4 VECTORS

ABSTRACT:

This chapter begins with the definition of the concept of vector, and then the basic operations including vectors are defined. Scalar, vector and mixed product as well as their applications will be presented. Exercises are also included, with the answers at the end of each task.

AIM: To acquire skills in solving tasks with vectors, and also to understand concepts standing behind those calculations. Vectors are also important for a number of real-life applications in the maritime field, which can be seen from the last part.

Learning outcomes of the lesson

1. View vectors geometrically
2. Determine equal and opposite vectors, magnitude and unit vector
3. Express vector as linear combination of vectors
4. Demonstrate vectors in the coordinate system
5. Calculate with vectors algebraically and graphically
6. Apply scalar, vector and mixed vector product.

Previous knowledge of mathematics: determinants, geometry in the plane R^2 and space R^3

Relatedness with solving problems in the maritime field: The real-world problems encountered most often with vectors are navigation problems. These navigation problems use variables like speed or velocity of vessel and direction or course to form vectors for computation. Some navigation problems ask us to find the actual (ground) speed of a vessel in wind situation using the combined forces of the wind and the vessel's velocity. Additionally, problems such as plotting, effect of ocean current on navigation, Cremona diagram used in statics of trusses to determine the forces in members can be solved by vector algebra.

Contents:

4.1 The concept of vector

Definition (vector):

A **vector** is a directed line segment which has *magnitude, direction and orientation*. Examples of vectors include force, time, distance, velocity, acceleration…

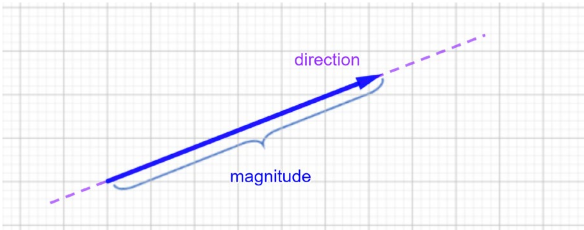


Figure 4.1 A vector

Vector vs Scalar

Numbers are called **scalars**. A scalar is a quantity which has magnitude only, but it does not any direction.

Notation

We denote vectors by labels of their initial and terminal points letters including arrows above the label, such as $\overrightarrow{AB}$ or small letters with arrows above the letter, such as $\vec{a}$.

The notation $\overrightarrow{AB}$ is useful because it indicates the orientation and location of the vector.

A Geometric View of Vectors

Geometrically, a vector or directed line segment is any ordered couple (A, B) of two arbitrary points A and B from space R^3, where A is the initial point, and B is the terminal point of this line segment.

A vector can be drawn as a directed line segment (as an **arrow**), whose length is the magnitude of the vector and with an arrow indicating the direction (orientation).

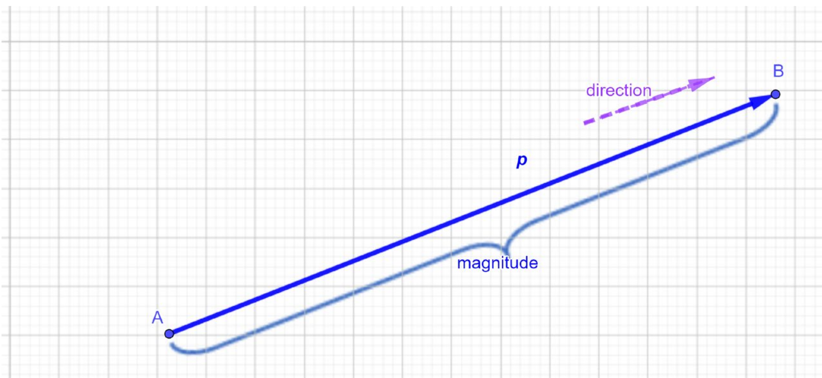


Figure 4.2 A vector is represented by a directed line segment from its initial point A to its

terminal point **B**

Line p on which the vector $\overrightarrow{AB}$ lies is called ***direction of this vector***. The ***direction*** of the vector is from its initial point A to its terminal point B.

Definition (magnitude or norm):

The ***magnitude*** or ***norm*** of the vector is the distance (the length of the line) between the initial and terminal points of the vector $\overrightarrow{AB}$ and is denoted as $|\overrightarrow{AB}|$.

Thus, if the initial point of a vector $\overrightarrow{AB}$ is $A(x_A, y_A)$ and the terminal point is $B(x_B, y_B)$, then the magnitude of a vector can be found using **Pythagoras's theorem**:

$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

IMPORTANT NOTE

A magnitude is always a positive number. It is a scalar.

Types of Vectors

$\overrightarrow{AA} = \vec{0}$ is called ***zero - vector*** and its norm is equal to 0. The zero-vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Unit vector of the vector $\vec{a}$ is the vector

$\vec{a}_0 = \frac{\vec{a}}{|\vec{a}|}$, $\vec{a} \neq \vec{O}$ i $|\vec{a}_0| = 1$.

To every point in a plane there can be assigned a unique vector whose initial point is in the origin O and the terminal point is in the given point $P(x_P, y_P)$. This vector is called the ***position*** or ***radius*** vector and it may be represented as (x_P, y_P) .

Example:

In the following figure, point A has the position vector $\vec{u}$ and point C has the position vector $\vec{v}$.

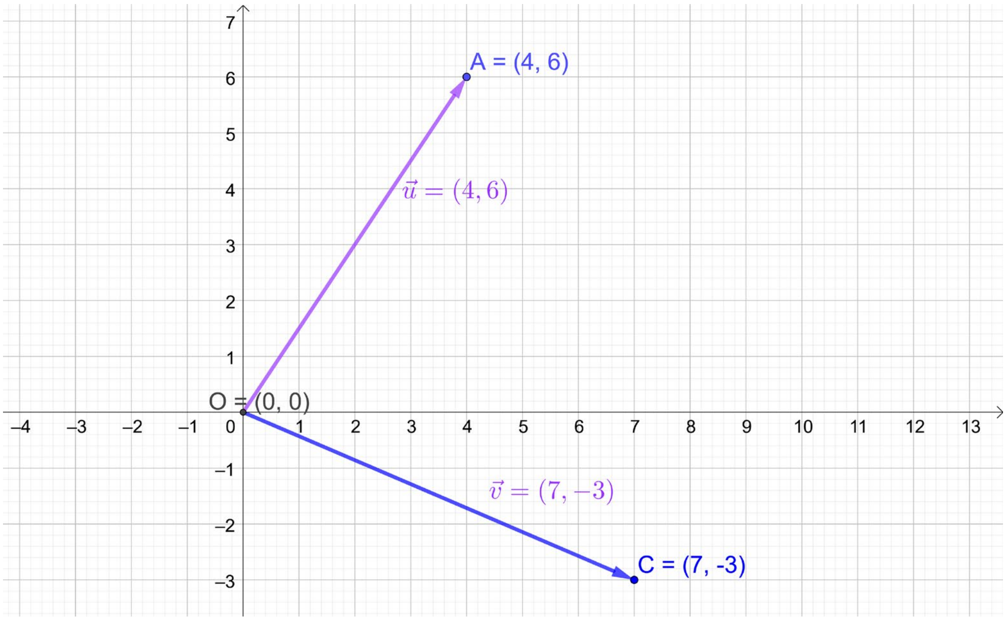


Figure 4.3

Vector $\overrightarrow{BA}$ is ***the opposite vector*** to vector $\overrightarrow{AB}$ and is designated $\overrightarrow{BA} = -\overrightarrow{AB}$.

The direction (orientation) of vectors

If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ that lie on the same line or a parallel line to the same, then their direction (orientation) can only be equal or opposite. The direction of the vector is shown by the arrow at the end of the vector.

For two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lying on the same line or the parallel lines, it holds:

1) they are of the same orientation if points *A* and *B* lie on the same side with regard to point *O*;

2) they are of the opposite orientation if the points *A* and *B* lie on different sides with regard to point *O*.

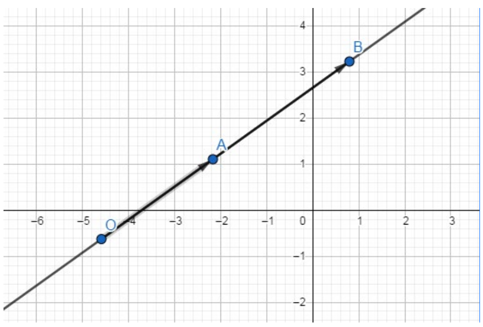


Figure 4.4 These position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ have the same orientation

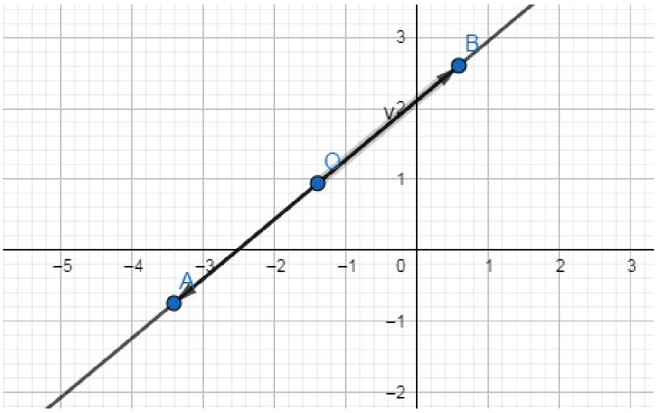


Figure 4.5 Vectors $\overrightarrow{\boldsymbol{OA}}$ and $\overrightarrow{\boldsymbol{OB}}$ are of the opposite orientation

Example

In **Figure 4.6** vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and have the same orientation.

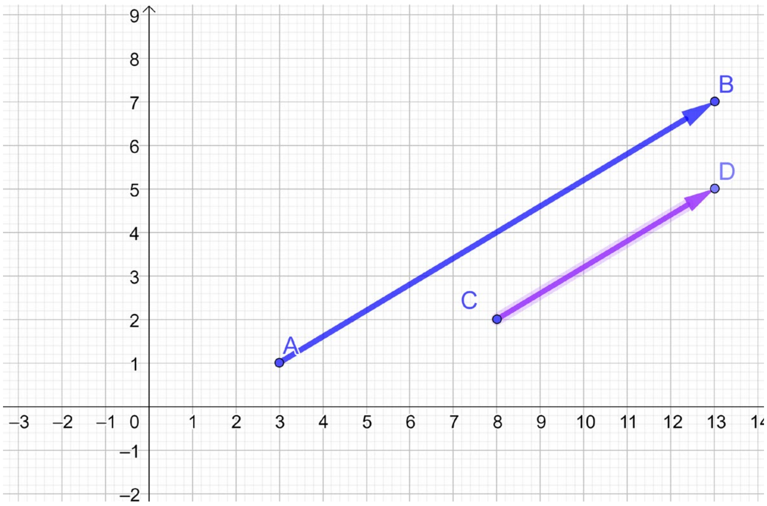


Figure 4.6 The vectors with the same orientation

In **Figure 4.7** vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel and are going in the opposite orientation.

