n}(x)e^{\alpha x} \cdot x^s$$

where

a) $\widetilde{P_n}(x)$ is a polynomial of order *n* with unknown coefficients, i.e.

if n=0, then $\widetilde{P_0}(x) = A$;

if n=1, then $\widetilde{P_1}(x) = Ax + B$;

if n=2, then $\widetilde{P_2}(x) = Ax^2 + Bx + C$;

and so on.

b) To find the power s of factor x^s ,we compare the coefficient α in the power of the exponential function with the roots $k_1$and k_2 of the auxiliary equation:

if $\alpha \neq k_1$ and $\alpha \neq k_2$ then $\underline{s = 0}$;

if $\alpha = k_1$ and $\alpha \neq k_2$ OR $\alpha \neq k_1$ and $\alpha = k_2$ then $\underline{s = 1}$;

if $\alpha = k_1 = k_2$ then $\underline{s = 2}$;

2) $f(x) = e^{\alpha x}(N\cos(\beta x) + M\sin(\beta x))$, where *N, M* are constants

$f(x) = e^{\alpha x}N\cos(\beta x) \iff f(x) = e^{\alpha x}(N\cos(\beta x) + \mathbf{0} \cdot \sin(\beta x))$ *(M=0)*

$f(x) = e^{\alpha x}M\sin(\beta x) \iff f(x) = e^{\alpha x}(\mathbf{0} \cdot \cos(\beta x) + M \cdot \sin(\beta x))$ *(N=0)*

The particular solution has the same form as f(x) only instead of constants *N and M* we write unknown coefficients. Furthermore, if the number $\alpha + \beta i$ *coincides with a root* of the auxiliary (characteristic) equation, the particular solution will contain the additional multiplier x^s, where s is the order of the root $\alpha + \beta i$ in the characteristic equation.

This means that the particular solution *Y* is written down in the form

$$Y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \cdot x^s$$

where A and B are unknown coefficients.

To find the power s of multiplier x^s , we compare the number $\alpha + \beta i$ with the roots $k_1$and k_2 of the auxiliary equation:

if $\alpha + i\beta \neq k_1$ and $\alpha + i\beta \neq k_2$ then $\underline{s = 0}$;

if $\alpha + i\beta = k_1$ or $\alpha + i\beta = k_2$ then $\underline{s = 1}$.

3) $f(x) = e^{\alpha x}(P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x))$

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$f(x) = e^{\alpha x} P_n(x) \cos(\beta x) \iff f(x) = e^{\alpha x}(P_n(x)\cos(\beta x) + \mathbf{0} \cdot \sin(\beta x))$

$f(x) = e^{\alpha x} Q_n(x) \sin(\beta x) \iff f(x) = e^{\alpha x}(\mathbf{0} \cdot \cos(\beta x) + Q_n(x) \cdot \sin(\beta x))$

where $P_n(x)$ and $Q_m(x)$ are polynomials of order n and m respectively.

In these cases the particular solution is found in the form

$$Y = e^{\alpha x}\left(\widetilde{P_k}(x)\cos(\beta x) + \widetilde{Q_k}(x)\sin(\beta x)\right) \cdot x^s$$

where $\widetilde{P_k}(x)$ and $\widetilde{Q_k}(x)$ are *polynomials* of order k *with unknown coefficients* and $k = \max(n, m)$.

To find the power s of multiplier x^s , we compare the number $\alpha + \beta i$ with the roots $k_1$and k_2 of the auxiliary equation:

if $\alpha + i\beta \neq k_1$ and $\alpha + i\beta \neq k_2$ then $\underline{s = 0}$;

if $\alpha + i\beta = k_1$ or $\alpha + i\beta = k_2$ then $\underline{s = 1}$.

The unknown coefficients are determined by substitution of the expected type of the particular solution into the original nonhomogeneous differential equation.

<u>Scheme of solving:</u>

1) Solve the corresponding homogeneous differential equation $\mathrm{a}_1\mathrm{y}'' + \mathrm{a}_2\mathrm{y}' + \mathrm{a}_3\mathrm{y} = 0$;

2) By the form of function f(x) on the right-hand side of the equation, write down the form of a particular solution Y with undefined coefficients;

3) Find Y’ and Y’’;

4) Determine the undefined coefficients A, B, C by substitution of the particular solution Y and its derivatives into the given original nonhomogeneous differential equation.

5) Substitute obtained coefficients into the form of the particular solution Y.

6) Write down the general solution of the given nonhomogeneous differential equation as

$$\mathrm{y} = \mathrm{y_c}(\mathrm{x}) + Y(\mathrm{x})$$

where $\mathrm{y_c}(\mathrm{x})$ is the general solution of the related homogeneous equation, and $Y(\mathrm{x})$ is a particular solution of the given nonhomogeneous equation.

Example 8.18

Let us solve the equation:

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$$y'' - 2y' = x^2 + 5x - 1$$

1) First, we solve the associated ***homogeneous equation***:

$$y'' - 2y' = 0$$

The auxiliary equation for this equation is:

$$k^2 - 2k = 0 \quad => \quad k \cdot (k-2) = 0$$

The roots of the auxiliary equation are real and distinct:

$$k_1 = \mathbf{0} \quad \text{and} \quad k_2 = 2$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1 e^{0 \cdot x} + C_2 e^{2 \cdot x}$$

or

$$y_c = C_1 + C_2 e^{2x}$$

where C_1 and C_2 are constants.

2) We write down the form for the particular solution Y, taking into account form of function $\underline{f(x) = x^2 + 5x - 1}$ on the right-hand side of the equation.

The function $f(x)$ can be written in the form: $f(x) = (x^2 + 5x - 1) \cdot e^{\mathbf{0} \cdot x}$.

In this case the particular solution Y has the form: $Y = \widetilde{P_n}(x) e^{\alpha x} \cdot x^s$.

a) The function $f(x)$ has a polynomial with degree 2 before the exponential function (n=2), therefore the polynomial $\widetilde{P_n}(x)$ must also be a polynomial with degree 2, but with undefined coefficients: $\widetilde{P_2}(x) = Ax^2 + Bx + C$

b) The coefficient in the argument of the exponential function is ***α=0***. It coincides with one root of the auxiliary (characteristic) equation: $\alpha = k_1 = 0$, therefore $s = 1$ and the particular solution will contain the additional factor x^1 .

Thus, the particular solution of the differential equation *Y* has the form:

$$Y = (Ax^2 + Bx + C) \cdot e^{\mathbf{0} \cdot x} \cdot x^1 = (Ax^2 + Bx + C) \cdot x$$

or

$$Y = Ax^3 + Bx^2 + Cx$$

3) We find first- and second-order derivatives for Y:

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$$Y' = (Ax^3 + Bx^2 + Cx)' = 3Ax^2 + 2Bx + C$$

$$Y'' = (3Ax^2 + 2Bx + C)' = 6Ax + 2B$$

4) We substitute them into the given nonhomogeneous differential equation:

$$y'' - 2y' = x^2 + 5x - 1.$$

As a result, we have:

$$6Ax + 2B - 2(3Ax^2 + 2Bx + C) = x^2 + 5x - 1$$

We simplify the left-hand expression:

$$-6Ax^2 + \ 6Ax - 4Bx + 2B - 2C = x^2 + 5x - 1$$

We group coefficients with the same powers of x on the left-hand side of the equation:

$$-6Ax^2 + \ (6A - 4B)x + 2B - 2C = 1 \cdot x^2 + 5x + (-1)$$

The right and left sides of the equation are equal for every $x \in R$**.** It would be possible only if the coefficients at the same powers of x on the right and left sides of the equation are equal:

The coefficients at x^2: $-6A = 1$;

The coefficients at x: $6A - 4B = 5$;

The coefficients at x^0: $2B - 2C = -1$.

Solve the obtained system of equations: $\begin{cases} -6A = 1 \\ 6A - 4B = 5 \\ 2B - 2C = -1 \end{cases}$

From the first equation of the system we have: $A = -\frac{1}{6}$.

From the second equation: $6A - 4B = 5$ ➔ $6 \cdot \left(-\frac{1}{6}\right) - 4B = 5$ ➔ $-4B = 6$

$$B = -\frac{3}{2}$$

From the third equation: $2B - 2C = -1$ ➔ $2 \cdot \left(-\frac{3}{2}\right) - 2C = -1$ ➔ $-2C = 2$

$$C = -1$$

Substitute the obtained coefficients into the form of the particular solution Y:

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$$Y = -\frac{1}{6}x^3 - \frac{3}{2}x^2 - x$$

As a result, the general solution of the given nonhomogeneous equation is :

$$\mathrm{y} = \mathrm{y_c} + \mathrm{Y} = \mathrm{C_1} + \mathrm{C_2} e^{2x} - \frac{1}{6}x^3 + \frac{3}{2}x^2 + 2x$$

Example 8.19

Solve the equation:

$$y'' + y' - 2y = xe^{2x}$$

1) The associated homogeneous equation is

$$y'' + y' - 2y = 0$$

Its auxiliary equation is

$$k^2 + k - 2 = 0$$

The roots for this equation are real and distinct:

$$\mathrm{k_1} = 1 \quad \text{and} \quad \mathrm{k_2} = -2$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$\mathrm{y_c} = \mathrm{C_1 e^{1\cdot x}} + \mathrm{C_2 e^{-2\cdot x}}$$

where $\mathrm{C_1}$ and $\mathrm{C_2}$ are constants.

