Applications of complex numbers

AIMS:

1)Students know different applications of complex numbers in maritime studies

2) Students know concept of phasors and alternating current

3) Students are introduced to different components of AC circuits

1. Complex numbers in electrical circuits

1.1. Introduction into alternating current and phasors

There are two types of electrical power supplies used both on board of ships and on the mainland: direct current (DC) and alternating current (AC) (Fig. 1.1, *a*, *b*). Let us assume that we apply either DC or AC voltage to a resistor R. Resistive element R models transformation of electrical energy into any other form of energy: mechanical, thermal, lighting, chemical, etc.

If a direct voltage U is applied to a resistor with resistance R (Fig. 1.1, *c*) it will cause a direct current I flowing in the circuit (Fig. 1.1, *d*). Both voltage and current are unipolar and related to each other by means of Ohm's law

$$I = \frac{U}{R}.$$

The power dissipating in the resistor R can be found by using Joule-Lenz law

$$P = UI = I^2 R.$$

The alternating voltage is usually a sine wave with the maximum value or amplitude U_m that can mathematically be described as follows

$$u = U_m \sin 2\pi f t = U_m \sin \omega t,$$

where u is the instantaneous value of the voltage (V) at the time instant t (s), f is the frequency of sine wave (Hz), $\omega = 2\pi f$ is the angular frequency (rad/s). The frequency f can be found via the period T (s) of the sine wave (Fig. 1.2) in the next way

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$$f = \frac{1}{T}.$$

Such a sine voltage being applied to a resistor *R* causes an alternating current *i* (Fig. 1.1, *e*)

$$i = I_m \sin 2\pi f t = I_m \sin \omega t,$$

where *i* is the instantaneous current value (A) at the time instant *t* (s), I_m is the amplitude of the current.

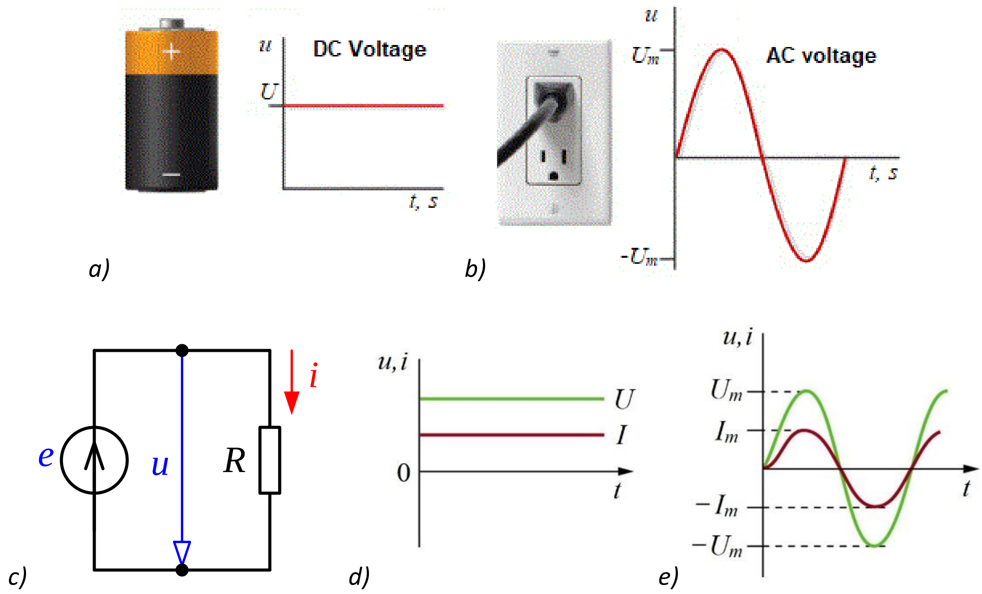

Fig. 1.1. DC and AC circuits: *a* – DC voltage source (battery), *b* – AC voltage source (an electrical network output: a socket), *c* – resistor connected to either DC or AC source, *d* – direct current caused by direct voltage, *e* – alternating current caused by alternating voltage.