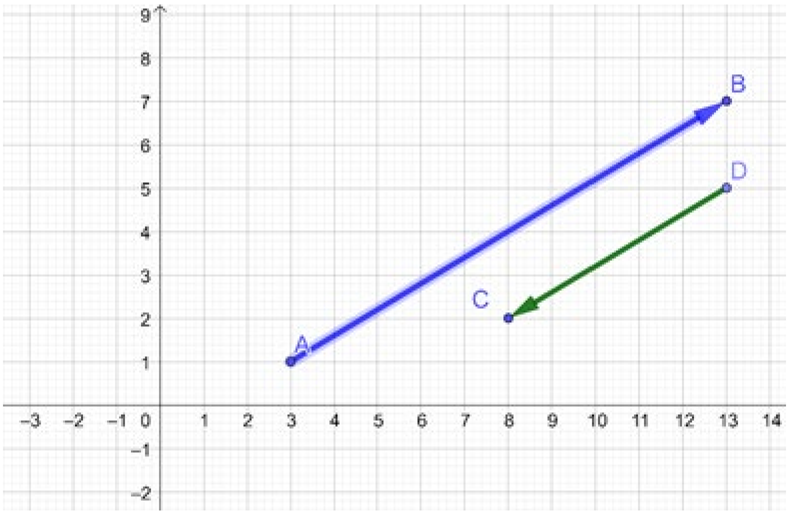


Figure 4.7 The vectors with the opposite orientation

Vectors $\vec{a}$ and $\vec{b}$ are mutually ***collinear*** if they lie in the same or in parallel lines, i.e. if there exists the number $\lambda \in$ R such that $\vec{\boldsymbol{a}} = \lambda\vec{\boldsymbol{b}}$. According to the agreement, zero - vector is ***collinear*** with any vector.

Example

All vectors on the figure below lie on the same line. The vectors are collinear vectors.

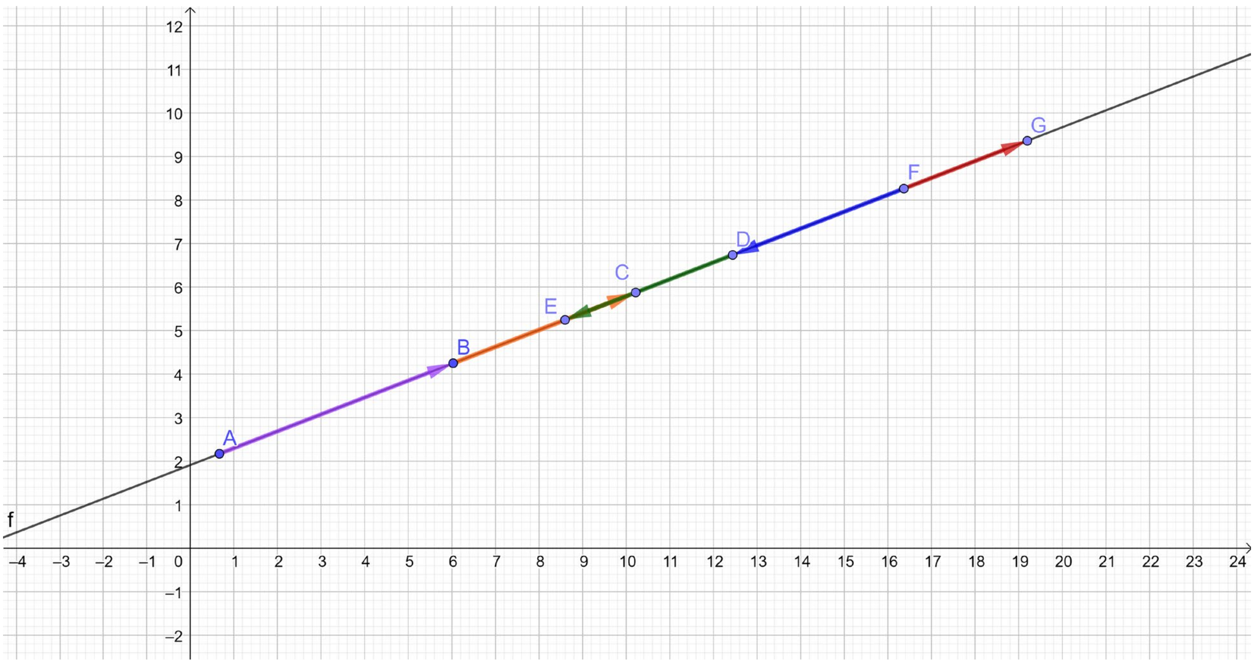


Figure 4.8 The example of collinear vectors

The following vectors on Figure 4.8 *with the same direction (orientation) are:*

- $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{FG}$
- $\overrightarrow{DE}, \overrightarrow{FD}$.

Vector coplanarity – a concept in geometry referring to the position of points and vectors. Four points in the space are ***coplanar*** if they lie in the same plane. Three vectors are ***coplanar*** (or linearly dependent) if they lie in a single plane (or parallel planes). Coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ may be represented as

$$\vec{c} = k\vec{a} + l\vec{b}, \qquad k.l \in \Re$$

Two vectors are ***equal*** and designated as $\vec{a} = \vec{b}$ if:

(1) they have the same length (magnitude)

(2) they are collinear vectors

(3) they have the same direction.

Example

Are the vectors $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ from the figure equal?

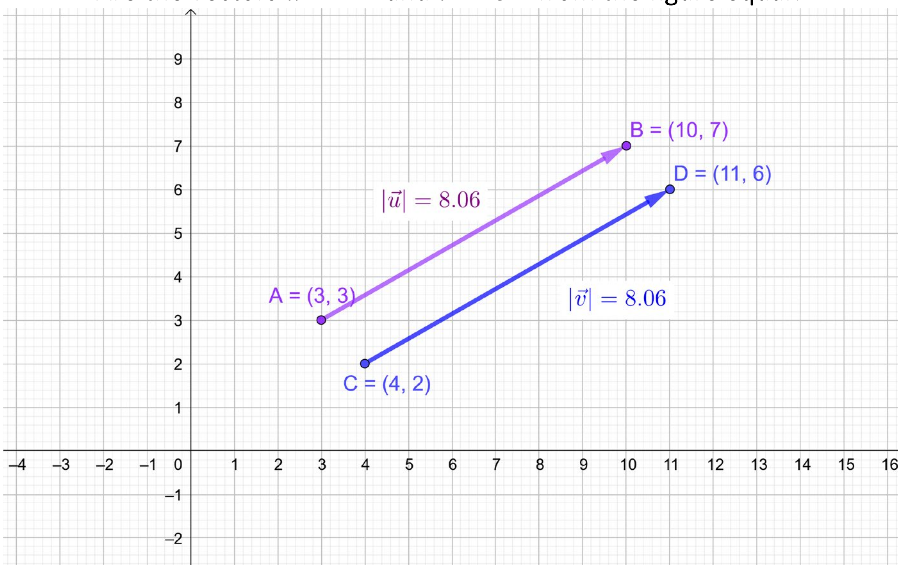


Figure 4.9

Solution:

Equal vectors have the same magnitude, the same direction and they are collinear vectors.

The vectors $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ **have the same magnitude**. The magnitude of the vector $\vec{u}$ is $|\vec{u}| = 8.06$ and of the vector $\vec{v}$ is $|\vec{v}| = 8.06$.

The vectors have the same directions and they lie in parallel lines.

<u>Answer</u>: The vectors $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$ are equal vectors.

Exercices 4.1

For the isosceles trapezoid in **Figure 4.10** find:

a) Equal vectors

b) Opposite vectors
c) Collinear vectors

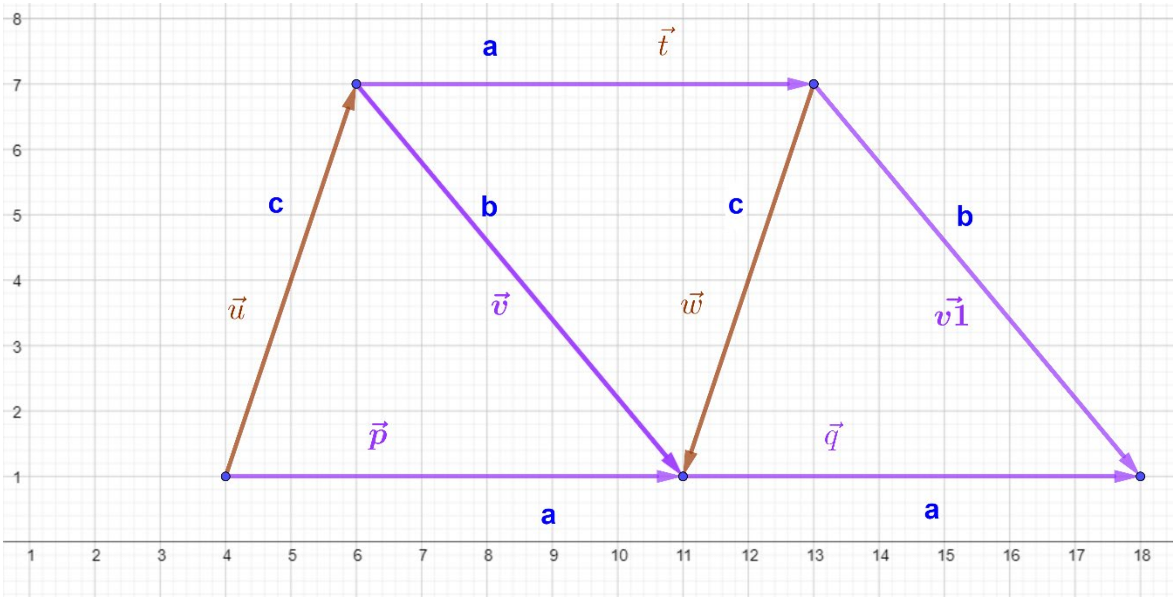


Figure 4.10

Solution:

d) Equal vectors: $\vec{p}=\vec{q}=\vec{t}$
e) Opposite vectors: $\vec{p}$ and $\vec{t}$, $\vec{q}$ and $\vec{t}$, $\vec{u}$ and $\vec{w}$, $\vec{v}$ and $\vec{v1}$
f) Collinear vectors: $\vec{p}, \vec{q}$ and $\vec{t}$, $\vec{u}$ and $\vec{w}$, $\vec{v}$ and $\vec{v1}$

Vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are ***linearly independent*** if from the equation

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \cdots + \lambda_n\vec{a_n} = \vec{0} \text{ it follows that je } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Projection of vector $\vec{b}$ onto vector $\vec{a}$ is calculated according to the formula:

$$proj_a\,\vec{b} = \left|\vec{b}\right| \cos\varphi, \text{ where } \varphi \text{ is the angle between vectors } \vec{a} \text{ and } \vec{b}.$$

4.2 Three basic vector operations

Vector addition

The sum of two vectors $\vec{a}$ and $\vec{b}$ is the vector starting in the initial point of the first vector, and finishing in the final point of the second vector and is designated as $\vec{a} + \vec{b}$ (Figure 4.11). This approach is called the ***triangle method***.

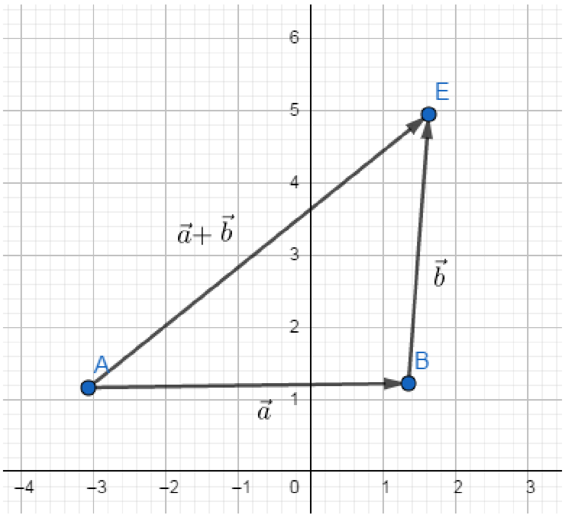


Figure 4.11 Adding vectors by the triangle method

The following features are valid:

(1) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(2) $\vec{a} + \vec{O} = \vec{O} + \vec{a} = \vec{a}$

(3) For each vector $\vec{a}$ there is the vector $\overrightarrow{a'}$ such that

$\vec{a} + \overrightarrow{a'} = \vec{a} + \overrightarrow{a'} = \vec{O}$.