2) We construct the form of a particular solution *Y* by taking into account the form of the function on the right-hand side of the equation $(x) = xe^{2x}$.

For such function (x) , the particular solution *Y* has the form: $Y = \widetilde{P_n}(x)e^{\alpha x} \cdot x^s$.

a) The function $f(x)$ has a polynomial with degree 1 before the exponential function (n=1), therefore the polynomial with undefined coefficients $\widetilde{P_n}(x)$ must also be a polynomial with degree 1: $\widetilde{P_1}(x) = Ax + B$.

b) The coefficient in the power of the exponential function is $\alpha = 2$. It ***does not* coincide with any root** of the auxiliary (characteristic) equation: $\alpha \neq k_1$ and $\alpha \neq k_2$, therefore, $s = 0$ and the particular solution does not contain an additional factor.

Thus, a particular solution Y of the differential equation Y has the form

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$$Y = (Ax + B) \cdot e^{2 \cdot x} \cdot x^0 = (Ax + B) \cdot e^{2 \cdot x}$$

3) We find first- and second-order derivatives for Y:

$$Y' = \left((Ax + B)e^{2 \cdot x}\right)' = (Ax + B)' \cdot e^{2x} + (Ax + B) \cdot (e^{2 \cdot x})' = Ae^{2x} + (Ax + B) \cdot 2e^{2 \cdot x}$$

It can be also written as $Y' = (2Ax + 2B + A)e^{2x}$.

$$Y" = \left((2Ax + 2B + A)e^{2x}\right)' = (2Ax + 2B + A)' \cdot e^{2x} + (2Ax + 2B + A) \cdot (e^{2x})' =$$

$$= 2Ae^{2x} + (2Ax + 2B + A) \cdot 2e^{2x} = (4Ax + 4B + 4A) \cdot e^{2x}$$

So, $Y" = (4Ax + 4B + 4A) \cdot e^{2x}$.

4) Substitute the obtained Y”, Y’ and Y into the given nonhomogeneous differential equation:

$$y'' + y' - 2y = xe^{2x}.$$

As a result, we have:

$$(4Ax + 4B + 4A) \cdot e^{2x} + (2Ax + 2B + A)e^{2x} - 2(Ax + B) \cdot e^{2 \cdot x} = xe^{2x}$$

We simplify the expression:

$$(4Ax + 4B + 4A + 2Ax + 2B + A - 2Ax - 2B) \cdot e^{2x} = xe^{2x}$$

$$4Ax + 4B + 5A = x$$

or $$4Ax + 4B + 5A = 1x + 0.$$

The right and left sides of the equation are equal for every $\forall x \in R$. That would only be possible if the coefficients at the same powers of x on the right-hand side and left-hand side of the equation are equal.

The coefficients at x: $4A = 1$;

The coefficients at x^0: $4B + 5A = 0$.

That leads us to solving the system: $\begin{cases} 4A = 1 \\ 4B + 5A = 0 \end{cases}$.

It follows from the first equation of the system: $A = \frac{1}{4}$.

It follows from the second equation: $4B + 5 \cdot \frac{1}{4} = 0 \rightarrow 4B = -\frac{5}{4} \rightarrow B = -\frac{5}{16}$.

We substitute the obtained coefficients into the form of the particular solution Y:

$$Y = (Ax + B) \cdot e^{2x} = \left(\frac{1}{4}x - \frac{5}{16}\right) \cdot e^{2x}$$

As a result, the general solution of the given nonhomogeneous equation is

$$y = y_c + Y = C_1 e^x + C_2 e^{-2x} + \left(\frac{1}{4}x - \frac{5}{16}\right) \cdot e^{2x}$$

Example 8.20

Solve the equation

$$y'' + y = 3\cos x + 2\sin x.$$

1) The associated homogeneous equation is

$$y'' + y = 0$$

Its auxiliary equation is

$$k^2 + 1 = 0 \;\; \rightarrow \;\; k^2 = -1 \;\; \rightarrow \;\; k^2 = \pm\sqrt{-1} \;\; = \pm i$$

The roots of the auxiliary equation are complex and conjugated:

$$k_1 = i = 0 + 1 \cdot i \quad \text{and} \quad k_2 = -i = 0 - 1 \cdot i$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1 e^{0 \cdot x} \cos(1 \cdot x) + C_2 e^{0 \cdot x} \sin(1 \cdot x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

2) The function on the right-hand side of the equation is $\underline{f(x) = 3cosx + 2sinx}$.

The function $f(x)$ can be written as $f(x) = e^{0x}\big(3cos(1 \cdot x) + 2sin(1 \cdot x)\big)$.

For such function $f(x)$, the particular solution Y has the form:

$$Y = e^{\alpha x}\big(A\cos(\beta x) + B\sin(\beta x)\big) \cdot x^s.$$

The power of the exponential function in function $f(x)$ is $\boldsymbol{\alpha} = 0$ and the coefficient before *x* in the argument of cosine and sine is $\boldsymbol{\beta = 1}$.

The number $\boldsymbol{\alpha + i\beta = 0 + 1 \cdot i = i}$ **coincides with one root** of the auxiliary (characteristic) equation: $\alpha + i\beta = k_1$, therefore, $s = 1$ and the particular solution contains the factor x^1.

Thus, the particular solution Y of the differential equation Y has the form:

$$Y = e^{0\cdot x}(\mathrm{A}\cos x + \mathrm{B}\sin x)\cdot x^1 = (\mathrm{A}\cos x + \mathrm{B}\sin x)\cdot \mathrm{x}$$

3) We find first and second-order derivatives for Y:

$$Y' = \left((\mathrm{A}\cos x + \mathrm{B}\sin x)\cdot \mathrm{x}\right)' = (\mathrm{A}\cos x + \mathrm{B}\sin x)'\cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x)\cdot (\mathrm{x})' =$$

$$= (-A\sin x + B\cos x)\cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x)$$

$$Y'' = \left((-A\sin x + B\cos x)\cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x)\right)' =$$

$$= (-A\sin x + B\cos x)'\cdot \mathrm{x} + (-A\sin x + B\cos x)\cdot \mathrm{x}' + (\mathrm{A}\cos x + \mathrm{B}\sin x)' =$$

$$= (-A\cos x - B\sin x)\cdot \mathrm{x} + (-A\sin x + B\cos x) - A\sin x + B\cos x =$$

$$= (-A\cos x - B\sin x)\cdot \mathrm{x} - 2A\sin x + 2B\cos x$$

4) We substitute Y” and Y into the given nonhomogeneous differential equation:

As a result, we have:

$$(-A\cos x - B\sin x)\cdot \mathrm{x} - 2A\sin x + 2B\cos x + (\mathrm{A}\cos x + \mathrm{B}\sin x)\cdot \mathrm{x} = 3\cos x + 2\sin x$$

We simplify the obtained expression:

$$-2A\sin x + 2B\cos x = 3\cos x + 2\sin x$$

The right and left sides of the equation are equal for every $x \in R.$ It would be possible only if the coefficients at $sinx$ and $cosx$ on the right-hand side and left-hand side of the equation are equal:

The coefficients at $cosx$: $2B = 3 \Longrightarrow B = \frac{3}{2}$

The coefficients at $sinx$: $-2A = 2 \Longrightarrow A = -1$

We substitute the obtained coefficients into the form of a particular solution Y:

$$Y = \left(-1\cdot \cos x + \frac{3}{2}\cdot \sin x\right)\cdot \mathrm{x}$$

As a result, the general solution of the given nonhomogeneous equation is:

$$\mathrm{y} = \mathrm{y_c} + \mathrm{Y} = \mathrm{C_1}\cos x + \mathrm{C_2}\sin x + \left(-\cos x + \frac{3}{2}\sin x\right)\cdot \mathrm{x}$$

Superposition Principle

If the right side of a nonhomogeneous equation is the sum of several functions such as

$f(x) = P_n(x)e^{\alpha x}$ and $f(x) = (P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)) \cdot e^{\alpha x}$,

then a particular solution of the differential equation is also the sum of particular solutions constructed separately for each such function on the right-hand side expression.