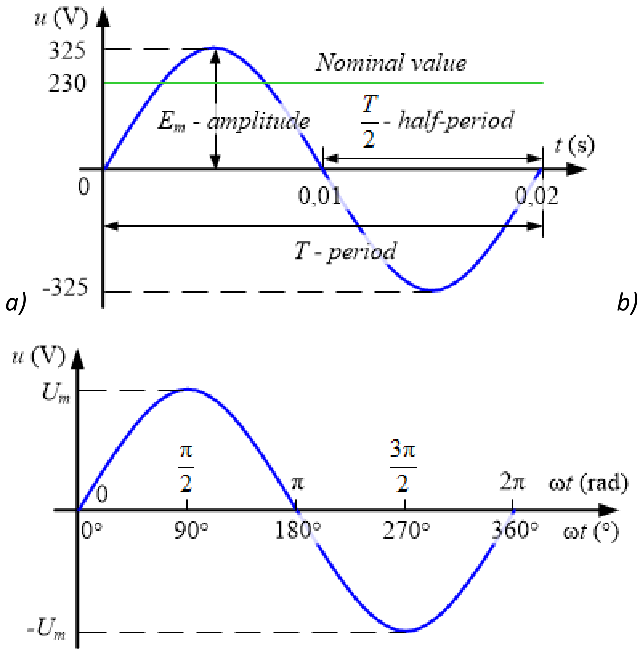

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Fig. 1.2. Sine wave of instantaneous voltage values alternating along the axis of time (*a*) and angle (b).

At each instant of time, there is relationship between instantaneous values of the voltage and current according to Ohm's law

$$i = \frac{u}{R}.$$

Since maximum values are also instant ones, there is valid

$$I_m = \frac{U_m}{R}.$$

Let us find the amplitudes of the alternating current and voltage that produce the same power *P* dissipation as happens in the resistance *R* under the direct voltage *U* and current *I*. In electrical engineering theory, it is found that the sinusoidal voltage *u* and current *i* with amplitudes

$$U_m = \sqrt{2}U \text{ and } I_m = \sqrt{2}I$$

produce the power *P* in the AC circuit in Fig. 1.3, *b* equal to the DC power caused by *U* and *I* in the DC circuit Fig. 1.3, *a*

$$P = UI = I^2 R.$$

The values of alternating voltage and current that are equivalents to DC values and found as follows

$$U = \frac{U_m}{\sqrt{2}} \text{ and } I = \frac{I_m}{\sqrt{2}}.$$

and they are named rms or effective values.

When one is telling that an AC socket voltage is 230 volts or a current in an AC cable is 100 amperes then there are mentioned effective values that produce the same power as a direct voltage of 230 V or a direct current of 100 A. For example, if there is a direct current of 10 A flowing through a resistance *R* of 100 Ω in Fig. 1.3, *a* then it produces the power $P = I^2R = 10^2 \cdot 100 = 10000$ W = 10 kW. To get the same power *P* dissipated on the resistor *R* of 100 Ω in Fig. 1.3, *b*, there should be applied the alternating current *i* with the effective value of 100 A, i.e. with the amplitude $I_m = 10\sqrt{2} = 14{,}1\ \mathrm{A}$.

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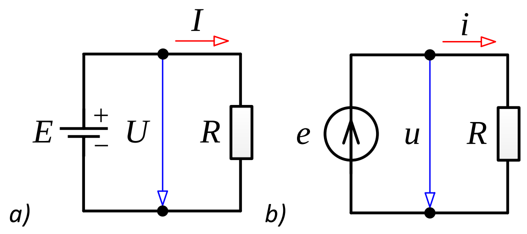

Fig. 1.3. DC (*a*) and AC (*b*) circuits.

1.2. Phasors of sine wave

There is a set of mathematical tools developed for DC circuits that could be applied to AC circuits. The application of Ohm's and Kirchhoff's laws to each instant of an alternating sine wave is a vast and complicated task. One way to facilitate the use of DC circuit laws in AC circuits is the application of effective values instead of amplitudes. However, it is not enough. The alternating nature of sine waves creates complex electromagnetic field around AC circuits that creates time (angle) shifts between voltages and currents. To represent and calculate those shifts, there was proposed the substitution of sine values with their rotating vectors also known as phasors. Let us consider how to substitute a sinusoidal current $i = I_m \sin\omega t$ (Fig. 1.4) with its phasor.

If a vector with the magnitude I_m is placed on a Cartesian plane and rotates counter clockwise around the point *O* with the angular speed ω then it can be described in polar and trigonometrical form as

$$\underline{I}_m = I_m e^{j\omega t} = I_m \cos\omega t + jI_m \sin\omega t,$$

where $\underline{I}_m$ is the complex amplitude, *j* is the electrical notation for the imaginary quantity $\sqrt{-1}$ denoted in mathematics as *i*, $\omega = \frac{2\pi}{T} = 2\pi f$ is the rotational speed that is equal to the angular frequency of the alternating current.

