Introduction to Indefinite Integrals

1. Conceptions of the antiderivative and definition of the indefinite integral

In the previous chapters we learned the differentiation of continuous functions. If function $F(x)$ is given, its derivative is function $f(x)$

$$F'(x) = f(x)$$

or in the form of differentials

$$\frac{d(F(x)+C)}{dx} = f(x).$$

We are interested in the reverse procedure: what function have we differentiated to get function $f(x)$? For instance, $f(x) = cosx$.

According to the derivation formulas

$$sin'x = cosx$$

Taking into account that the derivative of the constant number is zero

$$C' = 0,$$

we can determine several functions whose derivative is cosine function

$$(sinx + 1)' = cosx; \quad (sinx - 1.5)' = cosx; \quad (sinx + 3)' = cosx$$

We can conclude that all sinus functions plus an arbitrary constant number are the prime functions of the cosine. We determine the family of prime functions of function $cosx$ for all real numbers C

$$cosx = (sinx + C), C \in R.$$

Definition 1.1 For any given function $f(x)$, function $F(x)$ is the ***prime function or antiderivative of*** $\boldsymbol{f(x)}$ if $F'(x) = f(x)$.

The process of finding antiderivatives is the reverse procedure of derivation. We call this process ***integration***.

Definition 1.2 The ***Indefinite integral*** of a given function $f(x)$ is the set of all antiderivatives $F(x) + C$ of the function $f(x)$ and it is denoted

$$\int f(x)dx = F(x) + C,$$

where

the sign $\int$ is called the *integral symbol*,

$f(x)$ is called the *integrand*,

x is called the *integration variable*,

C is called the *integration constant*.

The above-mentioned example can be written

$$\int cosx\, dx = sinx + C$$

2. Geometric interpretation of the indefinite integral

Knowing the geometric meaning of the derivative of a function, the given function $f(x)$ expresses the rate of change of some prime function. Geometric solution of integration of $f(x)$ presents a set of graphs that completely cover the plane. For instance, representatives of the whole family of antiderivatives $F(x) = e^x + C$ are shown in figure 2.1.

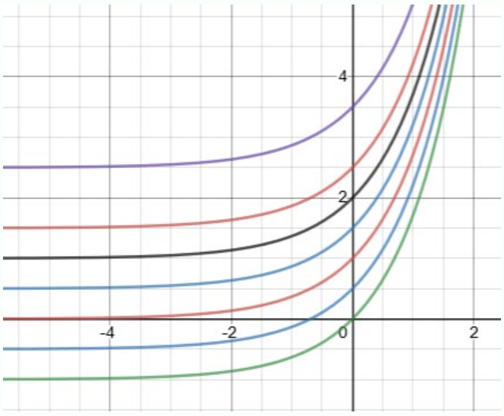

Figure 2.1 The family of antiderivatives $F(x) = e^x + C$

We can get one definite function of the set of answers if some initial condition is given: that is, we have the coordinates of the point belonging to the curve.

Example 2.1 Find function $\omega(x)$ whose rate of change is $\omega'(x) = cosx$ and the point $(0,2)$ belongs to the graph of the function.

Solution We will solve this problem in two steps.

Step 1. Find antiderivatives of the function $cosx$

$$\omega(x) = \int cosx\, dx = sinx + C$$

Step 2. Calculate the definite value of constant C according to the value of the function at the point $(0,2)$

$$\omega(0) = sin0 + C = C$$

$$\omega(0) = 2; \;\; C = 2$$

Answer $\omega(x) = sinx + 2$.

The graph of this function belongs to the family of functions $\omega(x) = sinx + C$. The y-intercept is the point (0, 2) where the graph of function $\omega(x) = sinx + 2$ crosses the y-axis (see Figure 2.2).

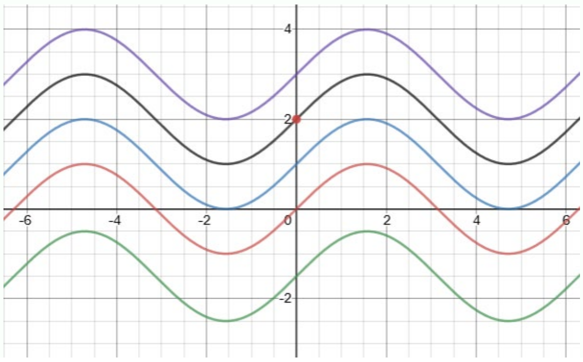

Figure 2.2 The family of functions $\omega(x) = sinx + C$

An indefinite integral can be used to express the functional relations of physical processes.

