5. Polar Form of a Complex Number

The complex number $z = a + bi$ is written in ***polar form*** as

$$z = r(\cos\varphi + i\sin\varphi)$$

where $a = r\cos\varphi$, $b = r\sin\varphi$, $r = \sqrt{a^2 + b^2}$ and

$$\varphi = \begin{cases} 2\pi - arctan\left|\frac{b}{a}\right|, \ if\ a > 0,\ b < 0 \\ arctan\left|\frac{b}{a}\right|, \ if\ a > 0, b \geq 0 \\ \pi - arctan\frac{b}{a}, \ if\ a < 0,\ b > 0 \\ \pi + arctan\frac{b}{a}, \ if\ a < 0, b < 0 \\ \frac{\pi}{2}, if\ b > 0,\ a = 0 \\ \frac{3\pi}{2}, if\ b < 0,\ a = 0 \end{cases}$$

The value of ***r*** is called the modulus of the complex number and the angle ϕ is called the *argument* of the complex number z with $0 \leq \varphi < 2\pi$.

EXAMPLE 3: Write z =$1 + \sqrt{3}i$ in polar form. a =1, b = $\sqrt{3}$

1) $r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$
2) a > 0, b > 0, $\varphi = arctan\left|\frac{b}{a}\right| = arctan\left|\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$
3) z = 2(cos $\frac{\pi}{3}$ + *i*sin$\frac{\pi}{3}$)

6. Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ be two complex numbers in polar form.

Their product, $z_1 z_2$ is

$$z_1 z_2 = r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$$

To multiply two complex numbers, multiply moduli and add arguments.

EXAMPLE 4: Find $z_1 z_2$, if $z_1 = 4(\cos 30° + i\sin 30°)$ and $z_2 = 2(\cos 60° + i\sin 60°)$.

$$z_1 z_2 = 4(\cos 30° + i\sin 30°) \cdot 2(\cos 60° + i\sin 60°)$$
$$= 4 \cdot 2[\cos(30° + 60°) + i\sin(30° + 60°)] = 8(\cos 90° + i\sin 90°)$$

7. Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$ be two complex numbers in polar form.

Their quotient, $\frac{z_1}{z_2}$ is $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)]$.

To divide two complex numbers, divide moduli and subtract arguments.

EXAMPLE 5: Find $\frac{z_1}{z_2}$, if $z_1 = 10(\cos 58° + i\sin 58°)$ and $z_2 = 2(\cos 30° + i\sin 30°)$.

$$\frac{z_1}{z_2} = \frac{10(\cos 58° + i\sin 58°)}{2(\cos 30° + i\sin 30°)} = 5[\cos(58° - 30°) + i\sin(58° - 30°)]$$
$$= 5(\cos 28° + i\sin 28°)$$