Thus, a complex number $\underline{I}_m$ is obtained. Using the complex plane, we can envision the behaviour of this complex parameter. The magnitude of the complex amplitude is *OA*, and the initial angle is equal to zero at $t = 0$ s. As time increases, the locus of points traced by the complex amplitude creates a circle with the constant magnitude of *OA*. The number of times per second $\underline{I}_m$ goes around the circle equals to the frequency *f*. The time taken to go around the circle once is period *T*.

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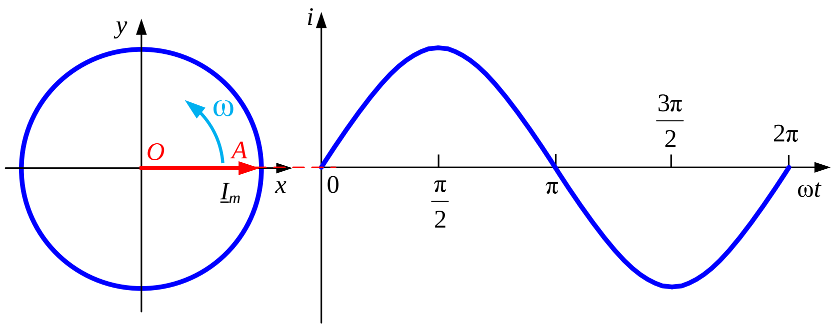
The projections of $\underline{I}_m$ onto the real x and imaginary y-axes give us the real and imaginary parts of the complex amplitude. It is seen from Fig. 1.4. There is a remarkable interest towards the imaginary part of the complex amplitude in electrical engineering.

The projection of the $\underline{I}_m$ onto the imaginary y-axis is equal to zero at the initial time instant t_0=0 s (Fig. 1.4, *a*). It also corresponds to zero on the plane i-ωt on the right side of the Fig. 1.4, *a*. When the time t_1 is passed then the phasor $\underline{OA}$ is shifted by the angle ωt_1 (Fig. 1.4, *b*). Its projection onto the axis y is equal to the value OA sinωt_1, i.e. it gives instantaneous current value $i_1 = I_m$ sinωt_1. At another time instant t_2 the phasor $\underline{OA}$ is under angle ωt_2 to the x-axis and its projection onto y-axis is OA sinωt_2 (Fig. 1.4, *c*), that gives the current value $i_2 = I_m$ sinωt_2 on the right side diagram. When the quarter of period T is passed. i.e. $t_3 = T/4$ (Fig. 1.4, *d*), then the phasor $\underline{OA}$ is in its vertical position and its projection onto the y axis is equal to the vector full length

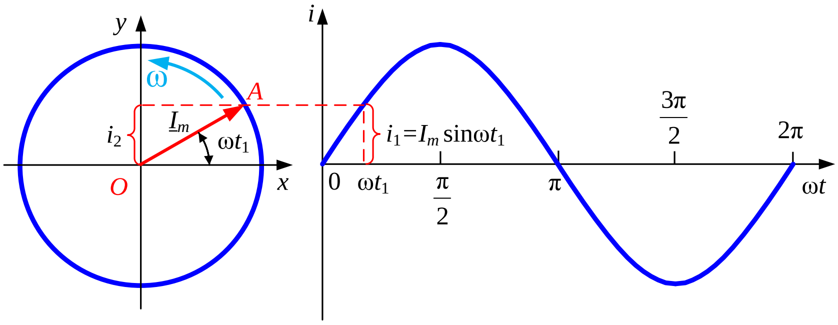
$$OA\sin\omega t_3 = OA\sin\left(\frac{2\pi}{T}\cdot\frac{T}{4}\right) = OA\sin\frac{\pi}{2} = OA\,.$$

The current value i on the right side of the diagram Fig. 1.4, *d* reaches its maximum value I_m. The further rotation of the vector $\underline{OA}$ leads to the decreasing of the y-axis projection OA sinωt (Fig. 1.4, *e*, time instants t_4 and t_5) until the reaching of zero value at the time instant $t_6 = T/2$, i.e. at the angle $\omega t_6 = \pi$. The sine wave is again equal to zero. After that the y-projection $\underline{OA}$ and the instantaneous current i obtain negative values (time instants t_7, t_8, t_9) and then returns to its initial position at the time instant $t_3 = T$ giving the angle $\omega t = 2\pi$.