Example 8.21

Find a general solution of the differential equation:

$$y'' - 2y' + y = e^x + 5\cos 3x.$$

1) The associated homogeneous equation:

$$y'' - 2y' + y = 0$$

The auxiliary equation for this equation is $k^2 - 2k + 1 = 0$

The roots of this equation are real and repeated: $k_1 = k_2 = 1$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1e^x + C_2xe^x$$

2) We see that the right-hand side of the given equation is the sum of two functions $f_1(x) = e^x$ and $f_2(x) = 5\cos 3x$. According to the superposition principle, a particular solution is a sum of particular solutions so that it can be expressed

$$Y = Y_1 + Y_2$$

where Y_1 is a particular solution for the differential equation $y'' - 2y' + y = e^x$

and Y_2 is a particular solution for the equation $y'' - 2y' + y = 5\cos 3x$.

a) First, we determine the function Y_1. In this case $\boldsymbol{f_1(x) = e^x}$ and we will be looking for a solution in the form

$$Y_1 = Ae^{\alpha x} \cdot x^s$$

The power of the exponential function is $\boldsymbol{\alpha = 1}$ and it coincides **with two roots** of the auxiliary (characteristic) equation: $\alpha = k_1 = k_2$, therefore, $s = 2$ and the particular solution Y_1 contains the factor x^2 .

Thus, the particular solution Y_1 of the first differential equation has the form

$$Y_1 = Ae^x \cdot x^2 = Ax^2e^x$$

3) We find first- and second-order derivatives for Y_1:

$$Y_1' = (Ax^2e^x)' = (Ax^2)' \cdot e^x + (Ax^2) \cdot (e^x)' = 2Ax \cdot e^x + (Ax^2) \cdot e^x = (2Ax + Ax^2) \cdot e^x$$

$$Y_1'' = \left((2Ax + Ax^2) \cdot e^x\right)' = (2Ax + Ax^2)' \cdot e^x + (2Ax + Ax^2) \cdot (e^x)' =$$

$$= (2A + 2Ax) \cdot e^x + (2Ax + Ax^2) \cdot e^x = (2A + 4Ax + Ax^2) \cdot e^x$$

Substitute Y_1' , Y_1'' and Y_1 into the corresponding nonhomogeneous differential equation:

$$y'' - 2y' + y = e^x,$$

we have

$$(2A + 4Ax + Ax^2) \cdot e^x - 2(2Ax + Ax^2) \cdot e^x + Ax^2e^x = e^x$$

We simplify the obtained expression:

$$(2A + 4Ax + Ax^2 - 4Ax - 2Ax^2 + Ax^2)e^x = e^x$$

and get: $2A = 1$ ➔ $A = 1/2$.

Then

$$Y_1 = \frac{1}{2}x^2e^x$$

b) <u>We determine the function Y_2.</u>

Due to the form of the function $\boldsymbol{f_2(x) = 5\cos 3x = e^{0x}(5 \cdot \cos 3x + 0 \cdot \sin 3x)}$, we seek for a solution in the form

$$Y_2 = e^{ox}(C \cdot \cos 3x + D \cdot \sin 3x) \cdot x^s$$

The power of the exponential function is $\alpha = 0$ and the coefficient before x in the argument of cosine and sine is $\boldsymbol{\beta = 3}$. The number $\boldsymbol{\alpha + i\beta = 0 + 3i = 3i}$ does not coincide with any root of the auxiliary equation, therefore, $s = 0$.

$$Y_2 = C \cdot \cos 3x + D \cdot \sin 3x$$

We find first- and second-order derivatives for Y_2:

$$Y_2' = (C\cos 3x + D\sin 3x)' = -3C\sin 3x + 3D\cos 3x$$

$Y_2'' = (-3C\sin 3x + 3D\cos 3x)' = -9C\cos 3x - 9D\sin 3x$.

After substituting $Y_2{}'$, $Y_2{}''$ and Y_2 into the corresponding nonhomogeneous differential equation:

$$y'' - 2y' + y = 5\cos 3x,$$

We have:

$$-9C\cos 3x - 9D\sin 3x - 2(-3C\sin 3x + 3D\cos 3x) + C\cos 3x + D\sin 3x = 5\cos 3x$$

$$(-8C - 6D)\cos 3x + (-8D + 6C)\sin 3x = 5\cos 3x$$

The coefficients at $cos 3x$: $-8C - 6D = 5$

The coefficients at $sin 3x$: $-8D + 6C = 0 \implies C = \frac{4D}{3}$

It follows from the first equation: $-8 \cdot \left(\frac{4D}{3}\right) - 6D = 5 \;\Rightarrow\; \frac{-50D}{3} = 5 \;\Rightarrow D = -\frac{3}{10}$

and $$C = \frac{4D}{3} = -\frac{4 \cdot 3}{3 \cdot 10} = -\frac{2}{5}$$

As a result,

$$Y_2 = -\frac{2}{5} \cdot \cos 3x - \frac{3}{10} \cdot \sin 3x$$

As a result, the general solution of the given nonhomogeneous equation is:

$$y = y_c + Y = y_c + Y_1 + Y_2$$

Then the general solution of the given differential equation is:

$$y = C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x - \frac{2}{5}\cos 3x - \frac{3}{10} \cdot \sin 3x$$

8.3.5 *Exercises*

Exercise 8.7.

Find a general and particular solution of the differential equation:

$$y'' - 4y' + 13y = 0\,, \quad y(0) = 6, \quad y'(0) = 1$$

Solution:

The auxiliary equation for the given differential equation is

$$k^2 - 4k + 13 = 0$$

The discriminant of the quadratic equation is $D = -36 < 0$, therefore, the roots are complex and conjugated:

$$k_1 = \frac{4 + \sqrt{-36}}{2} = 2 + 3 \cdot i \quad \text{and} \quad k_2 = \frac{4 - \sqrt{-36}}{2} = 2 - 3 \cdot i$$

It means that the general solution of the given differential equation is

$$y = C_1 e^{2 \cdot x} \cos(3x) + C_2 e^{2 \cdot x} \sin(3x)$$

In order to find the particular solution that satisfies the given initial conditions,

1) We substitute $x = 0$ and $y = 6$ (i.e., the initial condition) into the general solution:

$$6 = C_1 e^{2 \cdot 0} \cos(0) + C_2 e^{2 \cdot 0} \sin(0)$$

$$6 = C_1 \cdot 1 + C_2 \cdot 0$$

$$C_1 = 6$$

2) We find a y'(x) derivative of the general solution y (x).

$$y' = \left(e^{2 \cdot x} \cdot (C_1 \cos 3x + C_2 \sin 3x)\right)' = (e^{2 \cdot x})'(C_1 \cos 3x + C_2 \sin 3x) + e^{2 \cdot x}(C_1 \cos 3x + C_2 \sin 3x)' =$$

$$= 2e^{2 \cdot x}(C_1 \cos 3x + C_2 \sin 3x) + e^{2 \cdot x}(-3C_1 \sin 3x + 3C_2 \cos 3x) =$$

$$= e^{2 \cdot x}(2C_1 \cos 3x + 2C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x)$$

Thus, the derivative of the general solution is

$$y' = e^{2 \cdot x}(2C_1 \cos 3x + 2C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x)$$

We substitute $x = 0$ and $y' = 1$ from the initial conditions into the obtained expression:

$$1 = e^{2 \cdot 0}(2C_1 \cos 0 + 2C_2 \sin 0 - 3C_1 \sin 0 + 3C_2 \cos 0) = 2C_1 + 0 - 0 + 3C_2$$

$$1 = 2C_1 + 3C_2$$

We substitute $C_1 = 6$ into the obtained expression, then

$$1 = 12 + 3C_2$$

$$C_2 = -\frac{11}{3}$$

We substitute the obtained constants into the general solution. As a result, the particular solution of the given differential equation is

$$y = 6e^{2x}\cos(3x) - \frac{11}{3}e^{2x}\sin(3x)$$

Exercise 8.8.

Solve the equation:

$$y'' + 2y' + y = 3e^{-x}\sqrt{x+1}$$

Solution:

We use the Method of Variation of Constants

1) We solve the associated homogeneous equation:

$$y'' + 2y' + y = 0$$

Its characteristic (auxiliary) equation is

$$k^2 + 2k + 1 = 0$$

$$(k+1)^2 = 0$$

The roots of this equation are real and repeated

$$k_1 = k_2 = -1$$

The general solution of the associated homogeneous differential equation is

$$y_0 = C_1e^{-x} + C_2xe^{-x}$$

where C_1 and C_2 are constants.