(4) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(5) $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Inequality (5) follows from the triangle method. These three vectors are sides of a triangle. Thus, it is true that the length of any one side is less than the sum of the lengths of remaining sides.

Vector subtraction

Subtracting a vector is the same as adding its negative.

The difference of the vectors $\vec{a}$ *and* $\vec{b}$ is the sum of $\vec{a}$ and $(-\vec{b})$.

We define:

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) = \vec{a} + (-1)\vec{b} \quad \text{or}$$

$$\overrightarrow{AB} - \overrightarrow{CD} = \overrightarrow{AB} + \left(-\overrightarrow{CD}\right) = \overrightarrow{\boldsymbol{AB}} + \overrightarrow{\boldsymbol{DC}}$$

The difference of two vectors is the vector starting in the initial point of the first vector, and finishing in the final point of the opposite vector of the second vector.

Example

Find $\vec{a} - \vec{b}$ for the vectors in **Figure 4.12**.

Solution:

<u>Step 1:</u> We reverse the direction of the vector $\vec{b}$ we want to subtract.

<u>Step 2:</u> Add the vector $\vec{a}$ and the opposite vector of the vector $\vec{b}$ as usual.

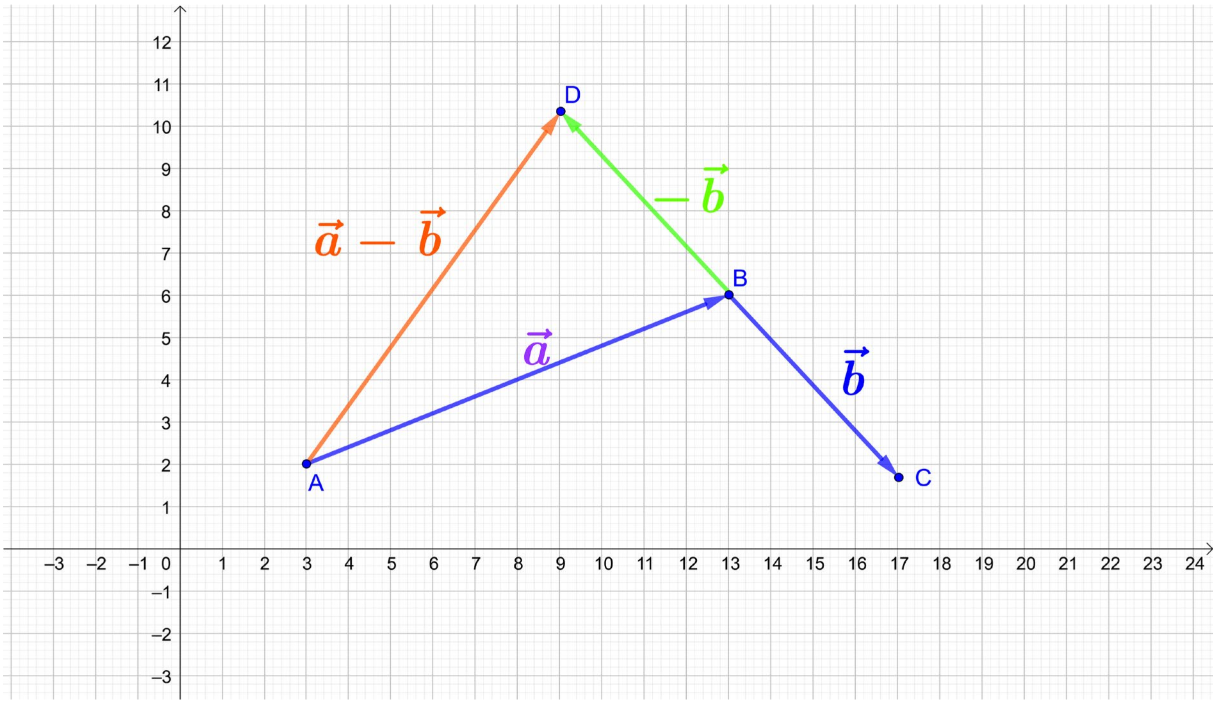


Figure 4.12 Vector subtraction

Multiplication of vector by a scalar (number)

With ***the multiplication by a scalar λ***, only the magnitude of the vector $\vec{a}$ is multiplied by the absolute value of the scalar λ. The result of the product $\lambda \cdot \vec{a}$ is the vector with a magnitude that is $|\lambda|$ times the magnitude of the vector $\vec{a}$ if $\lambda > 0$, and opposite direction if $\lambda < 0$.

The following applies:

1) vector norm $\lambda\vec{a}$ is equal to the product of absolute $|\lambda|$ and magnitude (norm) of $|\vec{a}\,|$;

$$|\lambda\vec{a}| = |\lambda||\vec{a}|.$$

2) vector $\lambda\vec{a}$ is collinear with vector $\vec{a}$.

The features of scalar multiplication by vector:

(1) $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$

(2) $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$

(3) $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$

(4) $1 \cdot \vec{a} = \vec{a}$

(5) $(-1)\cdot\vec{a} = -\vec{a}$

Note that the vector $-\vec{a}$ has the same magnitude as $\vec{a}$, but has the opposite direction.

4.3 Scalar, vector and mixed triple products

Scalar product or Dot product of vector $\vec{a}$ and vector $\vec{b}$ is equal to the product of their magnitudes and the cosine of the angle between these vectors.

$$\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos \sphericalangle (\vec{a}, \vec{b})$$

Vectors $\vec{a}$ and $\vec{b}$ are perpendicular ($\vec{a} \perp \vec{b}$) if and only if $\vec{a} \circ \vec{b} = 0$.

Vector product or Cross product

If $\vec{a}$ and $\vec{b}$ are two vectors in space, then their ***cross product:***

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\varphi$, where φ is the angle between vectors.

The Cross Product $\vec{a} \times \vec{b}$ of two vectors is another vector $\vec{c}$ that is at right angles to both.

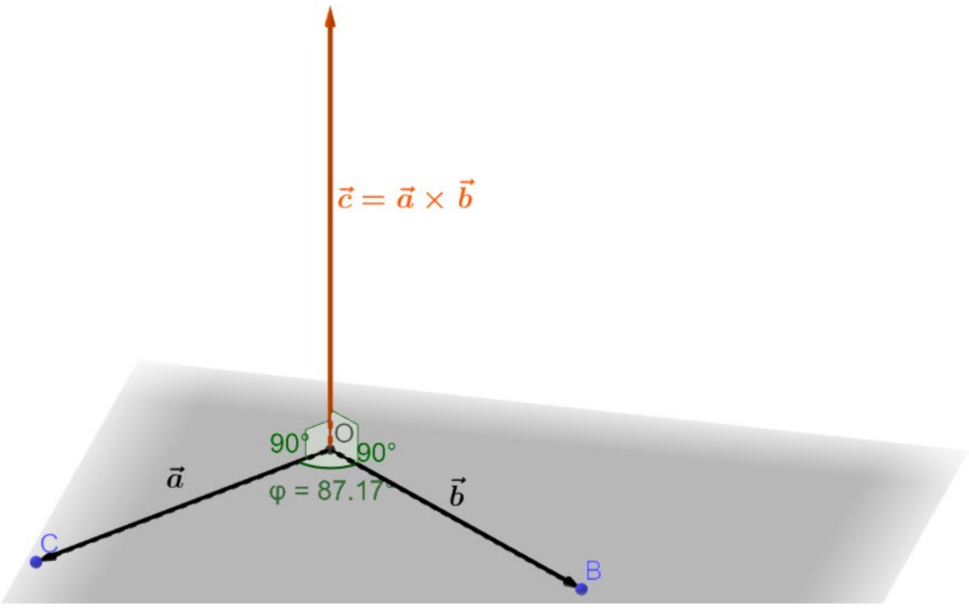


Figure 4.13 Vector product

The vector or cross product (*red*) is:

- zero in length when vectors $\vec{a}$ and $\vec{b}$ point in the same, or opposite, direction

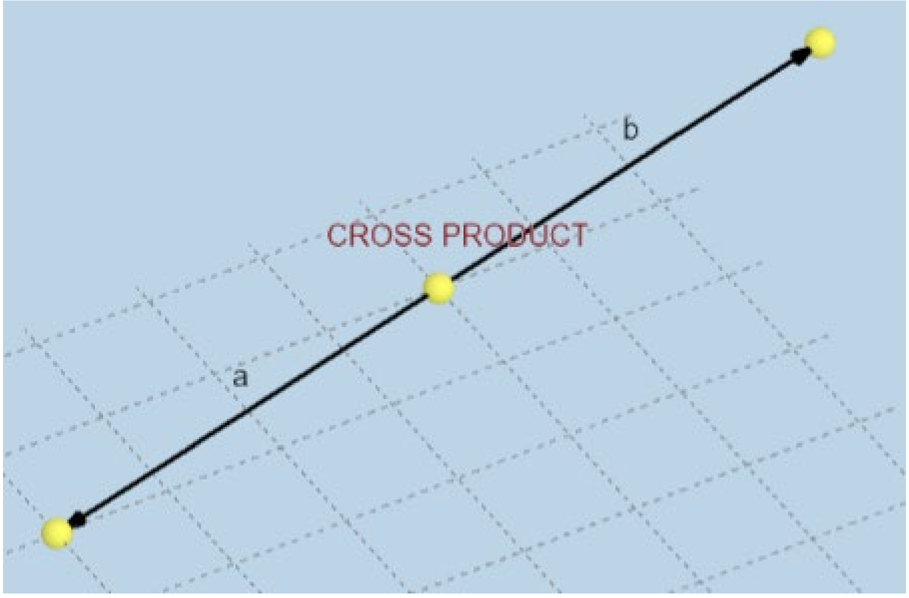


Figure 4.14 Vector product when vectors $\vec{a}$ and $\vec{b}$ point in the opposite direction

- reaches maximum length when vectors $\vec{a}$ and $\vec{b}$ are at right angles.

From the geometric perspective, $|\vec{a} \times \vec{b}|$ is the **area of the parallelogram spanned** by vectors $\vec{a}$ and $\vec{b}$**.**

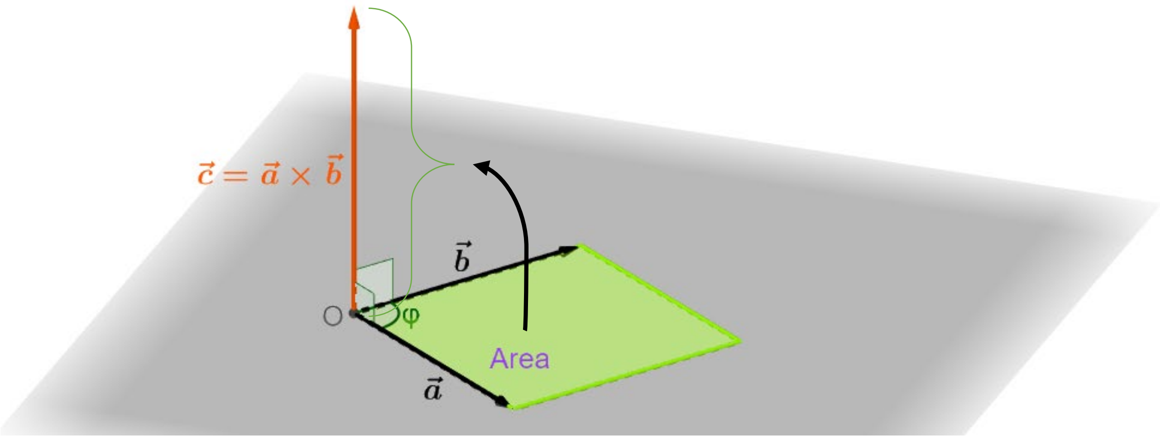


Figure 4.15 Area of the parallelogram spanned by vectors a ⃗ and b ⃗.