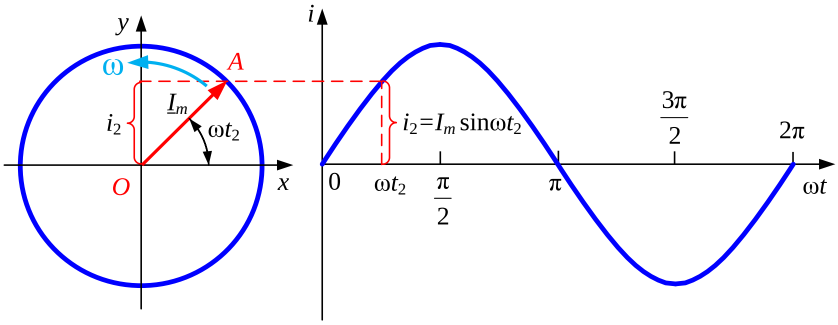
Therefore, the full cycle of phasor $\underline{OA}$ rotation gives us a full period of a sine wave. That means the phasor $\underline{I}_m = I_m\, e^{j\omega t}$ exactly describes the behaviour of the sinusoidal current i. It may be present shortly by the mathematical operation that extracts the imaginary part of a complex number



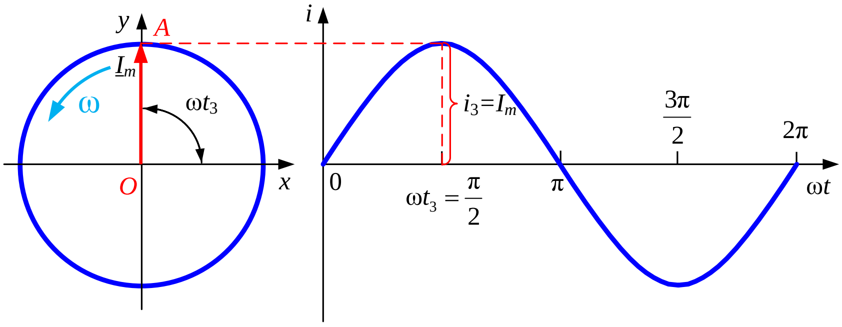
a)



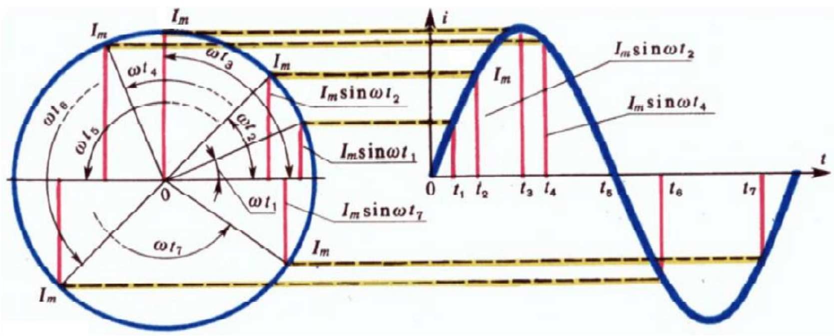
b)



c)



d)



e)

Fig. 1.4. Representation of a sine wave with a phasor (rotating vector).

$$\mathrm{Im}[I_m e^{j\omega t}] = \mathrm{Im}[I_m \cos\omega t + jI_m \sin\omega t,] = I_m \sin\omega t = i.$$

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If the vector $\underline{OA}$ is under the angle Ψ at the time instant $t_0 = 0$ s (Fig. 1.5) then its initial value is not equal to zero $i_0 = I_m \sin(\omega t_0 + \Psi) = I_m \sin\Psi$ (since $\omega t_0 = 0°$). The common expression for the sine waveform is as follows

$$i = I_m \sin(\omega t + \Psi),$$

where Ψ is the initial phase.

The corresponding phasor in a complex form in this case looks like

$$\underline{I}_m = I_m e^{j(\omega t + \Psi)} = I_m \cos(\omega t + \Psi) + jI_m \sin(\omega t + \Psi).$$

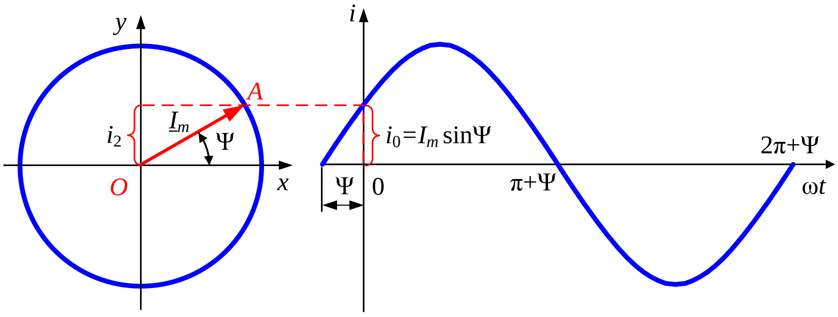

Fig. 1.5. A sine wave with the initial phase Ψ.

