4.3 Scalar, vector and mixed triple products

Scalar product or Dot product of vector $\vec{a}$ and vector $\vec{b}$ is equal to the product of their magnitudes and the cosine of the angle between these vectors.

$$\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \sphericalangle (\vec{a}, \vec{b})$$

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular ($\vec{a} \perp \vec{b}$) if and only if $\vec{a} \circ \vec{b} = 0$.

Vector product or Cross product

If $\vec{a}$ and $\vec{b}$ are two vectors in space, then their ***cross product:***

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\varphi,$$ where φ is the angle between vectors.

The Cross Product $\vec{a} \times \vec{b}$ of two vectors is another vector $\vec{c}$ that is at right angles to both.

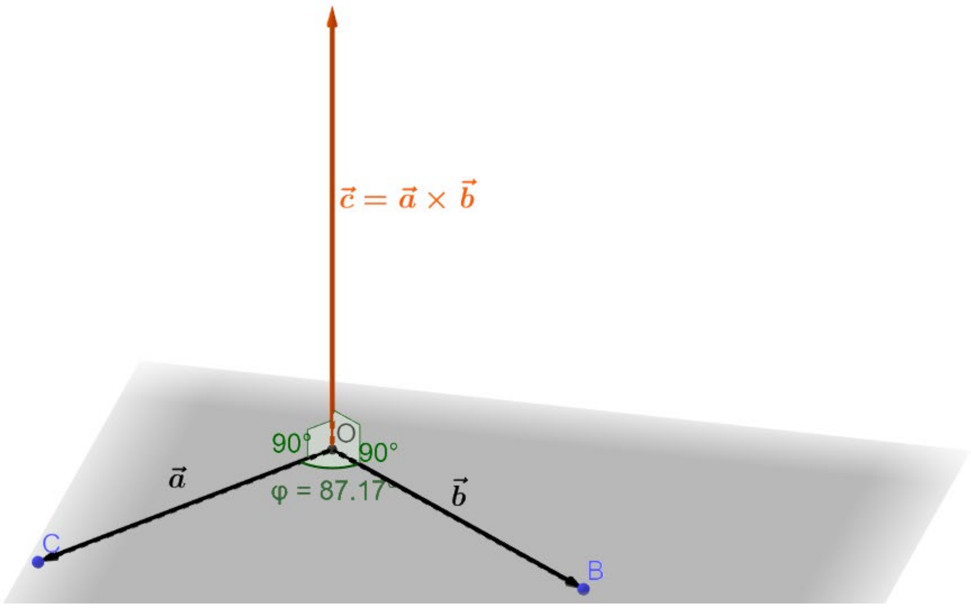

Figure 4.13 Vector product

The vector or cross product (*red*) is:

- zero in length when vectors $\vec{a}$ and $\vec{b}$ point in the same, or opposite, direction

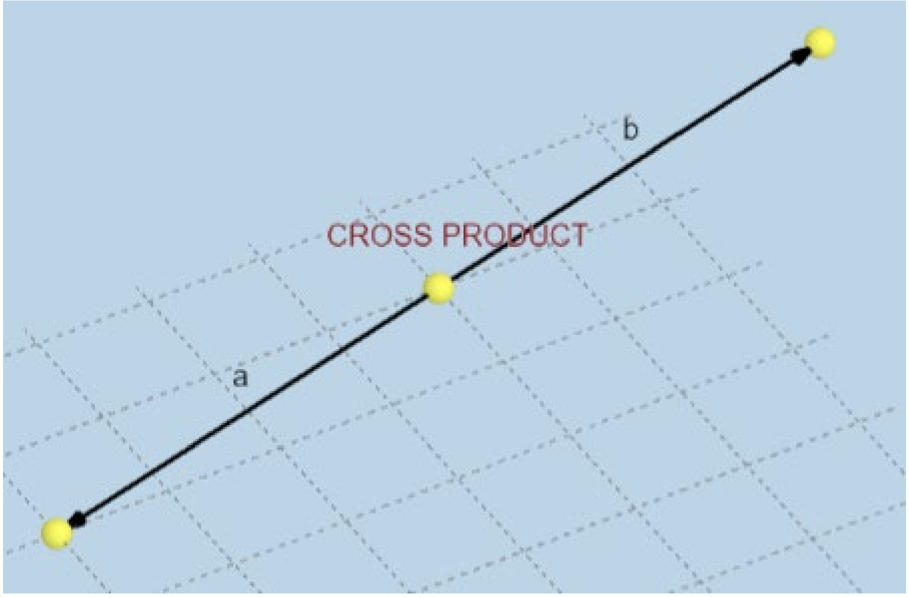

Figure 4.14 Vector product when vectors $\vec{a}$ and $\vec{b}$ point in the opposite direction

- reaches maximum length when vectors $\vec{a}$ and $\vec{b}$ are at right angles.

From the geometric perspective, $\left|\vec{a} \times \vec{b}\right|$ is the **area of the parallelogram spanned** by vectors $\vec{a}$ and $\vec{b}$.

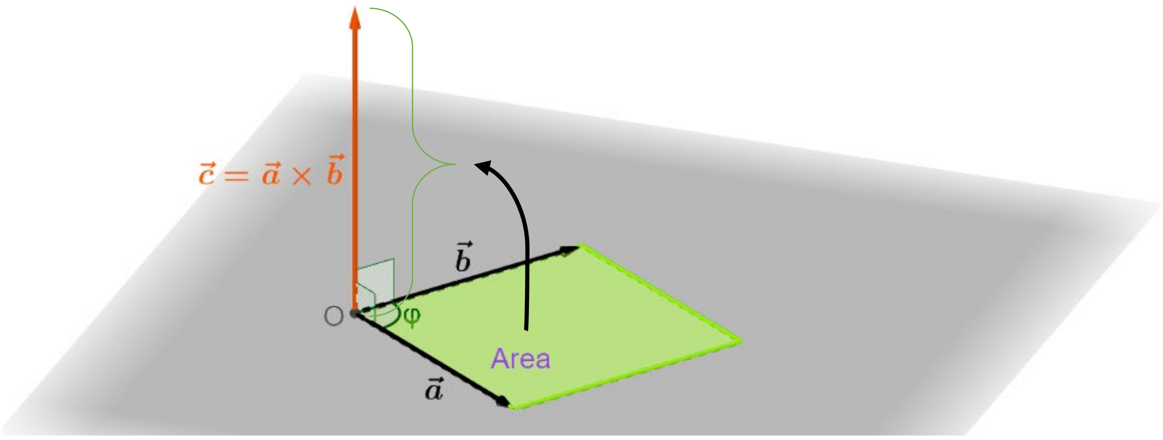

Figure 4.15 Area of the parallelogram spanned by vectors a ⃗ and b ⃗.

Vector $\vec{a} \times \vec{b}$ is perpendicular to vectors $\vec{a}$ and $\vec{b}$.

The features of vector product:

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

(3) $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

Mixed product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is scalar:

$(\vec{a} \times \vec{b}) \circ \vec{c} = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cos\varphi$, where φ is the angle between vectors $\vec{a} \times \vec{b}$ and $\vec{c}$.

Volume V of the parallelepiped determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is calculated as:

$$V = |(\vec{a} \times \vec{b}) \circ \vec{c}|.$$

It holds that:

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $\begin{cases} \vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \\ (\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) \end{cases}$

(3) $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$