# Application of Definite Integral. Area of Surface of Revolution

DETAILED DESCRIPTION:
Definite integrals can be applied to calculate the area of surface of revolution. The chapter demonstrate the application of a special formula of calculation of this area for cases if an arc revolves about the x-axis or about the y-axes. The formula can be transformed for curves that are given by the parametric equations or in polar form. The content is supplied by the examples with a graphs constructed with GeoGebra applet. The exercises that relay to the topic are attached at the end of the lesson.

AIM: to demonstrate the calculation of the area of surface of revolution in Cartesian coordinate system.

## Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can apply computer aids to construct geometric shapes and surfaces.
3. Students can calculate the area of surface of revolution.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of functions; the construction of the graphs of functions; the three dimensional construction of surfaces; algebra and trigonometry formulas.

Relationship to real maritime problems: Calculation of surface of revolution is an important part at the design of different parts of mechanical equipment. For instance, to increase the operational efficiency of centrifugal pump it is useful the calculation of surfaces of revolution for blade construction as an integral part of pump. The satellite dish has the shape of a solid of revolution. The calculation of its surface is necessary to detect the amount of paint required to cover the surface.

## Content

1. Formula for calculation of a surface of revolution
2. Revolution about the $y$-axis
3. The surface area of a solid of revolution for parametrically given curve
4. Exercises
5. Solutions

# Calculation of a surface of revolution

## 1. Formula for calculation of a surface of revolution

The function $y = f(x)$ is represent on the Cartesian coordinate plane (see figure 1.1). An arc of the function over the interval $[a, b]$ revolves about the $x$-axis and it forms the surface (see figure 1.2). The area of a surface of revolution we can calculate by the formula

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

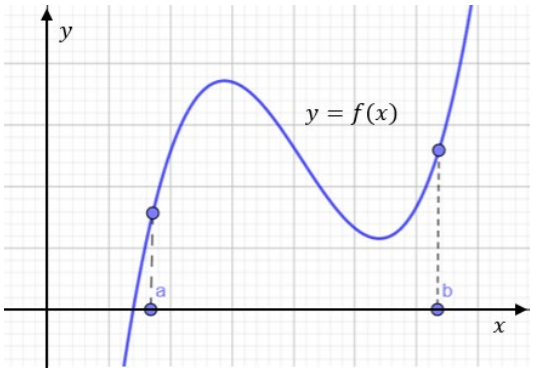


Figure 1.1

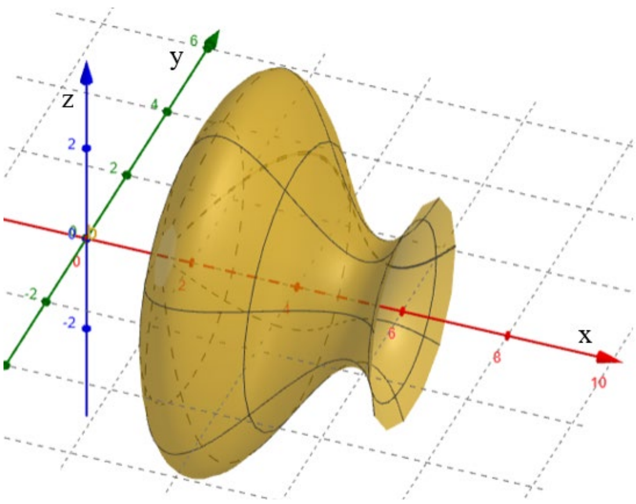


Figure 1.2

Example 1.1 A truncated cone is formed by the straight line $3x + 4y = 0$ revolving about the $x$-axis over the interval $[4, 8]$. Calculate the lateral surface area of the cone!

### Solution

We construct the graph of a function (see figure 1.3) and the cone (see figure 1.4).