2) We replace the constants C_1 and C_2 with arbitrary functions $C_1(x)$ and $C_2(x)$ and find the general solution of the given nonhomogeneous differential equation in the form

$$y = C_1e^{-x} + C_2xe^{-x}$$

3) To determine the unknown functions $C_1(x)$ and $C_2(x)$, we write a system of equations for derivatives of the unknown functions

$$\begin{cases} C'_1(x)\cdot e^{-x} + C'_2(x)\cdot xe^{-x} = 0 \\ C'_1(x)\cdot (e^{-x})' + C'_2(x)\cdot (xe^{-x})' = 3e^{-x}\sqrt{x+1} \end{cases}$$

After finding derivatives, we have

$$\begin{cases} C'_1(x)\cdot e^{-x} + C'_2(x)\cdot xe^{-x} = 0 \\ C'_1(x)\cdot (-e^{-x}) + C'_2(x)\cdot \left(1\cdot e^{-x} + x(-e^{-x})\right) = 3e^{-x}\sqrt{x+1} \end{cases}$$

The system can be written in the form

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$$\begin{cases} (C'_1(x) + C'_2(x) \cdot x)e^{-x} = 0 \\ \left(-C'_1(x) + C'_2(x) \cdot (1-x)\right) \cdot e^{-x} = 3e^{-x}\sqrt{x+1} \end{cases}$$

Let us simplify the system to the form

$$\begin{cases} C'_1(x) + C'_2(x) \cdot x = 0 \\ -C'_1(x) + C'_2(x) \cdot (1-x) = 3\sqrt{x+1} \end{cases}$$

It follows from the first equation of the system:

$$C'_1(x) = -C'_2(x) \cdot x$$

Substituting the obtained $C'_1(x)$ into the second equation of the system, it yields:

$$C'_2(x) \cdot x + C'_2(x) \cdot (1-x) = 3\sqrt{x+1}$$

As a result, we obtain $C'_2(x)$:

$$C'_2(x) = 3\sqrt{x+1}$$

Taking into account the expression for $C'_1(x)$, we have

$$C'_1(x) = -C'_2(x) \cdot x = -3x\sqrt{x+1}$$

4) We find the unknown functions $C_1(x)$ and $C_2(x)$ using integration

$$C_1(x) = \int C'_1(x)\,dx = \int -3x\sqrt{x+1}\,dx$$

To find this integral, we use the substitution $x + 1 = t^2$.

Then $x = t^2 - 1$ and $dx = (t^2 - 1)'dt = 2tdt$

$$\int -3x\sqrt{x+1}\,dx = -3\int (t^2-1)t \cdot 2tdt = -6\int (t^4 - t^2)dt = -6\left(\frac{t^5}{5} - \frac{t^3}{3}\right) + C_1$$
$$= -\frac{6}{5}\left(\sqrt{x+1}\right)^5 + 2\left(\sqrt{x+1}\right)^3 + C_1$$

As a result,

$$C_1(x) = -\frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + C_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int 3\sqrt{x+1}\ \ dx = 3\int (x+1)^{\frac{1}{2}}\ d(x+1) = 2(x+1)^{\frac{3}{2}} + C_2$$

As a result,

$$C_1(x) = -\frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + C_1 \quad \text{and} \quad C_2(x) = 2(x+1)^{\frac{3}{2}} + C_2$$

where C_1 and C_2 are constants.

6) We insert the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution:

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$$y=\left(-\frac{6}{5}(x+1)^{\frac{5}{2}}+2(x+1)^{\frac{3}{2}}+C_1\right)e^{-x}\ +\left(2(x+1)^{\frac{3}{2}}\ +C_2\right)xe^{-x}$$

Let us simplify the obtained expression:

$$y=e^{-x}\left(C_1+xC_2-\frac{6}{5}(x+1)^{\frac{5}{2}}+2(x+1)^{\frac{3}{2}}+2x(x+1)^{\frac{3}{2}}\right)\ =$$

$$=e^{-x}\left(C_1+xC_2-\frac{6}{5}(x+1)^{\frac{5}{2}}+2(x+1)^{\frac{3}{2}}(1+x)\right)=e^{-x}\left(C_1+xC_2-\frac{6}{5}(x+1)^{\frac{5}{2}}+2(x+1)^{\frac{5}{2}}\right)$$

As a result, the general solution of the given nonhomogenous differential equation is:

$$y=e^{-x}\left(C_1+xC_2+\frac{4}{5}(x+1)^{\frac{5}{2}}\right)$$

Exercise 8.9.

Solve the equation

$$y''-2y'=\frac{4e^{2x}}{1+e^{2x}}$$

Solution:

For this equation we use the Method of Variation of Constants, since the function on the right-hand side does not have a special form.

1) The associated homogeneous equation is

$$y''-2y'=0$$

The characteristic (auxiliary) equation is

$$k^2-2k=0$$

$$k(k-2)=0$$

The roots of the characteristic (auxiliary) equation are real and distinct:

$$k_1=0 \quad \text{and} \quad k_2=2$$

The general solution of the associated homogeneous differential equation is

$$y_c=C_1e^{0\cdot x}+C_2e^{2\cdot x}$$

or

$$y_c=C_1\cdot 1+C_2\cdot e^{2x}$$

where C_1 and C_2 are constants.

2) We replace the constants C_1 and C_2 with the arbitrary (but still unknown) functions $C_1(x)$ and $C_2(x)$ and find the general solution of the given nonhomogeneous differential equation in the form:

$$y=C_1(x)\cdot 1+C_2(x)\cdot e^{2x}$$

3) To determine the unknown functions $C_1(x)$ and $C_2(x)$, we write a system of equations for derivatives of the unknown functions

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$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_1(x)\cdot (1)' + C'_2(x)\cdot (e^{2x})' = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

The system can be written in the form

$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_1(x)\cdot 0 + C'_2(x)\cdot 2\cdot e^{2x} = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

or

$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_2(x)\cdot 2\cdot e^{2x} = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

4) From the second equation of the system we have:

$$C'_2(x) = \frac{2}{1+e^{2x}}$$

From the first equation of the system, it follows that

$$C'_1(x) = -C'_2(x)\cdot e^{2x} = -\frac{2\cdot e^{2x}}{1+e^{2x}}$$

5) We find the unknown functions $C_1(x)$ and $C_2(x)$ using integration

$$C_1(x) = \int C'_1(x)\,dx \quad \text{and} \quad C_2(x) = \int C'_2(x)\,dx\ .$$

$$C_1(x) = \int C'_1(x)\,dx = -\int \frac{2\cdot e^{2x}}{1+e^{2x}}dx = -\int \frac{d(e^{2x})}{1+e^{2x}} = -\int \frac{d(1+e^{2x})}{1+e^{2x}} = -\ln|1+e^{2x}| + C_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int \frac{2}{1+e^{2x}}dx = 2\int \frac{1+e^{2x}-e^{2x}}{1+e^{2x}}dx = 2\int \frac{1+e^{2x}}{1+e^{2x}}dx - 2\int \frac{e^{2x}}{1+e^{2x}}dx =$$

$$= 2\int 1\,dx - \int \frac{2e^{2x}}{1+e^{2x}}dx = 2x - \ln|1+e^{2x}| + C_2$$

As a result,

$$C_1(x) = -\ln|1+e^{2x}| + C_1 \quad \text{and} \quad C_2(x) = 2x - \ln|1+e^{2x}| + C_2$$

where C_1 and C_2 are constants.

6) Insert the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution:

$$y = (-\ln|1+e^{2x}| + C_1)\cdot 1 + (2x - \ln|1+e^{2x}| + C_2\ )e^{\,2x}$$

As the result, the general solution of the given nonhomogeneous differential equation is:

$$y = C_1 + C_2 e^{\,2x} - \ln|1+e^{2x}| + (2x - \ln|1+e^{2x}|)e^{\,2x}$$

It can be also written in the form

$$y = C_1 + C_2 e^{\,2x} + 2xe^{\,2x} - \ln|1+e^{2x}|\cdot(1+e^{\,2x})$$

Exercise 8.10.

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Find the general solution of the differential equation:

$$y'' - 9y = x + 2e^{-3x}$$

Solution:

1) The associated homogeneous equation is

$$y'' - 9y = 0$$

The auxiliary equation for this equation is $k^2 - 9 = 0$

The roots are real and distinct: $k_1 = 3\,,\; k_2 = -3$

Therefore, the general solution of the associated homogeneous differential equation is

$$y_c = C_1e^{3x} + C_2e^{-3x}$$

2) The right-hand side of the given equation is the sum of two functions:

$f_1(x) = x$ and $f_2(x) = 2e^{-3x}$.

According to the superposition principle, a particular solution is expressed by the formula

$$Y = Y_1 + Y_2$$

where Y_1 is a particular solution for the differential equation $y'' - 9y = x$

and Y_2 is a particular solution for the equation $y'' - 9y = 2e^{-3x}$.

a) First, we determine the function Y_1. The function $f_1(x)$ can be written as

$$f_1(x) = x = (x - 0) \cdot e^{0x}$$

In this case we will be looking for a solution in the form

$$Y_1 = (Ax + B)e^{\alpha x} \cdot x^s$$

The coefficient in the argument of the exponential function is $\alpha = 0$. It does not coincide with roots of the auxiliary (characteristic) equation: $k_1 = 3,\; k_2 = -3$, therefore $s = 0$ and the particular solution Y_1 does not contain any additional factor.