Therefore, any alternating current of a sinusoidal waveform may be described with a phasor rotating with the angular speed that equals to the angular frequency of the sine wave. Those phasor notations differ in different textbooks and could be, for example $\underline{I}_m$, $\bar{I}_m$, $\vec{I}_m$, $\dot{I}_m$. In electrical engineering, there is a traditional way to present a complex vector with its amplitude and initial phase keeping in mind that it rotates with the angular frequency ω

$$\underline{I}_m = I_m e^{j\Psi} = I_m \angle \Psi .$$

For example the notation $\underline{I}_m = 5\angle 30°$ A means the current with the amplitude of 5 A is shifted by the 30° in relation to the *y*-axis.

1.2.1. Addition of phasors

Phasors on a complex plane are called **vector or phasor diagram**. There two sinusoidal currents i_1 and i_2 on the right side in Fig. 1.6. Their phasors are presented on the left side. The *y*-axis of a complex plane is called imaginary axis and has notation "+*j*" while *x*-axis is called real one and denoted as "+1".

The sine waves i_1 and i_2 has different initial angles, i.e. their complex amplitudes may be expressed as

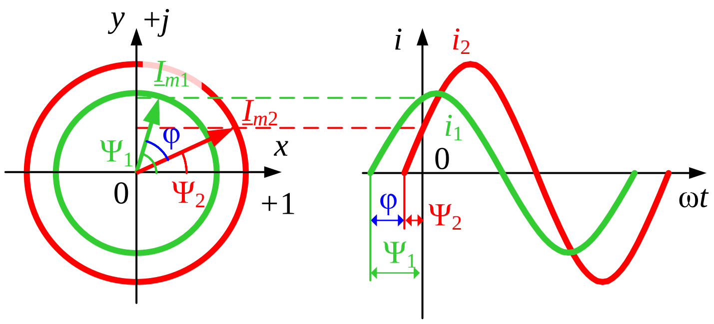

Fig. 1.6. Two currents represented as phasors.

$$\underline{I}_{m1} = I_{m1}\angle\Psi_1 \text{ and } \underline{I}_{m2} = I_{m2}\angle\Psi_2.$$

The current i_1 leads the current i_2 and there is a phase shift between them $\phi = \Psi_1 - \Psi_2$. This phase shift is also seen from their vector diagram, i.e. it is the angle between phasors $\underline{I}_{m1}$ and $\underline{I}_{m2}$.

Let us explore how one can use phasors in electrical calculations. There is a node between three branches with currents i_1, i_2 and i_3 (Fig. 1.7, *a*). For example, the currents i_1 and i_2 flow through two separate cables to some loads and i_1 is the current of a ship's main busbar. The currents i_1 and i_2 are known and the current i_3 is unknown. According to Kirchhoff's, law those node currents are interrelated at any time instant

$$i_3=i_1+i_2.$$

Instead of calculating the sum i_1+i_2 at each time instant the phasors $\underline{I}_{m1}$ and $\underline{I}_{m2}$ can be added to each other to get the resultant phasor $\underline{I}_{m3}$ (Fig. 1.7, *b*)

$$\underline{I}_{m3} = \underline{I}_{m1} + \underline{I}_{m2}.$$

The projections of the complex amplitude $\underline{I}_{m3}$ onto axes *x* (real axis +1) and *y* (imaginary axis +*j*) can be found by using trigonometrical functions

$$\underline{I}_{m3} = \underline{I}_{m1} + \underline{I}_{m2} = I_{m1}\cos(\omega t + \Psi_1) + I_{m2}\cos(\omega t + \Psi_2) + j\{\cos(\omega t + \Psi_1) + I_{m2}\cos(\omega t + \Psi_2)\} = I_{m3}\angle\Psi_3,$$

where the amplitude of the current i_3 is found by means of Pythagoras theorem

$$I_{m3} = \sqrt{\{I_{m1}\cos(\omega t+\Psi_1)+I_{m2}\cos(\omega t+\Psi_2)\}^2+\{I_{m1}\sin(\omega t+\Psi_1)+I_{m2}\sin(\omega t+\Psi_2)\}^2}$$