Example 2.2 A flare is ejected vertically upwards from the ground at 15 m/s. Find the height of the flare after 2.5 s.

Comment In the solution of this problem we suppose that it is not very hard to apply differentiation to find the function that gives the derivative $-9.8t + 15$

Solution The velocity of a given object can be expressed in terms of time according to gravity

$$v(t) = -9.8t + C$$

At the initial moment the velocity is 15 m/s ($t = 0$). We calculate $C = 15.$

The function of velocity in the given case is

$$v(t) = -9.8t + 15$$

To find the displacement $s(t)$ of the flare we integrate the function of velocity

$$s(t) = \int v(t)dt = \int (-9.8t + 15)dt =$$

$$= -4.9t^2 + 15t + C$$

At the initial position $t = 0, \;\; s = 0$ therefore $C = 0$. We calculate the height of the flare after 2.5 seconds

$$s(2.5) = -4.92.5^2 + 15 \cdot 2.5 = 6.875 \;\; m$$

3. Uniqueness of antiderivatives

A question arises when searching for the antiderivatives of the given function $f(x)$. How much these antiderivatives differ from one another? The following theorem states:

Theorem 2.1 If functions $F_1(x)$ and $F_2(x)$ are two different antiderivatives of the function $f(x)$ they differ only by a constant number.

It is given $[F_1(x)]' = f(x)$ and $[F_2(x)]' = f(x)$. Then the difference is

$$[F_1(x)]' - [F_2(x)]' = 0 \text{ or } [F_1(x) - F_2(x)]' = 0.$$

We conclude that $F_1(x) - F_2(x) = C$.

4. Exercises

Using the list of elementary derivatives, find the antiderivatives $f(x)$ of the given functions $f'(x)$ according to the initial conditions. Construct the graph of function $f(x)$.

1. $f'(x) = 3x^2; \quad f(0) = -1$
2. $f'(x) = e^x; \quad f(1) = e$
3. $f'(x) = \frac{1}{2x}; \quad f(1) = 1.5$
4. $f'(x) = 2sinx; \quad f\left(\frac{\pi}{3}\right) = -0.75$
5. $f'(x) = 4x - 3; \quad f(1) = 1$
6. Car starts form the origin and has acceleration $(t) = 2t - 5\ {}^{m}/_{s^2}$. Find the function of velocity of the car!

6. Solutions

Solution of exercise 1 We have the formula

$$(x^n)' = nx^{n-1}$$

From the given $f'(x) = 3x^2$ we can decide that $n = 3$ and we find $(x^3)' = 3x^2$.

Using integral we get the set of answers

$$\int f'(x)dx = \int 3x^2 dx = x^3 + C$$

Applying initial condition $x = 0; \quad y = 1$

$$0 + C = 1; \quad C = 1$$

Answer

$$f(x) = x^3 + 1$$

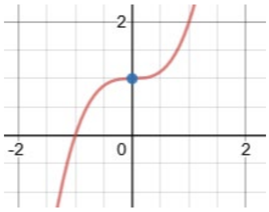

Figure 4.1 Function $f(x) = x^3 + 1$ passing through the point (0; 1).

Solution of exercise 2 We have the formula

$$(e^x)' = e^x$$

Using integral we get the set of answers

$$\int e^x dx = e^x + C$$

Applying initial condition $x = 1; \;\; y = e$

$$e^1 + C = e; \quad C = 0$$

Answer

$$f(x) = e^x$$

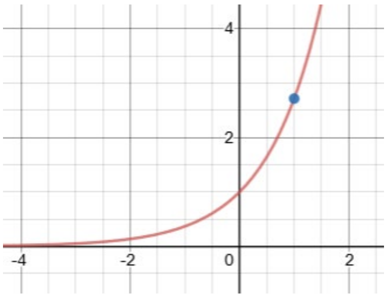

Figure 4.2 Function $f(x) = e^x$ passing through the point (1; e).