Vector $\vec{a} \times \vec{b}$ is perpendicular to vectors $\vec{a}$ and $\vec{b}$.

The features of vector product:

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

(3) $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

Mixed product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is scalar:

$(\vec{a} \times \vec{b}) \circ \vec{c} = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cos\varphi$, where φ is the angle between vectors $\vec{a} \times \vec{b}$ and $\vec{c}$.

Volume V of the parallelepiped determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is calculated as:

$$V = |(\vec{a} \times \vec{b}) \circ \vec{c}|.$$

It holds that:

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $\begin{cases} \vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \\ (\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) \end{cases}$

(3) $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

4.4 Vectors in rectangular coordinate system

Let *E* be the unit point on the *x*-axis, *F* the unit point on *y*-axis, *G* the unit point on *z*-axis and point *O* the origin in 3D-space R^3. Then radius vector $\overrightarrow{OE}$ is equal to unit vector $\vec{\imath}$, radius vector $\overrightarrow{OF}$ equal to unit vector $\vec{\jmath}$ and radius vector $\overrightarrow{OG}$ equal to unit vector $\vec{k}$.

The vectors $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ are the ***unit vectors*** in the positive *x*, *y*, and *z* direction, respectively. In terms of coordinates, we can write them as $\vec{\imath} = (1,0,0)$, $\vec{\jmath} = (0,1,0)$ and $\vec{k} = (0,0,1)$.

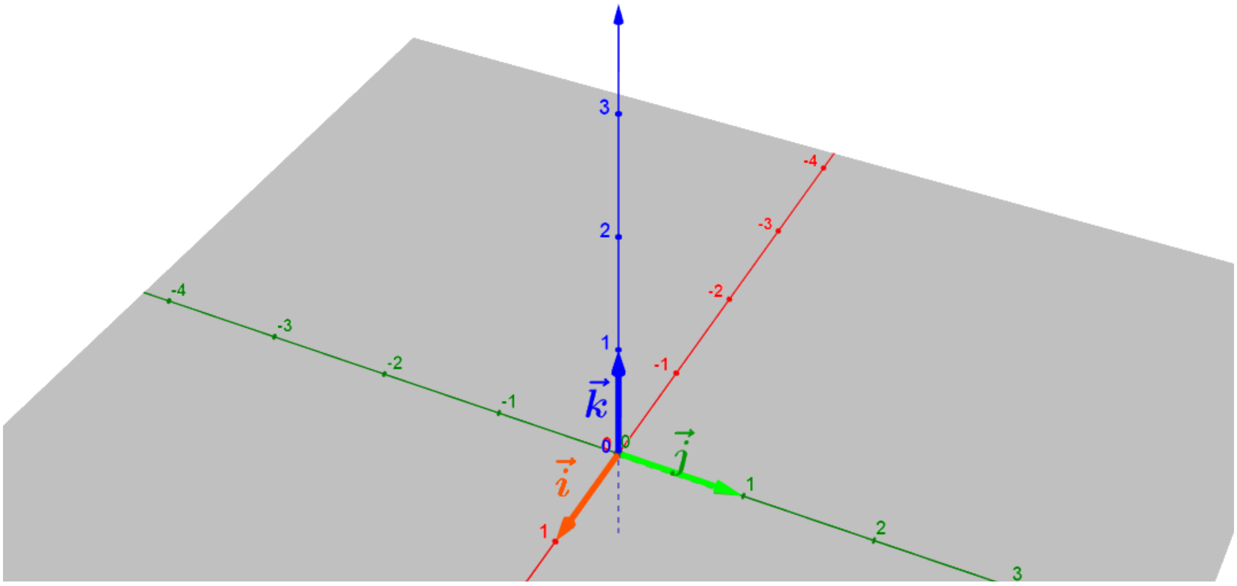


Figure 4.16 The standard unit vectors in three dimensions

A vector in three-dimensional space

Any point P in space can be assigned three coordinates $P = (a_x, a_y, a_z)$ and its position vector $\vec{a} = \overrightarrow{OP}$. In **Figure 4.17**, the vector $\vec{a}$ is drawn as the pink arrow with initial point fixed at the origin.

We assign coordinates of a vector $\vec{a}$ by orthogonal projecting the vector $\vec{a}$ on each axis x, y and z.

Black vectors $\overrightarrow{a_x} = \overrightarrow{OP_1}$, $\overrightarrow{a_y} = \overrightarrow{OP_2}$ and $\overrightarrow{a_z} = \overrightarrow{OP_3}$ show the projections of $\vec{a} = \overrightarrow{OP}$ on each axis and represent ***the scalar components or coordinates*** (a_x, a_y, a_z).

Any three-dimensional vector $\vec{a}$ can be represented as linear combination of three unit vectors $\vec{\imath}, \vec{\jmath}$, and $\vec{k}$ i.e. it can be expressed as the sum of the products of a scalar component and a unit vector lying on the corresponding coordinate axis in the form

$$\vec{a} = (a_x, a_y, a_z) = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$$

The magnitude of that position vector of point P is equal to: $|\vec{a}| = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}$.

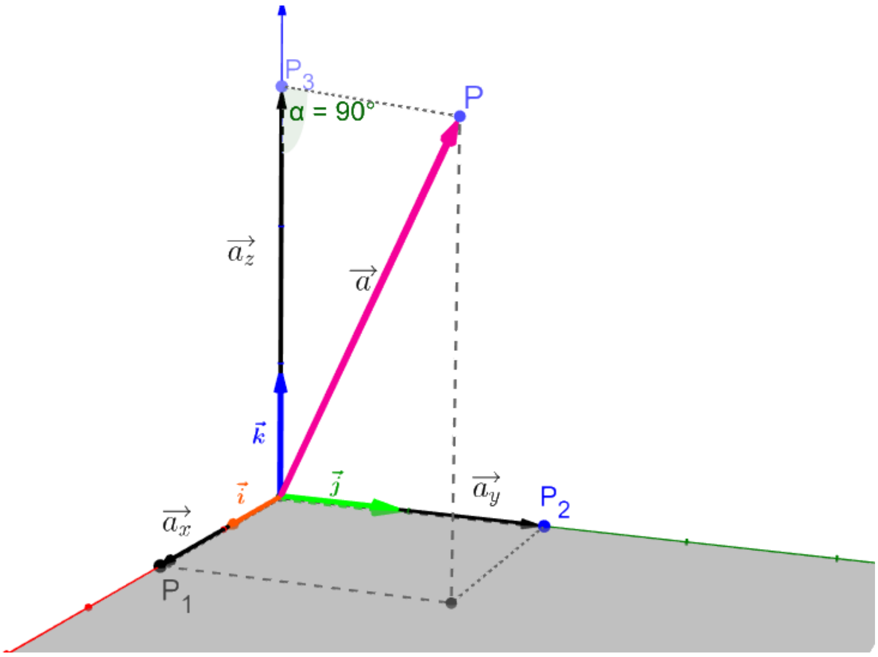


Figure 4.17 A vector $\vec{a}$ in three-dimensional space

Component Form of a Vector in three-dimensional space

Let be $\overrightarrow{AB}$ a vector with initial point $A(x_{i,}, y_{i,}, z_i)$ and terminal point $T(x_{t,}, y_{t,}, z_t)$. The component form of the vector $\overrightarrow{AB}$ can be expressed as $\overrightarrow{AB} = (x_{t,} - x_{i,},\ y_{t,} - y_{i,},\ z_{t,} - z_{i,})$.

The magnitude of that vector is equal to:

$$|\overrightarrow{AB}| = \sqrt{(x_{t,} - x_{i,})^2 + (y_{t,} - y_{i,})^2 + (z_{t,} - z_{i,})^2}.$$

A vector in the-plane

Each point P in the Cartesian system in the plane is identified with its x and y coordinates, $P(a_x, a_y)$.

Cartesian coordinates system in the plane is defined by an ordered triple $(O, \vec{\imath}, \vec{\jmath})$ where O is the origin, $\vec{\imath}$ and $\vec{\jmath}$ are two non-collinear unit vectors:

$\vec{\imath}$ - unit vector on the abscissa axis

$\vec{\jmath}$ - unit vector on the ordinate axis.

The position vector of the point P, $\overrightarrow{OP}$ may be represented as a linear combination of unit vectors:

$$\overrightarrow{OP} = a_x\vec{\imath} + a_y\vec{\jmath}$$

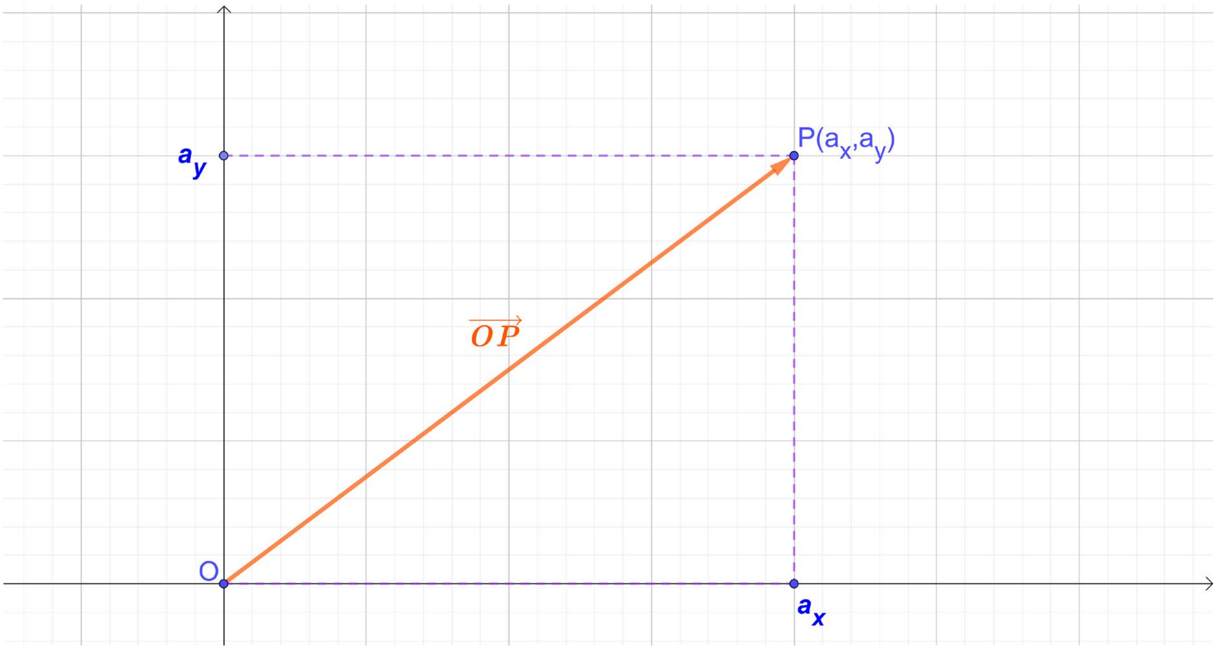


Figure 4.18 The components of a vector in the plane

Scalars a_x and a_y are called components of the vector $\overrightarrow{OP}$.