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and the phase angle – by means of some inverse trigonometrical function, e.g.

$$\Psi_3 = \tan^{-1}\left(\frac{I_{m1}\sin(\omega t+\Psi_1)+I_{m2}\sin(\omega t+\Psi_2)}{I_{m1}\cos(\omega t+\Psi_1)+I_{m2}\cos(\omega t+\Psi_2)}\right).$$

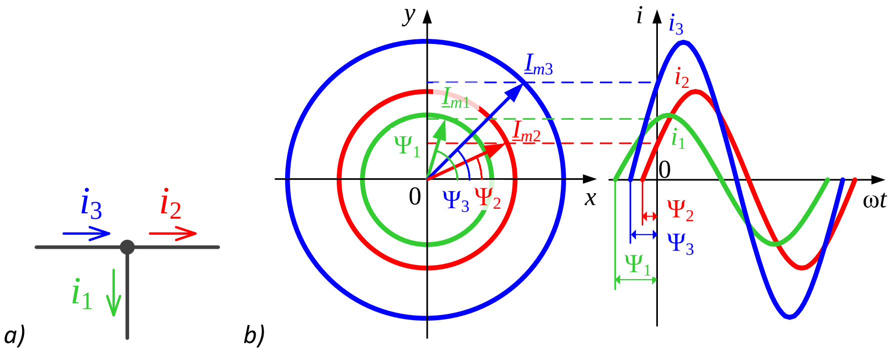

Fig. 1.7. Finding a sum of currents i_1 and i_2.

It must be noted, that in practical electrical calculations phasors are usually mot expressed by using amplitudes of electrical state parameteres, but with their effective (rms) values.

1.3. Passive components of AC circuits

If to compare AC circuits with DC circuits there are not only resistive passive components modelling electric power conversation into any other type of energy, e.g. heat, mechanical job, chemical energy and so on. AC circuits also contain inductive reactance storing magnetic energy and capacitive reactance storing electrical energy. The storing of electrical or magnetic energy causes time delays between the sine waves of instantaneous currents and voltages and corresponding phase shift between current and voltage phasors. It leads to the need to use complex values and proper mathematical tools in electrical calculations.

1.3.1. Resistance

If to apply a sinusoidal voltage *u* to a resistance *R* then a sine wave of a current *i* immediately appears without any time delay (Fig. 1.8, *c*) which value may be found by means of Ohm'i law

$$i = \frac{u}{R} = \frac{U_m \sin\omega t}{R} = I_m \sin\omega t\,.$$

Therefore, there is no phase shift between their vectors (Fig. 1.8, *b*) and Ohm's law can be rewritten either via amplitudes

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$$I_m = \frac{U_m}{R}$$

or, being divided by $\sqrt{2}$, via effective values like in DC circuits

$$I = \frac{U}{R}.$$

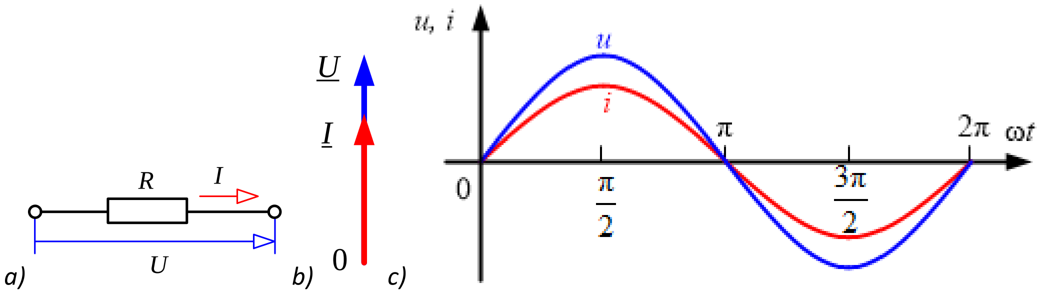

Fig. 1.8. AC circuit with resistance *R*: *a* – circuit diagram, *b* – phasors, *c* – instantaneous waveforms.

1.3.2. Inductive reactance

The ability of an AC current to store magnetic field energy around a conductor is characterised by its inductance *L* and inductive reactance X_L = ω*L* = 2π*f*. As a result, the current lags voltage by angle $\frac{\pi}{2}$ or 90° (Fig. 1.9). This lag can be briefly and clearly presented by means of complex values. If a voltage *u* is supposed to be with a phase of 0° then Ohm's law for effective complex values can be written like this

$$\underline{I} = \frac{\underline{U}}{jX_L} = -j\frac{U}{X_L} = I\mathrm{e}^{-j90^\circ},$$

where $-j = e^{-j90^\circ}$ is the operator shifting the complex current $\underline{I}$ back in relation to $\underline{U}$ by 90° or $\frac{\pi}{2}$.