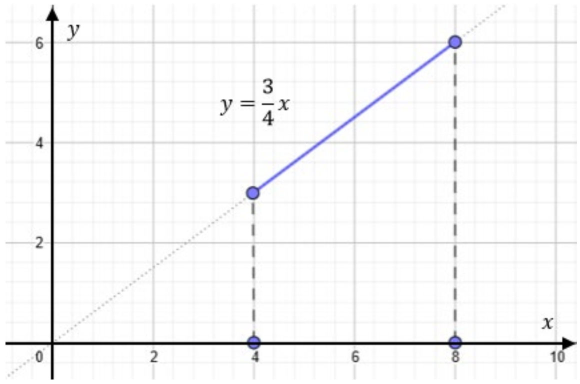


Figure 1.3

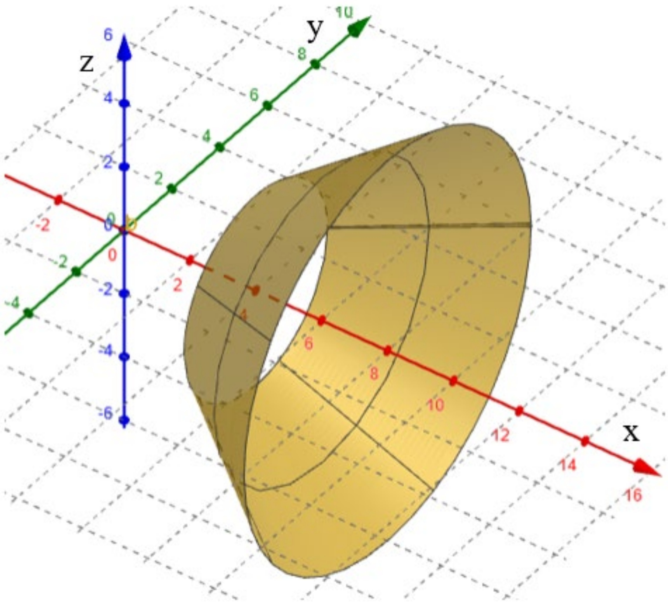


Figure 1.4

It is necessary to differentiate given function to apply the formula

$$y' = (\frac{3}{4}x)' = \frac{3}{4}$$

The lateral surface area of a cone is

$$S = 2\pi \int_4^8 \frac{3}{4}x\sqrt{1 + \left(\frac{3}{4}\right)^2}\,dx = \frac{3}{2}\pi \int_4^8 x\sqrt{\frac{25}{16}}\,dx =$$

$$= \frac{15}{8}\pi \int_4^8 x\,dx = \frac{15}{8}\pi \frac{x^2}{2}\Big|_4^8 = \frac{15}{16}\pi(64 - 16) = 5\pi$$

## 2. Revolution about the $y$-axis

If an arc of a function $y = f(x)$ (see figure 1.1) revolves about the $y$-axis, we have to express the inverse function with respect of the argument $y$, that is, $x = g(y)$. We detect the appropriate projection interval $[c, d]$ of the arc on the $y$-axis (see figure 2.1). Then the formula of the area of a surface of revolution (see figure 2.2) is