Using the Pythagorean Theorem, we can obtain an expression for the **magnitude** of a vector in terms of its components.

The magnitude of that position vector of point P is equal to:

$$|\overrightarrow{OP}| = \sqrt{(a_x)^2 + (a_y)^2}.$$

Component Form of a Vector in E^2

Let be $\vec{a}$ a vector with initial point $(x_{i,}, y_{i,})$ and terminal point $(x_{t,}, y_{t,})$. The component form of the vector $\vec{a}$ can be expressed as $\vec{a} = (x_{t,} - x_{i,},\ y_{t,} - y_{i,})$.

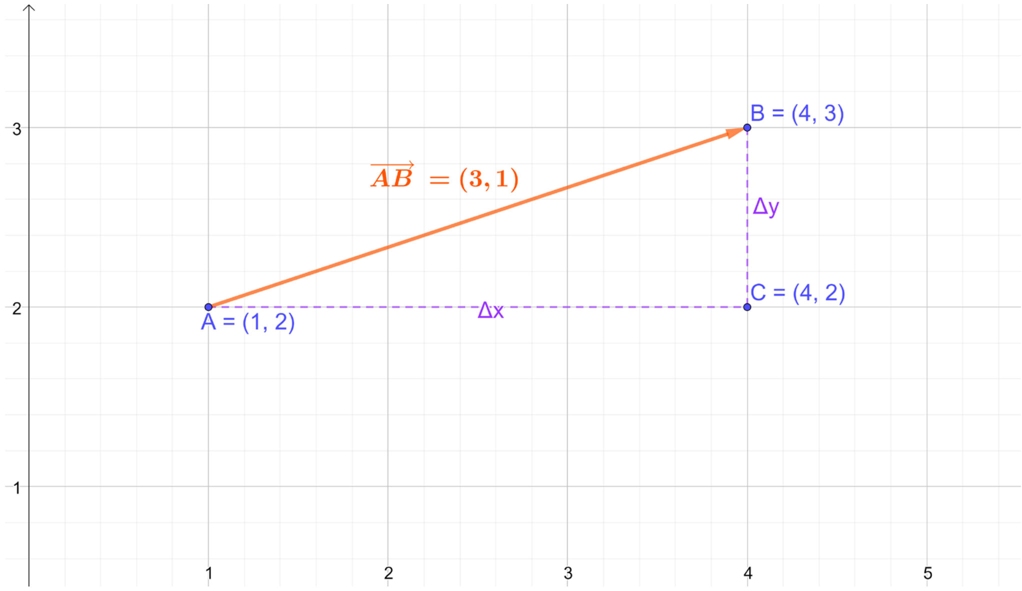
The magnitude of that vector is equal to:

$$|\vec{a}| = \sqrt{(x_{t,} - x_{i,})^2 + (\ y_{t,} - y_{i,})^2}.$$

Example:

Draw in the plane the vector $|\overrightarrow{AB}|$ whose initial point A is (1, 2) and terminal point B is (4, 3) and find its magnitude.

Solution:



$$|\overrightarrow{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

$$|\overrightarrow{AB}| = \sqrt{(4-1)^2 + (3-2)^2}$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + 1^2}$$

$$|\overrightarrow{AB}| = \sqrt{10} \approx 3.2$$

In some cases, only the magnitude and direction of a vector are known, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (Figure 4.19).

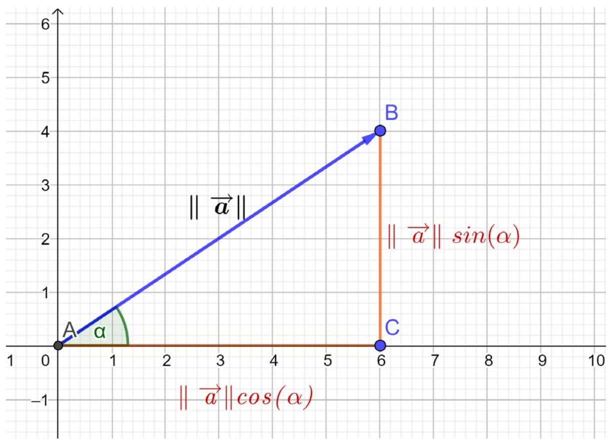


Figure 4.19 The components of a vector form the cathetus of a right triangle, with the vector as the hypotenuse

4.5 Performing Operations in Component Form

Scalar multiplication and Vector addition

- Scalar multiplication:

$$\lambda \cdot \vec{a} = (\lambda \cdot a_{x,}, \lambda \cdot a_{y,}, \lambda \cdot a_{z,})$$

- Vector addition:

$$\vec{a} + \vec{b} = (a_{x,}, a_{y,}, a_{z,}) + (b_{x,}, b_{y,}, b_{z,}) = (a_{x,} + b_{x,}, a_{y,} + b_{y,}, a_{z,} + b_{z,})$$

Example:

Let $\vec{a}$ be the vector with initial point (1, 1) and terminal point (3, -4) , and let $\vec{b} = (-1, 4)$.

a) Express $\vec{a}$ in component form and find $\|\vec{a}\|$. Then, using algebra, find
b) $\vec{a} + \vec{b}$
c) $3\vec{b}$
d) $\overrightarrow{2a} - \vec{b}$.

$Solution$:

a) $\vec{a}$ = (3-1, -4-1) =(2, -5)

$\vec{a} + \vec{b} = (2, -5) + (-1,4) = (2 + (-1), -5 + 4) = (1, -1)$ (orange vector on **Figure** 4.20)

b) $3\vec{b} = 3(-1,4) = (3 \cdot (-1), 3 \cdot 4) = (-3,12)$
c) $\overrightarrow{2a} - \vec{b} = 2 \cdot (2, -5) - (-1,4) = (4, -10) + (1, -4) = (4 + 1, -10 - 4) = (5, -14)$

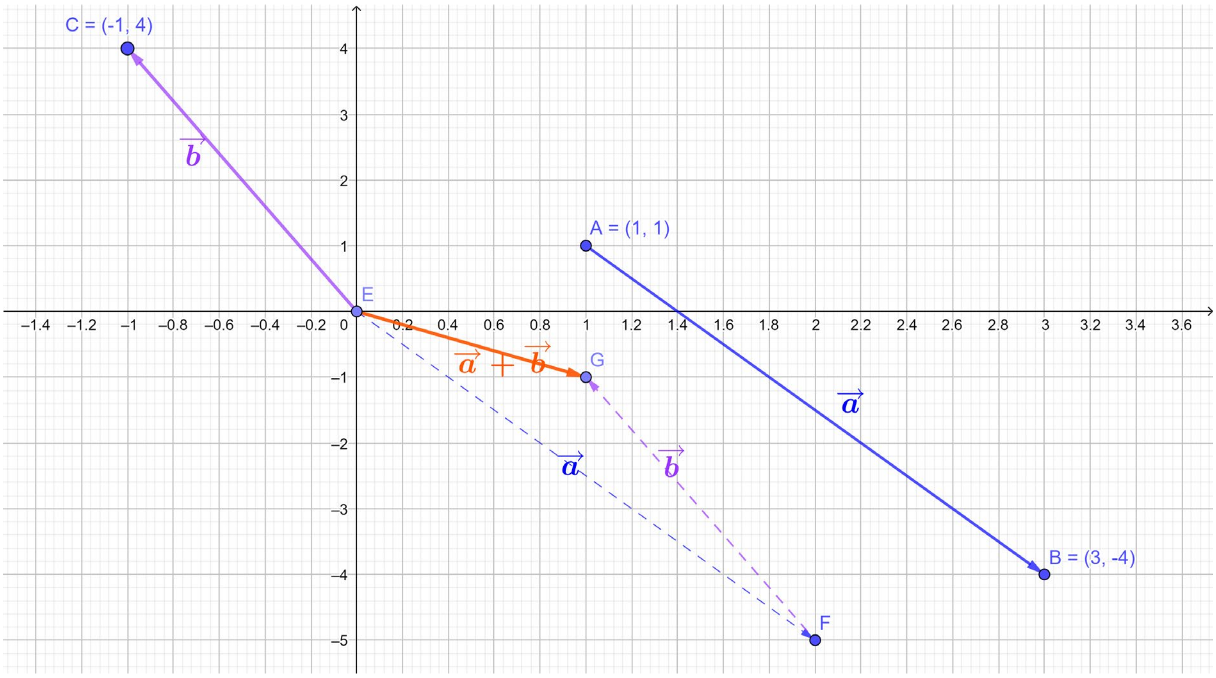


Figure 4.20 The component form of the vector $\vec{a}$ is $\vec{a} = (2, -5)$. In component form, $\vec{a} + \vec{b} = (1, -1)$

Dot or scalar product of vectors $\vec{a}$ and $\vec{b}$ is equal to:

$$\vec{a} \circ \vec{b} = a_x b_x + a_y b_y + a_z b_z .$$

The result from scalar product of two vectors is always a real number.

If the angle between two verctors $\vec{a}$ and $\vec{b}$ is 90°, then $\vec{a} \cdot \vec{b} = 0$, because $cos(90°) = 0$.

Angle between vectors $\vec{a}$ and $\vec{b}$ is calculated according to the formula for dot product:

$$cos \sphericalangle (\vec{a}, \vec{b}) = \frac{\vec{a} \circ \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}} .$$

Cross or vector product of vectors $\vec{a}$ and $\vec{b}$ can be calculated according to the formula

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} .$$

The result from cross or vector product of two vectors is always a vector.

Mixed triple product is calculated according to the formula:

$$\left(\vec{a}\times\vec{b}\right)\circ\vec{c}=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Example:

Let are given the vectors: $\vec{a}$ = (1, 0, 1) and $\vec{b}$ = (2, –2, 2). Determine the angle $\varphi = \sphericalangle(\vec{a}, \vec{b})$.

Solution:

$$cos\varphi = \frac{(1,0,1)\cdot(2,-2,2)}{\sqrt{1^2+0^2+1^2}\cdot\sqrt{2^2+(-2)^2+2^2}} = \frac{2+0+2}{\sqrt{2}\cdot\sqrt{12}} = \frac{4}{2\sqrt{6}}$$

$$\varphi = 35°15'52''$$

Example:

Examine whether the vectors $\vec{a}$ = (2, –3,1) and $\vec{b}$ = (3, 1, 0) are perpendicular to each other.

Solution:

Two vectors are perpendicular to each other if and only if scalar product of these vectors is zero. Therefore,

$$\vec{a}\circ\vec{b} = 6-3+0=3$$

Answer: The vectors are perpendicular to each other.

Example:

Determine area of a triangle that is spanned by vectors $\vec{a}$ = (–3, 2, –2) i $\vec{b}$ = (1, –4, 1).

