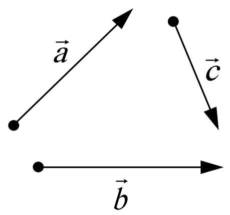
4.6 Exercises

Task 3.1.

Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are shown in the figure below. Construct vectors:

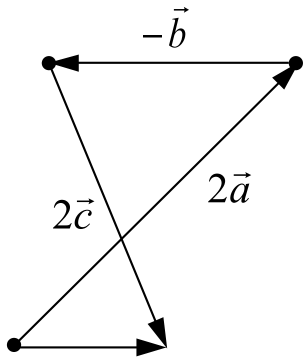
(1) $2\vec{a} - \vec{b} + 2\vec{c}$;

(2) $\vec{a} + \vec{b} + \vec{c}$

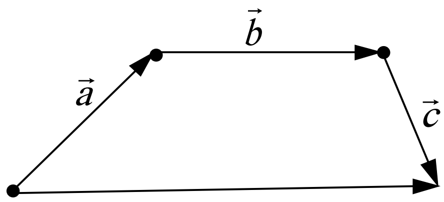


Solution:

(1)



(2)



Task 3.2.

If $\vec{e}_1$ and $\vec{e}_2$ are non-collinear vectors, determine the constant β so that vectors $\vec{a} = 2\vec{e}_1 - \vec{e}_2$ i $\vec{b} = \beta\vec{e}_1 + \vec{e}_2$ are collinear.

Solution:

$\vec{a}$ i $\vec{b}$ are collinear if there is the number $\lambda \in$ R such that $\vec{a} = \lambda\vec{b}$. It follows that:

$$3\vec{e}_1 - 2\vec{e}_2 = \lambda(\beta\vec{e}_1 - 3\vec{e}_2),$$

i.e.

$$3\vec{e}_1 - 2\vec{e}_2 = \lambda\beta\vec{e}_1 - 3\lambda\vec{e}_2.$$

It follows that:

$$\left.\begin{array}{r} 3 = \lambda\beta \\ -2 = -3\lambda \end{array}\right\} \Rightarrow \lambda = \frac{2}{3}, \quad \beta = \frac{9}{2}.$$

Vectors $\vec{a}$ and $\vec{b}$ are collinear for $\beta = \frac{9}{2}$.

Task 3.3.

Prove that vectors $\vec{a} = 5\vec{i} + 4\vec{j} + 3\vec{k}$, $\vec{b} = 3\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{c} = 8\vec{i} + \vec{j} + 3\vec{k}$ are coplanar.

Solution:

$8\vec{i} + \vec{j} + 3\vec{k} = \alpha(5\vec{i} + 4\vec{j} + 3\vec{k}) + \beta(3\vec{i} + 3\vec{j} + 2\vec{k})$, slijedi:

$$\left.\begin{array}{l} 5\alpha + 3\beta = 8, \\ 4\alpha + 3\beta = 1, \\ 3\alpha + 2\beta = 3, \end{array}\right\} \Rightarrow \quad \alpha = 7; \quad \beta = -9.$$

$\vec{c} = 7\vec{a} - 9\vec{b}$, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Task 3.4.

Given are vectors $\vec{a} = 4\vec{e}_1 - 3\vec{e}_2$, $\vec{b} = \vec{e}_1 - \vec{e}_2$ i $\vec{c} = 2\vec{e}_1 - 3\vec{e}_2$, in which the angle between the basis vector $\sphericalangle(\vec{e}_1, \vec{e}_2) = \frac{\pi}{3}$. Factorize vector $\vec{a}$ along the lines of vectors $\vec{b}$ i $\vec{c}$

Solution:

$$\vec{a} = \alpha\vec{b} + \beta\vec{c}.$$

It follows that:

$$4\vec{e}_1 - 3\vec{e}_2 = \alpha(\vec{e}_1 - \vec{e}_2) + \beta(2\vec{e}_1 - 3\vec{e}_2),$$

$$4\vec{e}_1 - 3\vec{e}_2 = (\alpha + 2\beta)\vec{e}_1 + (-\alpha - 3\beta)\vec{e}_2.$$

$$\left.\begin{array}{r} 4 = \alpha + 2\beta \\ -3 = -\alpha - 3\beta \end{array}\right\} \Rightarrow \beta = -1; \alpha = 6.$$

Therefore,

$$\vec{a} = 6\vec{b} - \vec{c}$$

Task 3.5.

Coordinates of the vertices ΔABC are: A (–2, 0, 4); B (4, 1, –2); C (2, –4, –4).

Calculate:

(1) Norm of vector $\overrightarrow{AB}$;

(2) unit vector of $\overrightarrow{AC}$.