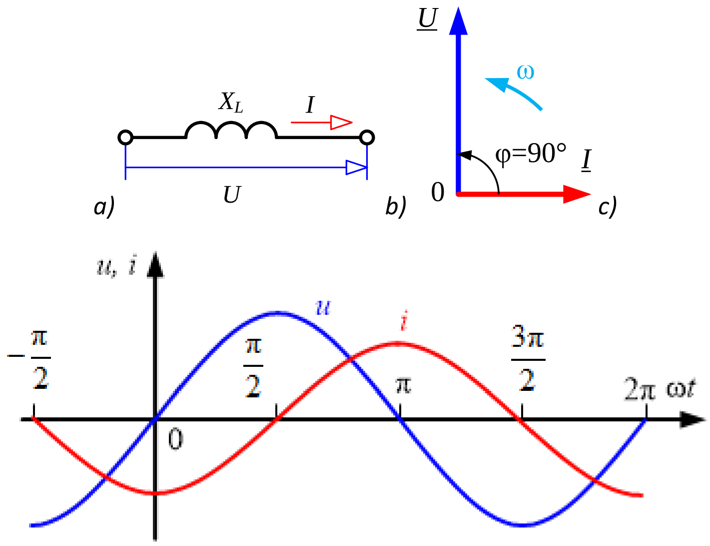

Fig. 1.9. AC circuit with inductive reactance X_L: *a* – circuit diagram, *b* – phasors, *c* – instantaneous waveforms.

1.3.3. Capacitive reactance

The ability of an AC current to store electric field energy around a conductor is characterised by its capacitance *C* and capacitive reactance $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$. As a result, the current *i* leads the voltage *u* by angle $\frac{\pi}{2}$ or 90° (Fig. 1.10). This phase shift can be clearly presented by means of complex values. If voltage *u* is supposed to be with a phase of 0° then Ohm's law for effective complex values can be written in the next way

$$\underline{I} = \frac{\underline{U}}{-jX_C} = j\frac{\underline{U}}{X_C} = Ie^{j90^\circ},$$

where $j = e^{j90^\circ}$ is the operator shifting complex current $\underline{I}$ in relation to $\underline{U}$ ahead by 90° or $\frac{\pi}{2}$.

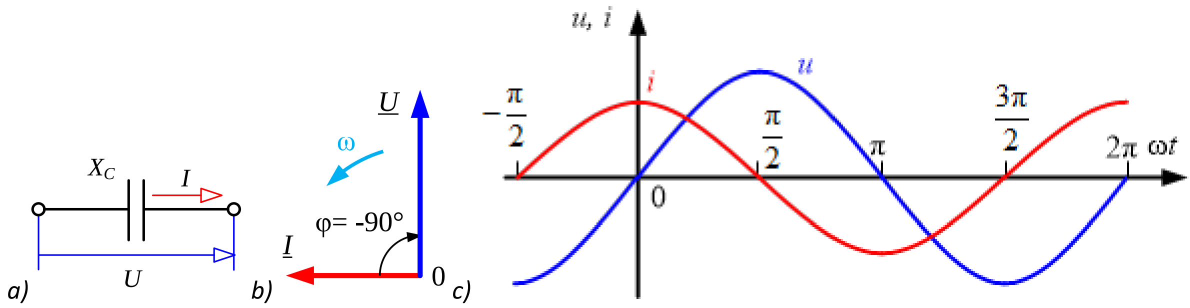

Fig. 1.10. AC circuit with capacitive reactance X_C: *a* – circuit diagram, *b* – phasors, *c* – instantaneous waveforms.

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1.3.4. Impedance

Practical loads on board of ships and in ports (electrical motors, lighting installations, etc.) may usually be modelled by a series combination of resistance *R* and inductive reactance X_L (Fig. 1.11, *a*).