Thus, the particular solution Y_1 of the differential equation has the form

$$Y_1 = (Ax + B)e^{0x} \cdot x^0 = Ax + B$$

3) We find first- and second-order derivatives for Y_1 :

$$Y_1{}' = (Ax + B)' = A$$

$$Y_1{}'' = (A)' = 0$$

We substitute $Y_1{}'$, $Y_1{}''$ and Y_1 into the corresponding nonhomogeneous differential equation

$$y'' - 9y = x,$$

As a result, we have:

$$0 - 9(Ax + B) = x$$

We simplify the obtained expression:

$$-9Ax - 9B = 1 \cdot x$$

The coefficients at x are $-9A = 1 \quad \Longrightarrow \quad A = -1/9$

The coefficients at x^0 are $-9B = 0 \quad \Longrightarrow \quad B = 0$

Then

$$Y_1 = -\frac{1}{9}x$$

b) We determine the function Y_2.

Due to function $f_2(x) = 2e^{-3x}$, we will construct the form of the particular solution as

$$Y_2 = Ce^{-3x} \cdot x^s$$

The coefficient in the argument of the exponential function is α =-3. It coincides with one root $k_2 = -3$ of the auxiliary equation, therefore $s = 1$ and the particular solution contains the factor x^1.

Thus, the particular solution Y_2 of the differential equation has the form:

$$Y_2 = Ce^{-3x} \cdot x$$

Find first- and second-order derivatives for Y:

$$Y_2' = (Ce^{-3x} \cdot x)' = -3Ce^{-3x}x + Ce^{-3x} = e^{-3x}(-3Cx + C)$$

$$Y_2'' = (e^{-3x} \cdot (-3Cx + C))' = -3e^{-3x} \cdot (-3Cx + C) + e^{-3x} \cdot (-3C) = e^{-3x} \cdot (9Cx - 6C)$$

After substituting Y_2' , Y_2'' and Y_2 into the corresponding nonhomogeneous differential equation

$$y'' - 9y = 2e^{-3x},$$

we obtain

$$e^{-3x} \cdot (9Cx - 6C) - 9Ce^{-3x} \cdot x = 2e^{-3x}$$

$$e^{-3x} \cdot (9Cx - 6C - 9Cx) = 2e^{-3x}$$

$$-6C = 2$$

$$C = -1/3$$

As a result,

$$Y_2 = -\frac{1}{3}e^{-3x} \cdot x$$

The general solution of the given nonhomogeneous equation is equal to

$$y = y_c + Y = y_c + Y_1 + Y_2$$

Therefore, the general solution of the given differential equation is:

$$y = C_1e^{3x} + C_2e^{-3x} - \frac{1}{9}x - \frac{1}{3}xe^{-3x}$$

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8.4 APPLICATION OF THE LAPLACE TRANSFORM FOR SOLVING DIFFERENTIAL EQUATIONS

In this chapter, we consider the solution of second-order linear nonhomogeneous differential equations by using the Laplace transform. Definition and properties of the Laplace transform also are considered in brief.

8.4.1 *The Laplace transform. Definition and main properties.*

The Laplace transform is one of the most popular solving methods of linear differential equations. It is widely used for solving both ordinary and partial differential equations. For linear ordinary differential equations, the Laplace transform is especially preferred in cases where the right-hand side function f(x) of the equation is not a continuous function of x. This kind of functions often occurs in applications in the electrical circuit theory, automatic control theory, signal theory and etc.

Definition: the Laplace Transform

Suppose that the real argument function f(t) satisfies the following three conditions:

1) $f(t)$ is defined at $t \geq 0$,

2) $f(t)$ is a continuous or piecewise continuous function (it has a finite number of the first-type break points) in the interval $t \in [0, +\infty)$,

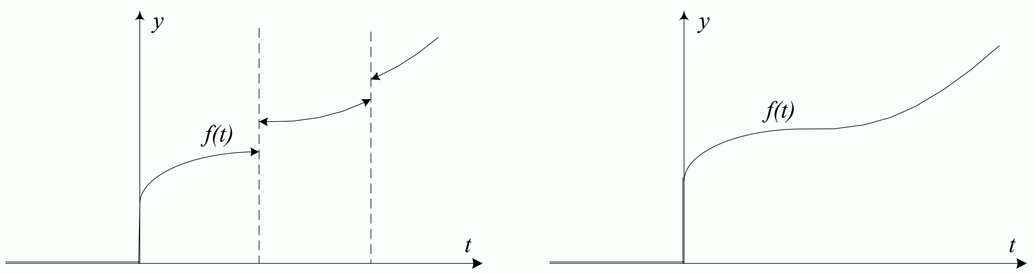

Figure 8.2

3) there exist such positive numbers $M = const$ and $S_0 = const$, that for all $t \geq 0$ holds

$$|f(t)| < Me^{S_0 t}.$$

In this case, the ***Laplace transform*** F(s) of a function f(t) is defined as the improper integral

$$\mathrm{F(s)} = \int_0^{+\infty} \mathrm{f(t)e^{-st}dt},$$

where *s* is a parameter (a complex number $s = \sigma + \omega i$ in the general case).

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The improper integral on the right-hand side is called a ***Laplace integral***.

The function ***f(t)*** is called an ***original*** and the function ***F(s)*** is called a ***transform***.

If the function F(s) is the transform of the function f(t), we use the notation

$$F(s) = L[f(t)] \qquad \text{or} \qquad f(t) \div F(s)$$

Theorem:

If the function satisfies the previously-mentioned conditions, then the Laplace integral exists provided that

$$\sigma = Re(s) > S_o$$

where $\sigma = Re(s)$ is a real part of the complex number $s = \sigma + \omega i$.

In the general case, the parameter *s* is a complex number, but here we assume that s is real.

In applications on solving physical problems by the Laplace transform method it is usually assumed that the function *f(t)* is equal to zero for *t <0*:

$$f(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

This assumption means that processes starting at the moment t = 0 are considered. This kind functions can also be defined by using the Heaviside function *H(t)*:

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

as a product of two functions *f(t)* and *H(t)*: $f(t) \cdot H(t)$

For example,

$$\sin(t) \cdot H(t) = \begin{cases} \text{sint}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Let us consider an example on finding the Laplace transform of the function *f(t)=1* $(t \geq 0)$, using the definition of the Laplace transform.

Example 8.22

Laplace transform of the function ***f(t)=1*** $(t \geq 0)$ is found as

$$L[1] = \mathrm{F(s)} = \int_0^{+\infty} \mathbf{1} \cdot \mathrm{e^{-st}dt} = \lim_{\mathrm{b}\to+\infty} \int_0^{\mathrm{b}} \mathrm{e^{-st}dt} = \lim_{\mathrm{b}\to+\infty} -\frac{1}{\mathrm{s}} \int_0^{\mathrm{b}} \mathrm{e^{-st}d(-s \cdot t)} =$$

$$= \lim_{\mathrm{b}\to+\infty} -\frac{1}{\mathrm{s}} \mathrm{e^{-st}} \Big|_0^{\mathrm{b}} = -\frac{1}{\mathrm{s}} \lim_{\mathrm{b}\to+\infty} \left(\mathrm{e^{-sb}} - \mathrm{e^0}\right) = -\frac{1}{\mathrm{s}}(0-1) = \frac{1}{\mathrm{s}}$$

Thus,

$$L[1] = \frac{1}{s}$$

or we can also write

$$1 \div \frac{1}{s}$$

Example 8.23

Let us find Laplace transform for the function $f(t) = e^t \ (t \geq 0)$:

$$L[e^t] = F(s) = \int_0^{+\infty} e^t \cdot e^{-st}dt = \lim_{b \to +\infty} \int_0^b e^{-(s-1)t}dt =$$

$$\lim_{b \to +\infty} -\frac{1}{s-1}\int_0^b e^{-(s-1)t}d(-(s-1)t) = \lim_{b \to +\infty} -\frac{1}{s-1}e^{-(s-1)t}\Big|_0^b =$$

$$= -\frac{1}{s-1}\lim_{b \to +\infty}\left(e^{-(s-1)b} - e^0\right) = -\frac{1}{s-1}(0-1) = \frac{1}{s-1}$$

Thus,

$$L[e^t] = \frac{1}{s-1}$$

In a similar way, the transforms for other elementary functions have been determined and summarized in special tables. Part of such a table is presented below.

f(t)	***F(s)***
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
sin(*at*)	$\frac{a}{s^2+a^2}$
cos(*at*)	$\frac{s}{s^2+a^2}$
$e^{\lambda t}\sin(at)$	$\frac{a}{(s-\lambda)^2+a^2}$

$e^{\lambda t}\cos(at)$	$\frac{s-\lambda}{(s-\lambda)^2+a^2}$
sinh*(at)*	$\frac{a}{s^2-a^2}$
cosh*(at)*	$\frac{s}{s^2-a^2}$
$e^{\lambda t}\sinh(at)$	$\frac{a}{(s-\lambda)^2-a^2}$
$e^{\lambda t}\cosh(at)$	$\frac{s-\lambda}{(s-\lambda)^2-a^2}$

In applications, exactly those summarized tables of elementary Laplace transforms and properties of Laplace transform are used in order to find Laplace transforms of necessary functions.

Properties of the Laplace transform

Let us consider only properties which are necessary for solving differential equations.