Solution:

$$P_\Delta = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -3 & 2 & -2 \\ 1 & -4 & 1 \end{vmatrix} = -6\vec{\imath}+\vec{\jmath}+10\vec{k}$$

$$P_\Delta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|-6\vec{i} + \vec{j} + 10\vec{k}\right| = \frac{1}{2}\sqrt{36+1+100} = \frac{1}{2}\sqrt{137}$$

Example:

Given are points $A\ (1,2,1);\ B\ (3,-2,1);\ C\ (1,4,3)\ i\ D\ (5,0,5)$. Determine volumen V of the parallelopiped determined by vectors $\overrightarrow{AB}, \overrightarrow{AC}\ i\ \overrightarrow{AD}$.

$Solution$:

$\vec{a} = \overrightarrow{AB} = (2,-4,0)$

$\vec{b} = \overrightarrow{AC} = (0,2,2)$

$\vec{c} = \overrightarrow{AD} = (4,-2,4)$

$$(\vec{a} \times \vec{b}) \circ \vec{c} = \begin{vmatrix} 2 & -4 & 0 \\ 0 & 2 & 2 \\ 4 & -2 & 4 \end{vmatrix} = 8$$

$V = |(\vec{a} \times \vec{b}) \circ \vec{c}| = |8| = 8$

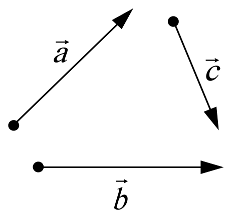
4.6 Exercises

Task 3.1.

Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are shown in the figure below. Construct vectors:

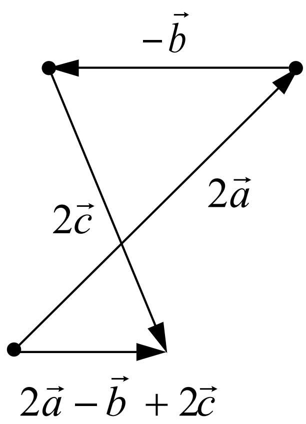
(1) $2\vec{a} - \vec{b} + 2\vec{c}$;

(2) $\vec{a} + \vec{b} + \vec{c}$

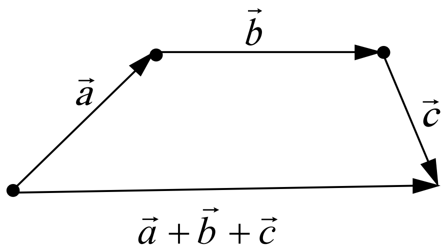


Solution:

(1)



(2)



Task 3.2.

If $\vec{e}_1$ and $\vec{e}_2$ are non-collinear vectors, determine the constant β so that vectors $\vec{a} = 2\vec{e}_1 - \vec{e}_2$ i $\vec{b} = \beta\vec{e}_1 + \vec{e}_2$ are collinear.

Solution:

$\vec{a}$ i $\vec{b}$ are collinear if there is the number $\lambda \in$ R such that $\vec{a} = \lambda\vec{b}$. It follows that:

$$3\vec{e}_1 - 2\vec{e}_2 = \lambda(\beta\vec{e}_1 - 3\vec{e}_2),$$

i.e.

$$3\vec{e}_1 - 2\vec{e}_2 = \lambda\beta\vec{e}_1 - 3\lambda\vec{e}_2.$$

It follows that:

$$\left.\begin{array}{l} 3 = \lambda\beta \\ -2 = -3\lambda \end{array}\right\} \Rightarrow \lambda = \frac{2}{3}, \quad \beta = \frac{9}{2}.$$

Vectors $\vec{a}$ and $\vec{b}$ are collinear for $\beta = \frac{9}{2}$.

Task 3.3.

Prove that vectors $\vec{a} = 5\vec{i} + 4\vec{j} + 3\vec{k}$, $\vec{b} = 3\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{c} = 8\vec{i} + \vec{j} + 3\vec{k}$ are coplanar.

Solution:

$8\vec{i} + \vec{j} + 3\vec{k} = \alpha(5\vec{i} + 4\vec{j} + 3\vec{k}) + \beta(3\vec{i} + 3\vec{j} + 2\vec{k})$, slijedi:

$$\left.\begin{array}{l} 5\alpha + 3\beta = 8, \\ 4\alpha + 3\beta = 1, \\ 3\alpha + 2\beta = 3, \end{array}\right\} \Rightarrow \alpha = 7; \quad \beta = -9.$$

$\vec{c} = 7\vec{a} - 9\vec{b}$, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Task 3.4.

Given are vectors $\vec{a} = 4\vec{e}_1 - 3\vec{e}_2$, $\vec{b} = \vec{e}_1 - \vec{e}_2$ i $\vec{c} = 2\vec{e}_1 - 3\vec{e}_2$, in which the angle between the basis vector $\sphericalangle(\vec{e}_1, \vec{e}_2) = \frac{\pi}{3}$. Factorize vector $\vec{a}$ along the lines of vectors $\vec{b}$ i $\vec{c}$

Solution:

$$\vec{a} = \alpha\vec{b} + \beta\vec{c}.$$

It follows that:

$$4\vec{e}_1 - 3\vec{e}_2 = \alpha(\vec{e}_1 - \vec{e}_2) + \beta(2\vec{e}_1 - 3\vec{e}_2),$$

$$4\vec{e}_1 - 3\vec{e}_2 = (\alpha + 2\beta)\vec{e}_1 + (-\alpha - 3\beta)\vec{e}_2.$$

$$\left.\begin{array}{l} 4 = \alpha + 2\beta \\ -3 = -\alpha - 3\beta \end{array}\right\} \Rightarrow \beta = -1; \alpha = 6.$$

Therefore,

$$\vec{a} = 6\vec{b} - \vec{c}$$

Task 3.5.

Coordinates of the vertices ΔABC are: A (–2, 0, 4); B (4, 1, –2); C (2, –4, –4).

Calculate:

(1) Norm of vector $\overrightarrow{AB}$;

(2) unit vector of $\overrightarrow{AC}$.

Solution:

(1) $\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k} = 6\vec{i} + \vec{j} - 6\vec{k}$;

$$|\overrightarrow{AB}| = \sqrt{6^2 + 1^2 + (-6)^2} = \sqrt{73}.$$

(2) $\vec{b} = \overrightarrow{AC} = 4\vec{i} - 4\vec{j} - 8\vec{k}$;

$$b_0 = \frac{\vec{b}}{|\vec{b}|} = \frac{4\vec{i} - 4\vec{j} - 8\vec{k}}{\sqrt{4^2 + (-4)^2 + (-8)^2}} = \frac{1}{4\sqrt{6}} \cdot (4\vec{i} - 4\vec{j} - 8\vec{k}) = \frac{\sqrt{6}}{6}(\vec{i} - \vec{j} - 2\vec{k}).$$

Task 3.6.

Find vector projection $\vec{a} = \vec{i} + \vec{j} - 4\vec{k}$ onto the line of vector $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$.

Solution:

$$proj_b\,\vec{a} = \frac{\vec{a} \circ \vec{b}}{|\vec{b}|} = \frac{1 \cdot 6 + 1 \cdot (-3) + (-4) \cdot 2}{\sqrt{6^2 + (-3)^2 + 2^2}} = -\frac{5}{7}.$$

Task 3.7.

Determine the angles ΔABC with vertices A (2, –1, 3), B (1, 1, 1) i C (0, 0, 5).

Solution:

$$\vec{c} = \overrightarrow{AB} = -\vec{i} + 2\vec{j} - 2\vec{k},$$

$$\vec{b} = \overrightarrow{AC} = -2\vec{i} + \vec{j} + 2\vec{k},$$

$\vec{a} = \overrightarrow{CB} = \vec{i} + \vec{j} - 4\vec{k}$.

$\vec{b} \circ \vec{c} = |\vec{b}| \cdot |\vec{c}| \cos\alpha \Rightarrow \cos\alpha = \dfrac{\vec{b} \circ \vec{c}}{|\vec{b}| \cdot |\vec{c}|}$.

$|\vec{b}| = \sqrt{4+1+4} = 3$

$|\vec{c}| = \sqrt{1+4+4} = 3 \quad \Rightarrow |\vec{b}| = |\vec{c}|$ (the triangle is isosceles), therefore $\beta = \gamma$.

As $\vec{b} \circ \vec{c} = (-1)(-2) + 1 \cdot 2 + 2 \cdot (-2) = 0$, so $\vec{b} \perp \vec{c}$, tj. $\alpha = 90°$. It is an isosceles right angle triangle, so $\beta = \gamma = 45°$.

Task 3.8.

Find the area of the triangle ΔABC if A (7,3,4); B (1,0,6) and C (4,5,-2) and the height $v = \overline{BD}$.

Solution:

$$\vec{a} = \overrightarrow{AB} = -6\vec{i} - 3\vec{j} + 2\vec{k} \quad \text{i}$$

$$\vec{b} = \overrightarrow{AC} = -3\vec{i} + 2\vec{j} - 6\vec{k}, \quad \text{to je}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & -3 & 2 \\ -3 & 2 & -6 \end{vmatrix} = \vec{i}(18-4) - \vec{j}(36+6) + \vec{k}(-12-9) = 14\vec{i} - 42\vec{j} - 21\vec{k}.$$

$$P_\Delta = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{196+1764+441} = \frac{49}{2}.$$

$$P_\Delta = \frac{|\vec{b}| \cdot v}{2} = \frac{49}{2} \quad \Rightarrow \quad v = \frac{49}{|\vec{b}|} = \frac{49}{\sqrt{9+4+36}} = 7.$$

Task 3.9.

Calculate the volume of the pyramide whose vertices are A (2,0,0), B (0,3,0), C (0,0,6) and D (2,3,8). Determine the height perpendicular to the base ABC.

Solution:

$V_T = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\circ\vec{c}\right|$, where V_T is the volume of the pyramide.

$\vec{a} = \overrightarrow{AB} = -2\vec{i} + 3\vec{j}$,
$\vec{b} = \overrightarrow{AC} = -2\vec{i} + 6\vec{k}$
$\vec{c} = \overrightarrow{AD} = 3\vec{j} + 8\vec{k}$.

$$\left(\vec{a}\times\vec{b}\right)\circ\vec{c} = \begin{vmatrix} -2 & 3 & 0 \\ -2 & 0 & 6 \\ 0 & 3 & 8 \end{vmatrix} = 36 + 48 = 84, \quad \text{so}$$

$$V_T = \frac{84}{6} = 14.$$

$$V_T = \frac{B\cdot h}{3} = 14 \Rightarrow h = \frac{42}{B} \quad \text{where } B = P_\Delta = \frac{\left|\vec{a}\times\vec{b}\right|}{2}.$$

Now

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18\vec{i} + 12\vec{j} + 6\vec{k}, \quad \text{so}$$

$$B = \frac{\sqrt{324+144+36}}{2} = \frac{1}{2}\sqrt{504} = \sqrt{126},$$

Thus,

$$h = \frac{42}{\sqrt{126}} = \frac{42}{\sqrt{9\cdot 14}} = \frac{42}{3\sqrt{14}} = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

4.7 Connectedness and application in the maritime field

Example 1: A problem involving the bearing (direction) of a boat. A boat is traveling at a speed of 25 mph. The vector that represents the velocity is (√2, -√2). Determine the bearing of the boat.

SOLUTION:

Let's start by sketching the situation described (Figure 3.8).