Solution of exercise 3 We have the formulas

$$(lnx)' = \frac{1}{x} \text{ and } (af(x))' = a(f(x))', \text{ where } a \text{ is a constant.}$$

Then

$$\left(\frac{1}{2}lnx\right)' = \frac{1}{2}(lnx)' = \frac{1}{2}\cdot\frac{1}{x}$$

Using integral

$$\int \frac{1}{2x}dx = \frac{lnx}{2} + C$$

Applying initial condition $x = 1;\ \ y = 1.5$

$$\frac{ln1}{2} + C = 0 + C = 1.5; \quad C = 1.5$$

Answer

$$f(x) = \frac{lnx}{2} + 1.5$$

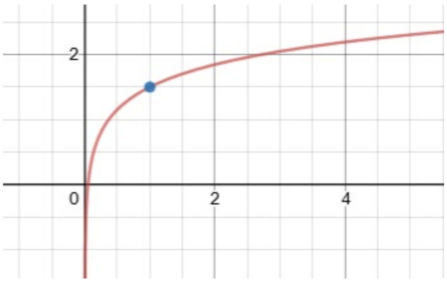

Figure 4.3 Function $f(x) = \frac{lnx}{2} + 1.5$ passing through the point (1; 1.5).

Solution of exercise 4 We have the formula

$$(cosx)' = -sinx$$

Using integral we get

$$\int 2sinx\,dx = 2\int sinx\,dx = -\,2cosx + C$$

Applying initial condition $x = \frac{\pi}{3};\ \ y = -0.75$

$$-2cos\frac{\pi}{3} + C = -2\cdot\frac{1}{2} + C = -0.75; \quad C = 0.25$$

Answer

$$f(x) = -2cosx + 0.25$$

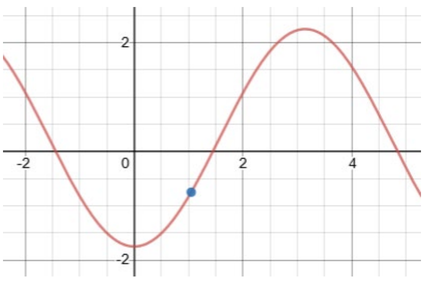

Figure 4.4 Function $f(x) = -2cosx + 0.25$ passing through the point $\left(\frac{\pi}{3}, -0.75\right)$.

Solution of exercise 5 We know that

$$(x^2)' = 2x; \quad (3x)' = 3$$

and

$$(2x^2 - 3x)' = 2(x^2)' - (3x)' = 4x - 3$$

Using integral we get

$$\int (4x - 3)dx = 2x^2 - 3x + C$$

Applying initial condition $x = 1; \ y = 1$

$$2 - 3 + C = 1; \quad C = 2$$

Answer

$$f(x) = 2x^2 - 3x + 2$$

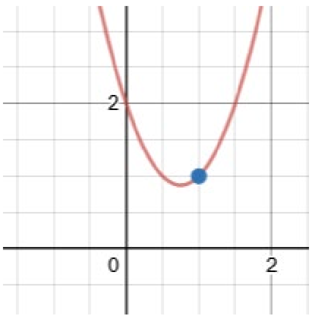

Figure 4.5 Function $f(x) = 2x^2 - 3x + 2$
passing through the point (1; 1).

6. Car starts form the origin and has the acceleration $(t) = 2t - 5 \ ^m/_{s^2}$. Find the function of velocity of the car!

Solution

Velocity can be determined

$$v(t) = \int a(t)dt$$

Applying the formula of differentiation of power function, we can detect that expression $2t - 5$ can be derived from the function t^2, and 5 from $5t$. Therefore, the antiderivative should be

$$F(x) = t^2 - 5t$$

Generally, $v(t) = t^2 - 5t + C$.

At the start $t = 0, v(0) = 0$, therefore $C = 0$. Therefore, the function of velocity is

$$v(t) = t^2 - 5t$$

This equation helps to detect the velocity of the car after a time moment, for instance, after 10 seconds

$$v(10) = 100 - 50 = 50\ m$$