Solution:

(1) $\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k} = 6\vec{i} + \vec{j} - 6\vec{k}$;

$$|\overrightarrow{AB}| = \sqrt{6^2 + 1^2 + (-6)^2} = \sqrt{73}.$$

(2) $\vec{b} = \overrightarrow{AC} = 4\vec{i} - 4\vec{j} - 8\vec{k}$;

$$b_0 = \frac{\vec{b}}{|\vec{b}|} = \frac{4\vec{i} - 4\vec{j} - 8\vec{k}}{\sqrt{4^2 + (-4)^2 + (-8)^2}} = \frac{1}{4\sqrt{6}} \cdot (4\vec{i} - 4\vec{j} - 8\vec{k}) = \frac{\sqrt{6}}{6}(\vec{i} - \vec{j} - 2\vec{k}).$$

Task 3.6.

Find vector projection $\vec{a} = \vec{i} + \vec{j} - 4\vec{k}$ onto the line of vector $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$.

Solution:

$$proj_b \vec{a} = \frac{\vec{a} \circ \vec{b}}{|\vec{b}|} = \frac{1 \cdot 6 + 1 \cdot (-3) + (-4) \cdot 2}{\sqrt{6^2 + (-3)^2 + 2^2}} = -\frac{5}{7}.$$

Task 3.7.

Determine the angles ΔABC with vertices A (2, –1, 3), B (1, 1, 1) i C (0, 0, 5).

Solution:

$\vec{c} = \overrightarrow{AB} = -\vec{i} + 2\vec{j} - 2\vec{k}$,

$\vec{b} = \overrightarrow{AC} = -2\vec{i} + \vec{j} + 2\vec{k}$,

$\vec{a} = \overrightarrow{CB} = \vec{i} + \vec{j} - 4\vec{k}$.

$\vec{b} \circ \vec{c} = |\vec{b}| \cdot |\vec{c}| \cos\alpha \Rightarrow \cos\alpha = \dfrac{\vec{b}\circ\vec{c}}{|\vec{b}|\cdot|\vec{c}|}$.

$|\vec{b}| = \sqrt{4+1+4} = 3$

$|\vec{c}| = \sqrt{1+4+4} = 3 \quad \Rightarrow |\vec{b}| = |\vec{c}|$ (the triangle is isosceles), therefore $\beta = \gamma$.

As $\vec{b} \circ \vec{c} = (-1)(-2) + 1 \cdot 2 + 2 \cdot (-2) = 0$, so $\vec{b} \perp \vec{c}$, tj. $\alpha = 90°$. It is an isosceles right angle triangle, so $\beta = \gamma = 45°$.

Task 3.8.

Find the area of the triangle ΔABC if *A* (7,3,4); *B* (1,0,6) and *C* (4,5,-2) and the height $v = \overline{BD}$.

Solution:

$$\vec{a} = \overrightarrow{AB} = -6\vec{i} - 3\vec{j} + 2\vec{k} \quad \text{i}$$

$$\vec{b} = \overrightarrow{AC} = -3\vec{i} + 2\vec{j} - 6\vec{k}, \quad \text{to je}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & -3 & 2 \\ -3 & 2 & -6 \end{vmatrix} = \vec{i}\,(18-4) - \vec{j}\,(36+6) + \vec{k}\,(-12-9) = 14\vec{i} - 42\vec{j} - 21\vec{k}.$$

$$P_\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{196+1764+441} = \frac{49}{2}.$$

$$P_\Delta = \frac{|\vec{b}| \cdot v}{2} = \frac{49}{2} \quad \Rightarrow \quad v = \frac{49}{|\vec{b}|} = \frac{49}{\sqrt{9+4+36}} = 7.$$

Task 3.9.

Calculate the volume of the pyramide whose vertices are *A* (2,0,0), *B* (0,3,0), *C* (0,0,6) and *D* (2,3,8). Determine the height perpendicular to the base *ABC*.

Solution:

$V_T = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\circ\vec{c}\right|$, where V_T is the volume of the pyramide.

$\vec{a} = \overrightarrow{AB} = -2\vec{i} + 3\vec{j}$,
$\vec{b} = \overrightarrow{AC} = -2\vec{i} + 6\vec{k}$
$\vec{c} = \overrightarrow{AD} = 3\vec{j} + 8\vec{k}$.

$$\left(\vec{a}\times\vec{b}\right)\circ\vec{c} = \begin{vmatrix} -2 & 3 & 0 \\ -2 & 0 & 6 \\ 0 & 3 & 8 \end{vmatrix} = 36 + 48 = 84,$$ so

$$V_T = \frac{84}{6} = 14.$$

$$V_T = \frac{B\cdot h}{3} = 14 \Rightarrow h = \frac{42}{B} \quad \text{where } B = P_\Delta = \frac{\left|\vec{a}\times\vec{b}\right|}{2}.$$

Now

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18\vec{i} + 12\vec{j} + 6\vec{k},$$ so

$$B = \frac{\sqrt{324+144+36}}{2} = \frac{1}{2}\sqrt{504} = \sqrt{126},$$

Thus,

$$h = \frac{42}{\sqrt{126}} = \frac{42}{\sqrt{9\cdot 14}} = \frac{42}{3\sqrt{14}} = \frac{14}{\sqrt{14}} = \sqrt{14}.$$