1) ***Linearity theorem*** (*C1, C2=const*):

$$L[C_1 f_1(t) \pm C_2 f_2(t)] = C_1 L[f_1(t)] \pm C_2 L[f_2(t)]$$

2) ***Theorem on a derivative of the original***

If $\mathrm{L}[f(\mathrm{t})] = \mathrm{F(s)}$, then

$$\mathrm{L}[f'(\mathrm{t})] = \mathrm{sF(s)} - f(0)$$

$$\mathrm{L}[f''(\mathrm{t})] = s^2\mathrm{F(s)} - \mathrm{s}f(0) - f'(0)$$

$$\mathrm{L}[f'''(\mathrm{t})] = s^3\mathrm{F(s)} - s^2 f(0) - sf'(0) - f''(0)$$

....

$$\mathrm{L}\left[f^{(\mathrm{n})}(\mathrm{t})\right] = s^{\mathrm{n}}\mathrm{F(s)} - s^{\mathrm{n}-1}f(0) - s^{\mathrm{n}-2}f'(0) - \cdots - f^{(\mathrm{n}-1)}(0)$$

where C1 and C2 are constants.

Example 23

Find the Laplace transform of the function $\mathrm{f(t)} = 2 - 3\sin 5\mathrm{t} + 4\mathrm{e}^{2\mathrm{t}} + \mathrm{t}^2$.

$$F(s) = L[\mathrm{f(t)}] = \mathrm{L}[2 - 3\sin 5\mathrm{t} + 4\mathrm{e}^{2\mathrm{t}} + \mathrm{t}^2] = 2\mathrm{L}[1] - 3\mathrm{L}[\sin 5\mathrm{t}] + 4\mathrm{L}[\mathrm{e}^{2\mathrm{t}}] + \mathrm{L}[\mathrm{t}^2] =$$

$$= 2\cdot\frac{1}{s} - 3\cdot\frac{5}{s^2+25} + 4\cdot\frac{1}{s-2} + \frac{2!}{s^3}$$

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As a result,

$$L[f(t)] = \frac{2}{s} - \frac{15}{s^2+1} + \frac{4}{s-2} + \frac{2}{s^3}$$

Definition: the Inverse Laplace Transform

If $L[f(t)] = F(s)$, then the inverse Laplace transform $f(t)$ of the function $F(s)$ is defined as the improper integral

$$f(t) = \int_0^{+\infty} F(t)e^{st}ds$$

It is often written as

$$f(t) = L^{-1}[F(s)]$$

It is to be noted that usually in applications the summarized tables of elementary Laplace transforms and properties of the Laplace transform are used in order to find originals.

In many practical problems the Laplace transform has the form of a rational fraction. In this case, the method of partial fractions can be useful in producing an expression; for those, the inverse Laplace transform can be easily found.

Example 8.24

Find the original of the function

$$F(s) = \frac{s+3}{s(s+1)}$$

i.e.

$$f(t) = L^{-1}\left[\frac{s+3}{s(s+1)}\right] = ?$$

We expand the given rational function into elementary fractions with undefined coefficients:

$$F(s) = \frac{s+3}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

In order to find the unknown coefficients, we find the least common denominator and equate the numerators of the functions on the right-hand side and left-hand side of the obtained expression:

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$$\frac{s+3}{s(s+1)} = \frac{A(s+1)+Bs}{s(s+1)}$$

$$s+3 = A(s+1)+Bs$$

$$s+3 = As+A+Bs$$

$$1\cdot s+3 = (A+B)s+A$$

The coefficients at s: 1=A+B,

The coefficient at s^0: $3 = A$ ➔ $A = 3$

It follows from the first equation that $B = 1 - A = 1 - 3 = -2$

As a result, we have

$$F(s) = \frac{3}{s} + \frac{-2}{s+1} = 3\cdot\frac{1}{s} - 2\cdot\frac{1}{s+1}$$

Using the linearity theorem and the table of Laplace transforms, we have

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[3\cdot\frac{1}{s} - 2\cdot\frac{1}{s+1}\right] = 3L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s+1}\right] = 3\cdot 1 - 2e^{-t}$$

So,

$$f(t) = L^{-1}[F(s)] = 3\cdot 1 - 2e^{-t}\,.$$

8.4.2 *Application of the Laplace transform for solving differential equations*

As was mentioned above, the Laplace transform is one of the most popular methods for solving differential equations. Here we consider the application of the Laplace transform for second-order linear differential equations with constant coefficients.
The Laplace transform can be only used for solving differential equations with given initial conditions at the point t=0, i.e. only for solving Cauchy problems.
Let us consider the linear differential equation with constant coefficients:

$$ay" + by' + cy = f(t)$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y_1$,

where a, b, c are constants, $y = y(t)$ is a function of t and $a \neq 0$.

The algorithm of solving a Cauchy problem by the Laplace transform is:

1) Apply the Laplace transform to both sides of the differential equation

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$$L[ay" + by' + cy] = L[f(t)]$$

2) Use the linearity theorem together with the *Theorem on a derivative of the original*

$$aL[y"] + bL[y'] + cL[y] = L[f(t)]$$

Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the *Theorem on a derivative of the original,* it yields

$$\mathrm{L}[y'(\mathrm{t})] = \mathrm{sY(s)} - y(0) = \mathrm{sY(s)} - y_0$$

$$\mathrm{L}[y''(\mathrm{t})] = \mathrm{s}^2\mathrm{Y(s)} - \mathrm{s}y(0) - y'(0) = \mathrm{s}^2\mathrm{Y(s)} - \mathrm{s}y_0 - y_1$$

On applying the Laplace transform to the given differential equation, we have got the algebraic equation for the unknown function Y(s):

$$\mathrm{a} \cdot (\mathrm{s}^2\mathrm{Y(s)} - \mathrm{s}y_0 - y_1) + b \cdot (\mathrm{sY(s)} - y_0) + c \cdot Y(s) = F(s)$$

where $F(s) = L[f(t)]$ is the Laplace transform of the right-hand side function.

3) Solve the obtained ***algebraic equation*** for the function Y(s):

$$\mathrm{Y(s)}(\mathrm{as}^2 + bs + c) = F(s) + asy_0 + by_0 + ay_1$$

$$\mathrm{Y(s)} = \frac{F(s) + asy_0 + by_0 + ay_1}{\mathrm{as}^2 + bs + c}$$

4) ***Find the original*** y(t) of the function Y(s) using properties of the Laplace transform and the table of Laplace transforms as

$$\mathrm{y(t)} = \mathrm{L}^{-1}[\mathrm{Y(s)}]$$

Example 0.25

Solve the Cauchy problem

$$y" + 9y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 2$$

1) We apply the Laplace transform to both sides of the given differential equation

$$\mathrm{L}[y" + 9y] = \mathrm{L}[e^{2t}]$$

$$\mathrm{L}[y"] + 9L[y] = \mathrm{L}[e^{2t}]$$

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Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the Theorem on a derivative of original, it yields

$$L[y'(t)] = sY(s) - y(0) = sY(s) - 1$$

$$L[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s \cdot 1 - 2$$

The Laplace transform of the right-hand side function is

$$L[e^{2t}] = \frac{1}{s-2}$$

After substituting $L[y''(t)]$, $L[y(t)]$ and $L[e^{2t}]$, we obtain the algebraic equation for the unknown function Y(s):

$$s^2 \cdot Y(s) - s - 2 + 9 \cdot Y(s) = \frac{1}{s-2}$$

3) We solve the obtained algebraic equation for the function Y(s):

$$Y(s) \cdot (s^2 + 9) = \frac{1}{s-2} + s + 2$$

$$Y(s) = \frac{1}{(s-2)(s^2+9)} + \frac{s}{(s^2+9)} + \frac{2}{(s^2+9)}$$

4) We find the original y(t) for the function Y(s).

a) First, we expand the first term on the right-hand side into elementary fractions with undefined coefficients:

$$\frac{1}{(s-2)(s^2+9)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9) + (Bs+C)(s-2)}{(s-2)(s^2+9)}$$

Thus, we get

$$1 = A(s^2+9) + (Bs+C)(s-2)$$

$$1 = As^2 + 9A + Bs^2 - 2Bs + Cs - 2C$$

$$0 \cdot s^2 + 0 \cdot s + 1 = (A+B)s^2 + (-2B+C)s + 9A - 2C$$

The coefficients at s^2 are $0 = A + B$

The coefficients at s are 0=-2B+C,

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coefficients at s^0 are $1 = 9A - 2C$

Solving the system of equations for the unknown coefficients A, B and C, we obtain:

$$A = \frac{1}{13}, \quad B = -\frac{1}{13}, \quad C = -\frac{2}{13}$$

As a result, we have

$$\frac{1}{(s-2)(s^2+9)} = \frac{\frac{1}{13}}{s-2} + \frac{-\frac{1}{13}s - \frac{2}{13}}{s^2+9} = \frac{1}{13}\cdot\frac{1}{s-2} - \frac{1}{13}\cdot\frac{s}{s^2+9} - \frac{2}{13}\cdot\frac{1}{s^2+9}$$