$$\boldsymbol{S = 2\pi \int_c^d g(y)\sqrt{1 + (g'(y))^2}\, dy}$$

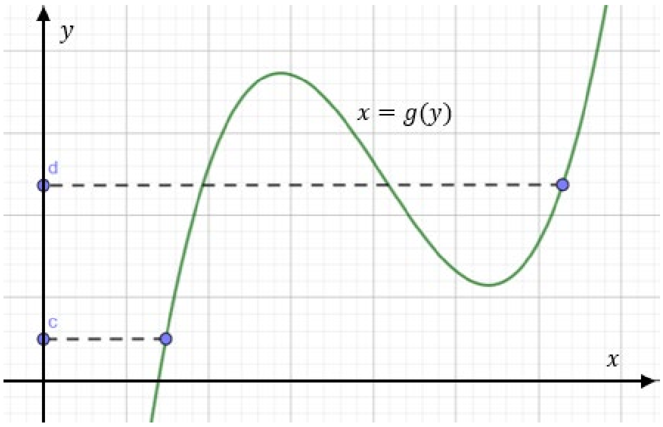


Figure 2.1

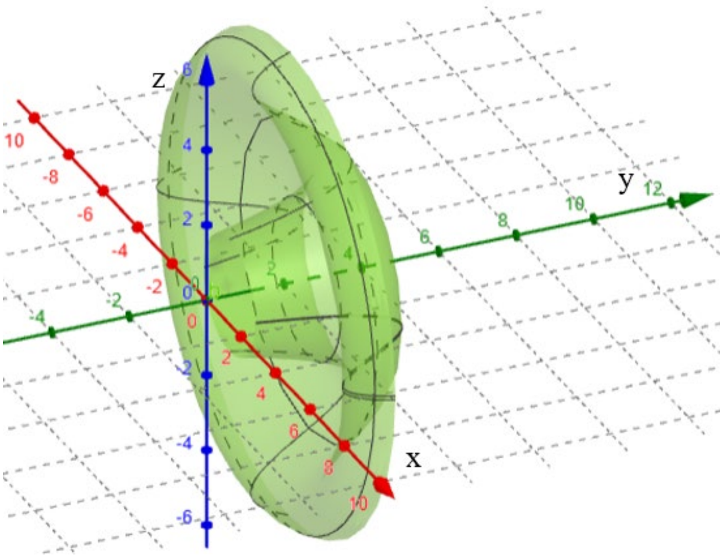


Figure 2.2

### Example 2.1

Find the area of the surface obtained by rotating the curve $y = x^2$ on the interval $[0, 2]$ around the $y$–axis.

### Solution

The arc and the surface are presented on figures 2.3 and 2.4. We rewrite the equation of the curve as a function with respect to the argument $y$.

$$x = g(y) = \sqrt{y}$$

The derivative of a function with respect to the argument $y$ is

$$x' = \frac{1}{2\sqrt{y}}$$

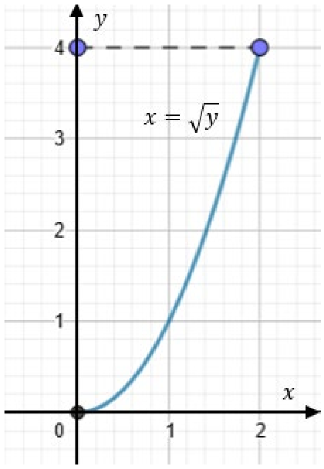


Figure 2.3

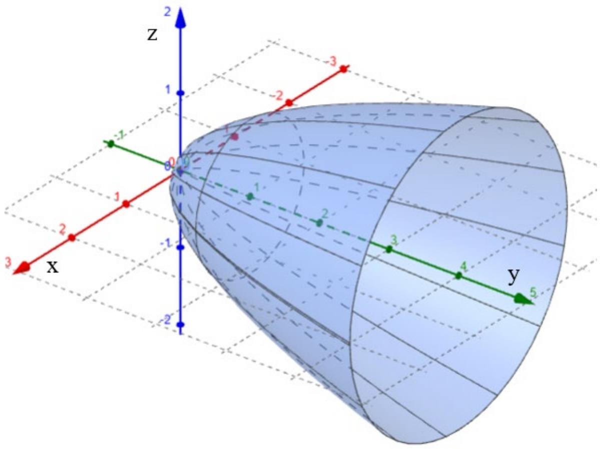


Figure 2.4

We detect the projection of the arc to the $y$-axis as interval [0, 4]. Then the integral for calculation of a surface of revolution is

$$S = 2\pi\int_0^4 \sqrt{y}\cdot\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy = 2\pi\int_0^4 \sqrt{y}\cdot\sqrt{1+\frac{1}{4y}}\,dy =$$

$$= 2\pi\int_0^4 \sqrt{y}\cdot\frac{\sqrt{4y+1}}{2\sqrt{y}}\,dy = \pi\int_0^4 \sqrt{4y+1}\,dy =$$

$$= \frac{\pi}{4}\int_0^4 \sqrt{4y+1}\,d(4y+1) = \frac{\pi}{4}\frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^4 = \frac{\pi}{6}\left(\sqrt{17}^3-1\right) \approx 36.18$$

## 3. The surface area of a solid of revolution for parametrically given curve

The formula for calculating the surface area in the Cartesian coordinates is

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

Let us recall the formula for calculation of the length of an arc that is defined by the function $y = f(x)$ above the interval $[a, b]$

$$L = \int_a^b ds$$

By expressing the differential of the arc $ds$ we have

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

So the formula for calculation of a surface of revolution we can rewrite in the following way

$$\boldsymbol{S = 2\pi \int_a^b f(x) ds}$$

Let the curve is defined by the parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Let the parameter $t$ belongs to the interval $t \in [\alpha, \beta]$. In this case the differential of the arc we can calculate in the following way

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

The formula for calculation the surface area when the curve is revolving around the $x$-axis is

$$\boldsymbol{S = 2\pi \int_\alpha^\beta y(t)\, ds = 2\pi \int_\alpha^\beta y(t) \sqrt{\dot{x}^2 + \dot{y}^2}\, dt}$$

### Example 3.1

The segment of a straight line $y = \frac{x}{2}$ above the interval $[2, 6]$ (see figure 3.1) is rotating around the $x$-axis. It forms the lateral surface of a truncated cone. Calculate the total surface of the cone!