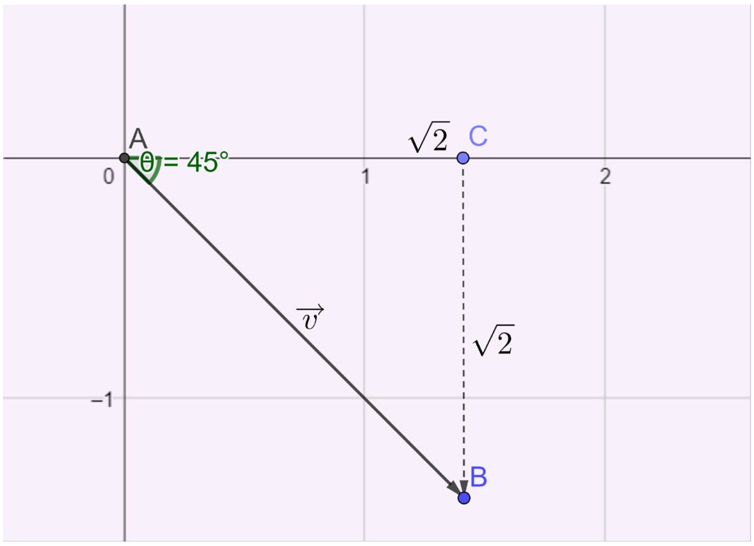


Figure 4.21 The component form of the velocity vector $\vec{v} = (\sqrt{2}, -\sqrt{2})$

Initially, the boat travels in the direction of velocity vector $v = (\sqrt{2}, \sqrt{2}) = \sqrt{2}\,\vec{\imath} - \sqrt{2}\,\vec{\jmath}$. Make a triangle ΔABC. It is the right triangle and the length of its cathetus $\overleftarrow{AC}$ and $\overleftarrow{AB}$ is $\sqrt{2}$.

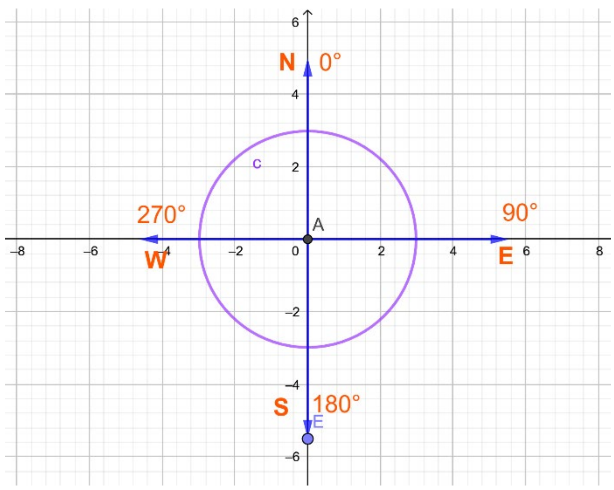
The **angle** θ is the bearing of the boat and it can be solved using tangent. Tangent is opposite cathetus over adjacent so we will get $\sqrt{2}$ over $\sqrt{2}$:

$$tg\theta = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\theta = arctg(1) = 45°$$

In navigationm, the bearing is described in the following way.

- Traveling due **north** it would be said the boat is at a bearing of **zero** degrees.
- If the boat travels due **east** it is at a bearing of **90** degrees.
- If the boat travels due **south** it is at a bearing of **180** degrees.



- If the boat travels due **west** it is at a bearing of **270** degrees.

So if the boat travels due east it would be at baring of 90 degrees. But actually, it travels at an extra $45°$ so $90° + 45° = 135°$. **The bearing $\boldsymbol{\theta}$ of the boat is 135°.**

Example 2: (https://www.geogebra.org/m/kmsTyU2S)

A ship leaves port on a bearing of 28° and travels 7.5 miles. The ship then turns due east and travels 4.1 miles. How far is the ship from the port and what is its bearing?

SOLUTION:

The following figure represents the described situation.

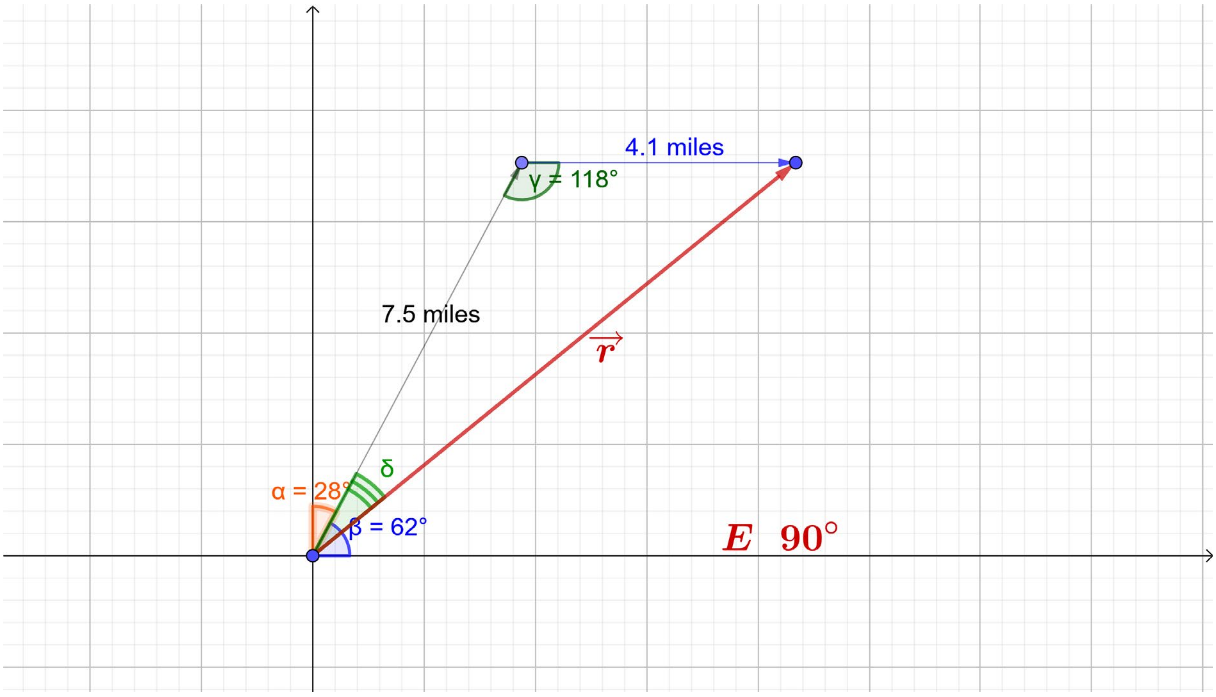


Figure 4.22 The component form of the resultant vector $\vec{r}$

The angle α is the bearing of the ship because the bearing is measured due to the north. In the direction α = 28°, the ship travels 7.5 miles and then it turns due east and travels in this direction for 4.1 miles. The distance between the current ship's position and the port would be determined as a magnitude of a resultant vector $\vec{r}$.

To solve the problem it is needed to determine some more details.

If α is 28°, it implies $\beta = 90° - \alpha = 62°$.

The angle $\boldsymbol{\gamma}$ would have to be $\gamma = 180° - 62° = 118°$

The magnitude of the vector would be determined by the cosine law:

$$|\vec{r}|^2 = (7.5)^2 + (4.1)^2 - 2 \cdot (7.5)(4.1)cos(118°)$$

$$|\vec{r}|^2 = (56.25) + (16.81) - 61.5 \cdot (-0.469)$$

$$|\vec{r}|^2 = 101.93$$

$$|\vec{r}| = \sqrt{101.93} = 10.09$$

The ship is far 10.1 miles from the port.

Ship's bearing would be determined as sum of the angles **α** and **δ.**

The angle **δ** would be calculated from the sinus law as follows:

$$\frac{sin(118°)}{10.1} = \frac{\boldsymbol{sin(\delta)}}{4.1}$$

$$\frac{0{,}8829}{10.1} = \frac{\boldsymbol{sin(\delta)}}{4.1}$$

$$0.0874 = \frac{\boldsymbol{sin(\delta)}}{4.1} \Rightarrow \boldsymbol{sin(\delta)} = 0.0874 \cdot 4.1 = 0.3584$$

$$\boldsymbol{\delta = arcsin(0.3584) = 21°}$$

The ship's bearing is $\boldsymbol{\alpha + \delta = 28° + 21° = 49°}$.

Example 3:

A sailboat under auxiliary power is sailing on a bearing 25° north of west at 6.25 mph. The wind is blowing in the direction 35° south of west at 15 mph.

SOLUTION:

This navigation problem uses variables like speed and direction to form vectors for computation. Like with an aircraft, some navigation problems ask us to find the ground speed of a boat using the combined the force of the wind which affects on the boat and the speed of the boat. For these problems it is important to understand the resultant of two forces and the components of force.

In our case, the sailboat is sailing in the direction 25° north of west (NW) or in direction of the speed vector $\vec{s}$ (sailboat vector). It is sailing at 6.26 mph which would have to be interpreted as the magnitude of the vector $\vec{s}$. There is a 15 mph wind's current blowing in the direction 35° south of west (SW). On the figure below this force is represented as the vector $\vec{w}$. If there is no wind the sailboat will be sailing in the direction of its course or 15 degrees.

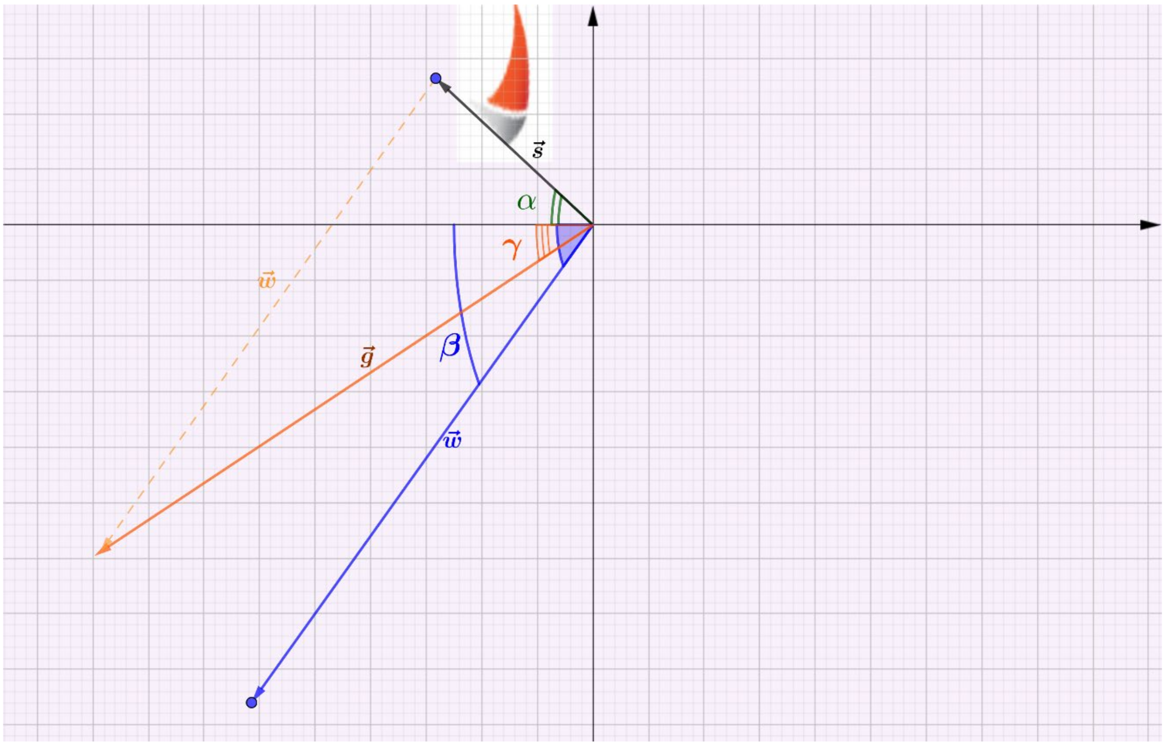


Figure 4.23 The vector of actual (ground) sailboat's velocity is denoted by $\vec{g}$ and it is the resultant of the sailboat vector plus the wind vector.