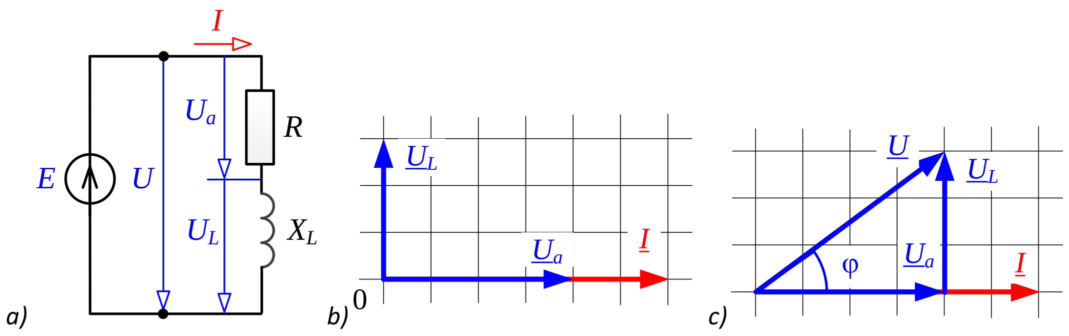

Fig. 1.11. *RL* circuit: *a* – circuit diagram, *b* – phasors of the current and voltage drops across the components, *c* – input voltage as a sum of voltage drops on the components.

In Fig. 1.7, there is no phase shift between the current and the voltage drop across the resistance *R*. Thus, the voltage drop $\underline{U}_a$ is in phase with the current $\underline{I}$ (Fig. 1.11, *b*). From Fig. 1.8, the voltage drop over the inductive reactance $\underline{U}_L$ leads the current $\underline{I}$ (Fig. 1.11, *b*).

According to Kirchhoff's second law, the input voltage $\underline{U}$ may be found as

$$\underline{U} = \underline{U}_a + \underline{U}_L.$$

If the phase of the current phasor $\underline{I}$ is equal to 0° then the voltages can be presented as follows:

a) Resistive voltage drop $\underline{U}_a = \underline{I}\cdot R = I\,R\,e^{j0^\circ} = U_a e^{j0^\circ}$,

b) Inductive voltage drop $\underline{U}_L = \underline{I}\cdot jX_L = I\,X_L e^{j90^\circ} = U_L e^{j90^\circ}$,

c) Input voltage $\underline{U} = \underline{I}(R+jX_L) = \underline{I}\cdot R + \underline{I}\cdot jX_L = U_a e^{j0^\circ} + U_L e^{j90^\circ} = U e^{j\phi}$.

Let us have a closer look at the last expression. The sum $R + jX_L$ is named impedance $\underline{Z} = Z\,e^{j\phi}$, which is a complex number with the magnitude $Z = \sqrt{R^2 + X_L^2}$ and the argument or phase $\varphi = \tan^{-1}\left(\frac{X_L}{R}\right)$. It is better seen from so-called impedance triangle constituted by resistance, reactance and impedance (Fig. 1.12).

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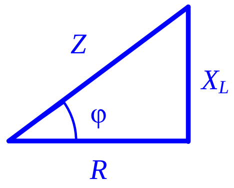

Fig. 1.12. Impedance triangle.

Therefore, Ohm's law in the most common form is as follows

$$\underline{U} = \underline{I}\cdot\underline{Z} = IZe^{j\phi} = Ue^{j\phi}.$$

That means it is the impedance phase ϕ defines phase shift between the terminal voltage *u* and the current *i*.

Example

As shown in Fig. 1.11, *a*, there are a resistor with resistance $R = 4\ \Omega$, an inductor with inductive reactance $X_L = 3\ \Omega$ and the current $I = 1$ A.

Find impedance $\underline{Z}$, voltages across resistance $\underline{U}_a$, reactance $\underline{U}_L$, and a terminal voltage $\underline{U}$.

Solution:

Impedance may be found as follows $\underline{Z} = R + jX_L = 4 + j3\ \Omega$.

According to the Pythagorean Theorem impedance magnitude $Z = \sqrt{R^2 + X_L^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5\,\Omega$.

Its argument $\varphi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$.

Hence, the complex value of impedance $\underline{Z} = Z\ e^{j\phi} = 5\ e^{j36.9°}\ \Omega$.

Voltages may be found by means of Ohm's law in a complex form.

The voltage across the resistor $\underline{U}_a = \underline{I}\cdot R = IR\ e^{j0°} = 1\cdot 4\ e^{j0°} = 4e^{j0°} = 4$ V.

The voltage across the reactance $\underline{U}_L = \underline{I}\cdot jX_L = IX_L\ e^{j90°} = 1\cdot 3e^{j90°} = 3e^{j90°}$ V.

The terminal or input voltage $\underline{U}_L = \underline{I}\cdot\underline{Z} = IZe^{j\phi} = 1\cdot 5e^{j36.9°} = 5e^{j36.9°}$ V.