We substitute the obtained expression into the expression for $Y(s)$ instead of the first term:

$$Y(s) = \frac{1}{13}\cdot\frac{1}{s-2} - \frac{1}{13}\cdot\frac{s}{s^2+9} - \frac{2}{13}\cdot\frac{1}{s^2+9} + \frac{s}{s^2+9} + \frac{2}{s^2+9}$$

We simplify the obtained expression as

$$Y(s) = \frac{1}{13}\cdot\frac{1}{s-2} + \frac{12}{13}\cdot\frac{s}{s^2+9} + \frac{24}{13}\cdot\frac{1}{s^2+9}$$

5) We find the original for the function $Y(s)$:

$$y(t) = L^{-1}[Y(\mathbf{s})] = L^{-1}\left[\frac{1}{13}\cdot\frac{1}{s-2} + \frac{12}{13}\cdot\frac{s}{s^2+9} + \frac{24}{13}\cdot\frac{1}{s^2+9}\right] =$$

$$= \frac{1}{13}\cdot L^{-1}\left[\frac{1}{s-2}\right] + \frac{12}{13}\cdot L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{24}{13}\cdot L^{-1}\left[\frac{1}{3}\cdot\frac{3}{s^2+9}\right] =$$

$$= \frac{1}{13}\cdot L^{-1}\left[\frac{1}{s-2}\right] + \frac{12}{13}\cdot L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{24}{13}\cdot\frac{1}{3}\cdot L^{-1}\left[\frac{3}{s^2+9}\right] =$$

$$= \frac{1}{13}e^{2t} + \frac{12}{13}\cos 3t + \frac{24}{39}\cdot\sin 3t$$

Thus, we have obtained the solution of the given Cauchy problem:

$$y(t) = L^{-1}[Y(s)] = \frac{1}{13}e^{2t} + \frac{12}{13}\cos 3t + \frac{24}{39}\sin 3t$$

8.4.3 *Exercises*

Exercise 8.11.

Solve the Cauchy problem using the Laplace transform

$$y'' + 4y' + 5y = 1 \ , \quad y(0) = 0, \quad y'(0) = 1$$

Solution:

1) We apply the Laplace transform to the given differential equation:

$$L[y'' + 4y' + 5y] = L[1]$$

$$L[y''] + 4L[y'] + 5L[y] = L[1]$$

Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the *Theorem on a derivative of original,* it yields

$$\mathrm{L}[y'(\mathrm{t})] = \mathrm{sY(s)} - y(0) = \mathrm{sY(s)} - 0 = \mathrm{sY(s)}$$

$$\mathrm{L}[y''(\mathrm{t})] = \mathrm{s^2Y(s)} - \mathrm{s}y(0) - y'(0) = \mathrm{s^2Y(s)} - \mathrm{s}\cdot 0 - 1 = \mathrm{s^2Y(s)} - 1$$

The result of application of the Laplace transform to the given differential equation gives us the algebraic equation for the unknown function Y(s):

$$\mathrm{s^2Y(s)} - 1 + 4\cdot \mathrm{sY(s)} + 5\cdot Y(s) = \frac{1}{s}$$

3) We solve the obtained algebraic equation for the function Y(s):

$$\mathrm{Y(s)}(\mathrm{s^2} + 4\mathrm{s} + 5) = \frac{1}{s} + 1$$

$$\mathrm{Y(s)}(\mathrm{s^2} + 4\mathrm{s} + 5) = \frac{1+s}{s}$$

$$\mathrm{Y(s)} = \frac{1+s}{s(\mathrm{s^2} + 4\mathrm{s} + 5)}$$

4) We find the original y(t) for the function Y(s) using properties of the Laplace transform and the table of Laplace transforms, as

$$\mathrm{y(t)} = \mathrm{L^{-1}[Y(s)]}$$

For this purpose, we expand the function on the right-hand side into elementary fractions with undefined coefficients:

$$\frac{1+s}{s(\mathrm{s^2} + 4\mathrm{s} + 5)} = \frac{\mathrm{A}}{\mathrm{s}} + \frac{\mathrm{Bs + C}}{\mathrm{s^2 + 4s + 5}} = \frac{\mathrm{A(s^2 + 4s + 5) + (Bs + C)s}}{\mathrm{s(s^2 + 4s + 5)}}$$

So that

$$s + 1 = A(s^2 + 4s + 5) + (Bs + C)s$$

$$s + 1 = As^2 + 4As + 5A + Bs^2 + Cs$$

$$0 \cdot s^2 + 1 \cdot s + 1 = (A + B)s^2 + (4A + C)s + 5A$$

The coefficients at s^2 : $0 = A + B$
The coefficients at s: $1 = 4A + C$,
The coefficients at s^0: $1 = 5A$

Solving the system of equations for the unknown coefficients, we have:

$$A = \frac{1}{5}, \quad B = -\frac{1}{5}, \quad C = \frac{1}{5}$$

As a result, we have

$$Y(s) = \frac{1 + s}{s(s^2 + 4s + 5)} = \frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s + \frac{1}{5}}{s^2 + 4s + 5} = \frac{1}{5} \cdot \frac{1}{s} - \frac{1}{5} \cdot \frac{s - 1}{s^2 + 4s + 5}$$

5) We find the original for the function $Y(s)$:

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{5} \cdot \frac{1}{s} - \frac{1}{5} \cdot \frac{s - 1}{s^2 + 4s + 5}\right] = \frac{1}{5} \cdot L^{-1}\left[\frac{1}{s}\right] - \frac{1}{5} \cdot L^{-1}\left[\frac{s - 1}{s^2 + 4s + 5}\right]$$

It follows from the Laplace transform table that $L^{-1}\left[\frac{1}{s}\right] = 1$,

However, for finding

$$L^{-1}\left[\frac{s - 1}{s^2 + 4s + 5}\right]$$

first, we should transform the fraction

$$\frac{s - 1}{s^2 + 4s + 5} = \frac{s - 1}{(s + 2)^2 + 1} = \frac{s + 2 - 3}{(s + 2)^2 + 1} = \frac{s + 2}{(s + 2)^2 + 1} - \frac{3}{(s + 2)^2 + 1}$$

Then

$$L^{-1}\left[\frac{s + 2}{(s + 2)^2 + 1} - \frac{3}{(s + 2)^2 + 1}\right] = L^{-1}\left[\frac{s + 2}{(s + 2)^2 + 1}\right] - 3 \cdot L^{-1}\left[\frac{1}{(s + 2)^2 + 1}\right] =$$

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$$= e^{-2t}\mathrm{cos}t - 3e^{-2t}\mathrm{sin}t$$

As a result, we have

$$\mathrm{y(t)} = \frac{1}{5}\cdot 1 - \frac{1}{5}\cdot\left(e^{-2t}\mathrm{cos}t - 3e^{-2t}\mathrm{sin}t\right) = \frac{1}{5} - \frac{e^{-2t}}{5}\cdot\left(\mathrm{cos}t - 3\mathrm{sin}t\right)$$

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8.5 CONNECTIONS AND APPLICATIONS

Example 1:

Ship stability is a maritime safety issue that needs to be explored even at the design stage. Rolling and pitching of the ship in the water are extremely important factors affecting the stability of a is the

1) Rolling of a vessel from one side to the another one, occurring in calm water without resistance is described by the following second-order differential equation:

$$\theta'' + {n_\theta}^2\theta = 0$$

where $\theta = \theta(t)$ is the rolling amplitude (Fig.3).

n_θ is the circular frequency of free (natural) vibrations during the rolling **without resistance**.

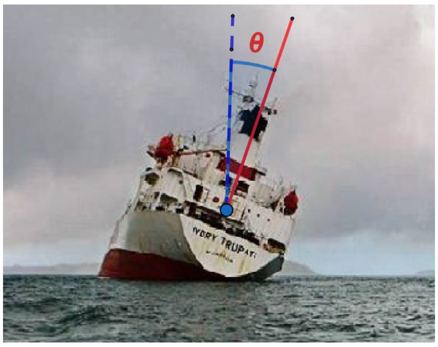

Figure 8.3 *Rolling of a vessel*

This equation is a second-order linear homogeneous differential equation with constants coefficients. Let us solve this equation.

The auxiliary equation is

$$k^2 + {n_\theta}^2 = 0,$$

whose roots are complex numbers

$$k_1 = n_\theta i \ , \qquad k_2 = -n_\theta i$$

The general solution of the equation is

$$\theta(t) = C_1\cos(n_\theta t) + C_2\sin(n_\theta t)$$

2) Taking into account the resistance during the rolling in calm water, the equation of motion of a vessel takes the form

$$\theta'' + 2\mu_\theta\theta' + {n_\theta}^2\theta = 0$$

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where μ_θ is the relative coefficient of resistance.