But the wind is blowing in the direction of the vector $\vec{w}$ and if you do not compensate sailboat's course, the sailboat will be pushed by the wind and its actual course would be to the down or SW of destination. Probably, it will affect how fast the sailboat is sailing.

Vectors will help us to determine exactly where the sailboat is sailing and how fast. It would be done through the vector addition.

If we consider adding the sailboat vector $\vec{s}$ to the wind vector $\vec{w}$, geometrically it means that it is possible to move wind vector $\vec{w}$ in the sense that these two vectors have the same start point. On the figure, this moved vector is noted by the orange colour and it is also wind vector. The resultant of these vectors (the red vector on the figure) is the sailboat vector plus the wind vector. This resultant vector is something what we are interested in right because this red vector is real direction where the sailboat is sailing. It is called the actual or ground course and it is noted by $\vec{g}$.

It is needed to find the speed of sailboat as the magnitude of the vector $\vec{g}$ and the direction of the sailboat by finding the angle γ.

The first part of problem is to determine the component of the sailboat vector $\vec{s}$ and the wind vector $\vec{w}$.

It can be done on the following way:

$$\vec{s} = \left(6.25 \cdot cos(180° - 25°), 6.25 \cdot sin(180° - 25°)\right).$$

Note that we must take the angle from second quadrant because the sailboat is sailing north of west. It is the reason why the angle of sailboat vector is determined as $180° - 25°$.

Calculating it is possible to find the components

$$\vec{s} = (-5.66, 2.64)$$

The wind vector lies in the third quadrant so its angle will be $180° + 35° = 215°$.

The components of the wind vector can be calculated as follows:

$$\vec{w} = \left(15 \cdot cos(215°), 15 \cdot sin(215°)\right) = (-12.29, -8.60)$$

$$\vec{g} = \vec{s} + \vec{w} = (-5.66, 2.64) + (-12.29. -8.60) = \left(-5.66 + (-12.29), 2.64 + (-8.60)\right)$$

$$\vec{g} = (-17.95, -5.96)$$

The speed of the sailboat is the same as the magnitude of the vector $\vec{g}$.

$$||\vec{g}|| = \sqrt{(-17.95)^2 + (-5.96)^2}$$

$||\vec{g}|| = \sqrt{322,2025 + 35,5216} = \sqrt{357,724} = 18.91359 \approx 18.91$ =

The actual (ground) speed of the sailboat is around 19.91 mph.

The second part of our problem is to determine the actual course (course over ground).

$$tg(\gamma) = \frac{-5.96}{-17.95} = 0.3320334 \;\; (III\ quadrant)$$

$\gamma = arctg(0.3320334) = 18.37° + 180° \; (III\ quadrant)$

We can say that the sailboat is sailing in the direction 18.37° south of the west.

Example 4:

A large ship has gone aground in the harbour and two tugboats, with cables attached, attempt to pull it free. If one tug pulls in the direction 38° north of east with a force of 2300 lbs and the second tug plus in the direction 9° south of the east with the force of 1900 lbs. Find the direction and magnitude of the resultant force.

SOLUTION:

The problem is sketched in the figure 3.11.

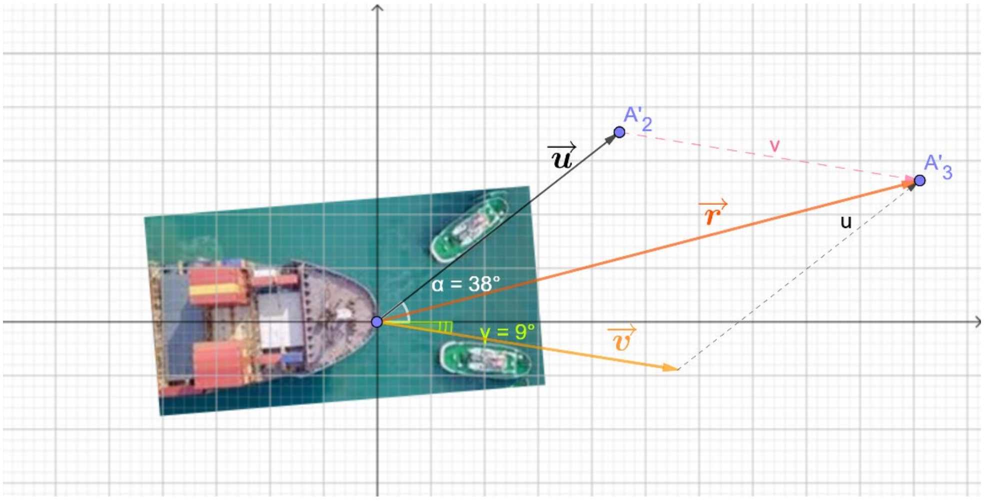


Figure 4.24 The tugboat vectors

The first step is to determine the components of tugboat vectors $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \left(2300 \cdot cos(38°), 2300 \cdot sin(38°)\right) = (1812.42, 1416.02\ )$$

$$\vec{v} = \left(1900 \cdot cos(-9°), 1900 \cdot sin(-9°)\right) = (1876.60, -297.23\ )$$

$$\vec{r}\ = \vec{u} + \vec{v} = (3689.028, 1118.79\ \ )$$

The next step is to determine the magnitude and the direction of vector $\vec{r}$.

Magnitude: $\overrightarrow{\|r\|}\ = \sqrt{3689.028^2\ +\ 1118.79^2}\ =\ \sqrt{14.860.551,52}$= 3854.938

Direction: $arctg\left(\frac{1118.79}{3689.028}\right) = arctg(0,3032750) = 16.87°\ north\ of\ east$

Example 5:

Navigational plotting

Ship A sails on course $K\ =\ 056°$, at the speed of 14.5 knots. In time $t\ =\ 13{:}16$ the officer of the watch observes on the radar display the reflection of another vessel B. The OOW proceeds with observing the ship B and obtains the following data:

tx = 13:20 taken the bearing to ship B $\boldsymbol{\omega 1}\ =\ \mathbf{093°}$ at a distance $\boldsymbol{D1}\ =\ \mathbf{8.5}\ \boldsymbol{NM}$,

tx = 13:26 taken the bearing to ship B $\boldsymbol{\omega 2}\ =\ \mathbf{093°}$ at a distance $\boldsymbol{D2}\ =\ \mathbf{8}\ \boldsymbol{NM}$.

The data recorded are represented in the picture below ($\boldsymbol{M}\ \mathbf{1}\ \boldsymbol{cm}\ =\ \mathbf{1}\boldsymbol{NM}\ =\ \mathbf{1}\ \boldsymbol{kt}$) :

Solution obtained by using true plotting:

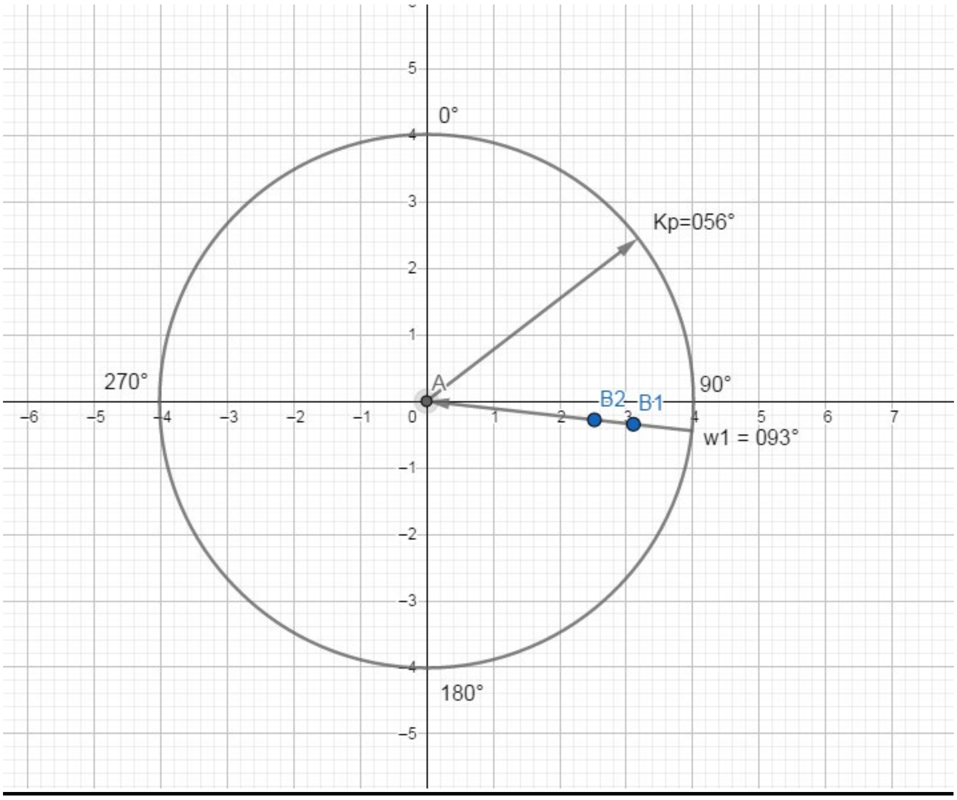


Figure 4.25

Ship (B) sails on course $K = 318°$, at the speed of $v = 14.2\,kt$
- ASPECT – the angle between the bearing line and the other ship's course line, $A = 44° = const.$, $\omega = 093° = const.$

Conclusion – since the recording bearing of ship B does not change in the time lapse of 6 minutes, and the distance between the ships A and B decreases, the ships are on collision courses, and if the avoiding manoeuvre prescribed in the Collision Regulations is not undertaken, the ships will collide.

Example 6:

Effect of ocean current on navigation

The figure shows the easiest way to solve the problem of navigation with the current. A ship sails from position $P1$ on course $Kp = 110º$ towards the port of Zadar. After having travelled 3 NM the ship was supposed to be in the DR position $Pz1$, but the measurements found that, due to drift, the ship was not in that position, but in position $P2$. The magnitude of the drift due to the ocean current can be calculated if the positions $Pz1$ and $P2$ are connected by vector showing by how much the ship went off course after travelling 3 NM. Although the course held on the compass was 110 º (course through the water), it can be seen that the ship was sailing in the true course 103 º (course over ground). If from position P_2 the course is plotted towards Zadar, it will be 117 º (planned course over ground), and if the ocean current had no effect, after travelling the next 3 NM the ship would be in position $Pz2$. However, since the current has an effect, the ship will again go off course and will find itself in position differing from $Pz2$ by the magnitude of the drift vector (drift vector is marked red in colour). Therefore, from $Pz2$ using the triangle and dividers, the drift vector is transferred, but in the opposite direction, and the course towards Zadar is obtained if the position P_2 is connected with the peak of the plotted drift vector. Thus, the course 123º is obtained (planned course through the water), and if the ship proceeds on the course planned, it will reach the planned point of arrival.

Direction and strength of the current can be detrmined so that with the use of nautical triangles the direction of the current is determined in degrees, and its strength can be detrmined so that the current vector is taken in the dividers and from the chart the strength in NM is read against the latitude scale.