The corresponding auxiliary equation is

$$k^2 + 2\mu_\theta k + {n_\theta}^2 = 0,$$

whose roots are

$$k_1 = -\mu_\theta + i\sqrt{{\mu_\theta}^2 - {n_\theta}^2}\ , \qquad k_2 = -\mu_\theta - i\sqrt{{\mu_\theta}^2 - {n_\theta}^2}$$

$$\theta(t) = C_1\, \mathrm{e}^{-\mu_\theta t}\cos(\omega_\theta \cdot t) + C_2\, \mathrm{e}^{-\mu_\theta t}\sin(\omega_\theta \cdot t)$$

where

$\omega_\theta = \sqrt{{\mu_\theta}^2 - {n_\theta}^2}$ is a natural (their own) frequency during the rolling with resistance.

It should be noted that similar differential equations describe also pitching and heaving motions of a vessel.

Example 2:

Any modern vessel is not complete without electrical and electro-mechanical systems. An alternating-current electrical circuit is a component of any such system. Transition processes in such electrical circuits that occur in a short period of time after switching on or off (after connecting the circuit to voltage or after disconnecting the circuit from voltage), as well as when the capacitive element is turned on or off, are described by the ordinary differential equations. As an example, we can consider one of the easiest electrical circuits: a resistor-inductor-capacitor circuit (RLC).

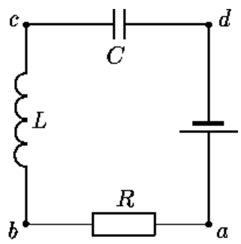

Figure 8.4 A resistor-inductor-capacitor circuit

1) For example, in the case of source of unchanging voltage, the following second-order differential equation describes the transition processes in RLC circuit:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = 0$$

where

t is the time,

i(t) is the current admitted through the circuit,

R is the effective resistance of the combined load, source, and components,

L is the inductance of the inductor component,

C is the capacitance of the capacitor component.

This is a homogeneous second-order ordinary differential equation whose *characteristic equation is*

$$Lk^2 + Rk + \frac{1}{C} = 0$$

or

$$k^2 + \frac{R}{L}k + \frac{1}{LC} = 0$$

The roots are

$$k_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad and \quad k_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The solution of the differential equation has the form

$$i(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

where C_1 and C_2 are terms of amplitude.

2) If a RL circuit with constant resistance R and inductance L at time t = 0 is connected to voltage U_0 (for example, battery), then the transition process within a short time period after switching on is described by the following 1st order linear inhomogeneous differential equation with constant coefficients

$$L\frac{di}{dt} + R \cdot i = U_0$$

Example 3:

Ships often carry containers with various liquids so that liquid leakage problems are essential. In this connection, we consider the problem of the liquid flowing out of a cylindrical tank of radius *R* through a small hole of radius *r* at the bottom of the container.

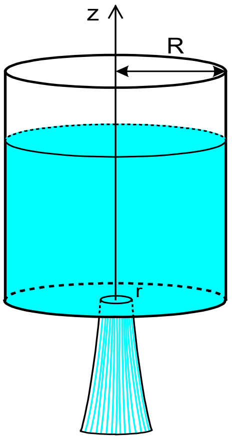

Figure 8.5

Liquid level in the tank at time moment *t* is a function of time which is described by the following differential equation:

$$R^2\frac{dz}{dt}+r^2k\sqrt{2gz}=0$$

where

t is time,

g is the gravitational acceleration (g=9.80665 m/s^2),

k is coefficient of the flow rate that depends on the viscosity of the liquid,

z=z(t) is the liquid level above the hole at time moment *t* .

Assuming, that in the initial time moment t=0 the liquid level was H, let us find:

a) *unknown function of liquid level in the tank z=z(t) ;*

b) *time T during which the liquid will completely drain out of the tank*.

In order to find unknown function of liquid level in the tank *z(t),* we solve the given differential equation. This is a separable-variables equation.

$$R^2\frac{dz}{dt}=-r^2\sqrt{2g}\cdot\sqrt{z}$$

$$\frac{dz}{\sqrt{z}}=-\frac{r^2}{R^2}\sqrt{2g}dt$$

$$\int\frac{dz}{\sqrt{z}}=-\frac{r^2}{R^2}\sqrt{2g}\int dt$$

$$2\sqrt{z}=-\frac{r^2}{R^2}\sqrt{2g}\cdot t+C$$

Taking into account, that at the initial time moment t=0 the height of the liquid in the container was H, we get

$$2\sqrt{H}=-\frac{r^2}{R^2}\sqrt{2g}\cdot 0+C$$

$$C=2\sqrt{H}$$

$$2\sqrt{z}=-\frac{r^2}{R^2}\sqrt{2g}\cdot t+2\sqrt{H}$$

As a result, we obtain the function z (t), which describes the liquid level in the tank at time moment t:

$$z(t) = \left(-\frac{r^2}{2R^2}\sqrt{2g}\cdot t + \sqrt{H}\right)^2$$

In order to find the time T during which the liquid will completely drain out of the tank, we take into account, that at the time moment t=T, the level of the liquid in the container will be z=0. Then we obtain the dependence of time on the height of the fluid

$$2\sqrt{0} = -\frac{r^2}{R^2}\sqrt{2g}\cdot T + 2\sqrt{H}$$

$$\frac{r^2}{R^2}\sqrt{2g}\cdot T = 2\sqrt{H}$$

Expressing T, we get the time during which the liquid will completely drain out of the tank.

$$T = \frac{R^2}{r^2}\sqrt{\frac{2H}{g}}$$

Example 4:

There are many marine ecological issues where differential equations are useful. For example, the mathematical modelling of propagation and extinction of fish population that is important for fish catch control.

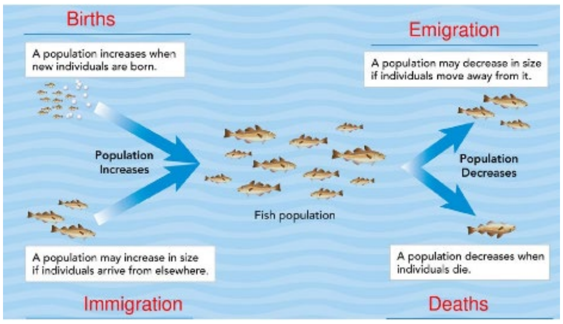

Figure 8.6

Fish population *P(t)* in the lake at the time moment *t* can be described by the first-order differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where

t is time, *k* is the growth parameter,

M is the carrying capacity, representing the largest population that the environment can support.

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If for some reason the population exceeds the carrying capacity, the population will decrease; and otherwise, as long as the population is less than the carrying capacity, the population will increase. This equation is known as the logistic equation.

The population *P(t)* of codfish in a certain marine fishery is modelled by a modified logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - H$$

where *H* is the rate at which fish are harvested.

An important question in this problem is how the fate of the fish population depends on the parameter H.

Example 5:

Differential equations are used in beam theory which is an important tool in the sciences, especially in structural and mechanical engineering. It is also very important in ship design. For example, we consider the Euler–Bernoulli equation which describes the relationship between the beam's deflection and the applied load. A beam is a constructive element capable of withstanding heavy loads in bending.

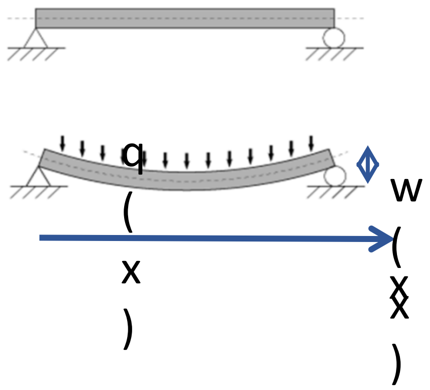

Figure 8.7

In the case of small deflections, the beam shape can be described by a fourth-order linear differential equation

$$E \cdot I \frac{d^4 w}{dx^4} = q(x)$$

where *q(x)* is external load acting on the beam,

E is the modulus of elasticity of the beam,

I is the second moment of area of the beam's cross-section.

The curve w(x) describes the deflection of the beam in the direction *z* at some position *x*.
Often, the product $E \cdot I$ is a constant, known as the flexural rigidity.
This equation under the appropriate boundary conditions determines the deflection of a loaded beam.

Example 6:

Ordinary differential equations are widely used for cooling/heating problems.

For example, consider a process of cooling down of a heated body placed in an environment. The temperature of a hot object decreases with the rate proportional to the difference between its temperature and the temperature of the surrounding environment. If the temperature of the environment is given by E(t), then the following differential equation describes the temperature of the body T(t) as the function of time:

$$\frac{dT}{dt} = -k(T(t) - E(t))$$

where *k>0* is a physical constant depending on the materials and sizes of the bodies.

If the object, whose temperature is being modelled, contains a source of heat, then the cooling of the body is described by the differential equation

$$\frac{dT}{dt} = -k(T(t) - E(t)) + mH(t)$$

where *m* is a positive constant, inversely proportional to the heat capacity of the object and H(t) denotes the rate that heat is generated within the object. (H(t) would be negative in some cases, such as air conditioning).

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