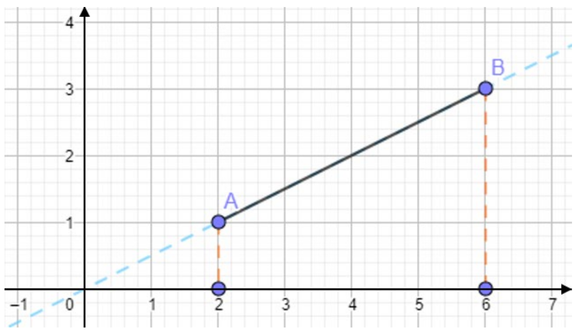


Figure 3.1

### Solution

To calculate the total area of a surface of truncated cone (see figure 3.2), it is necessary to calculate its lateral surface area and the area of the upper and lower bases. The upper and lower bases are the circles with the radius $r = 1$ and $R = 3$. The base area will be calculated using the circle area formula. The lateral surface area will be calculated by an integral.

Let us transform the expression of the function $y = \frac{x}{2}$ into a parametric form

$$\begin{cases} x = t \\ y = t/2 \end{cases}; \quad 2 \le t \le 6$$

We now differentiate both functions and compose the formula for calculation of the lateral area of the surface formed by segment rotating around the $x$ −axis (see figure 3.2)

$$\dot{x} = 1; \quad \dot{y} = \frac{1}{2}$$

$$S = 2\pi \int_2^6 \frac{t}{2} \sqrt{1 + \frac{1}{4}} dt$$

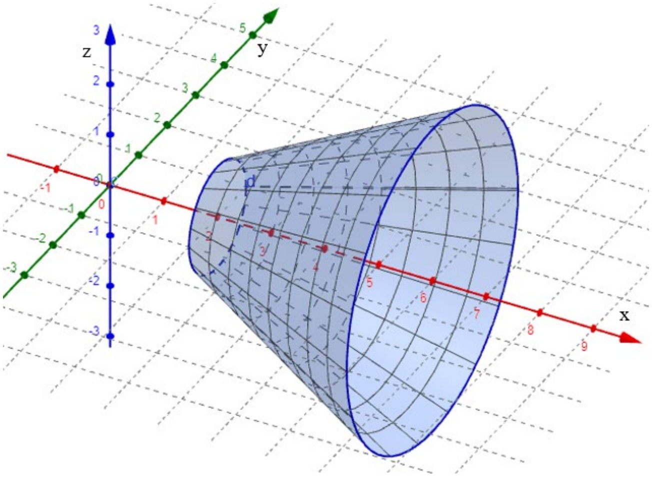


Figure 3.2

The area of a surface of revolution is

$$S = 2\pi \int_2^6 \frac{t}{2} \sqrt{\frac{5}{4}} dt = \frac{\sqrt{5}\pi}{2} \int_2^6 t dt = \frac{\sqrt{5}\pi}{2} \cdot \frac{t^2}{2} \Bigg|_2^6 = \frac{\sqrt{5}\pi}{4}(36 - 4) = 8\sqrt{5}\pi$$

The total surface area of a given cone is

$$S_T = S + \pi r^2 + \pi R^2 = 8\sqrt{5}\pi + 10\pi \approx 87.61$$

## 4. Exercises

1. Calculate the area of the surface obtained by the circle $x^2 + y^2 = 4$ rotating around the $x$-axis.
2. Calculate the area of the surface obtained by the arc of a function $y = e^{-x}$ about the interval $[0, 3]$ rotating around $x$-axis.
3. Find the area of the surface obtained by rotating the curve $y = \arccos x$ on the interval $[-1, 1]$ around the $y$–axis.
4. The arc of the cycloid rotates around the $x$-axis. Find the area of this surface if the parametric equations of the cycloid are

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

5. Find the area of the arc of astroid revolving around the $y$-axis

$$\begin{cases} x = 3\cos^3 t \\ y = 3\sin^3 t \end{cases}; \quad 0 \le t \le \frac{\pi}{4}$$

## 5. Solutions

1. Calculate the area of the surface obtained by the circle $x^2 + y^2 = 4$ rotating around the $x$-axis.

### Solution

We construct the circle.

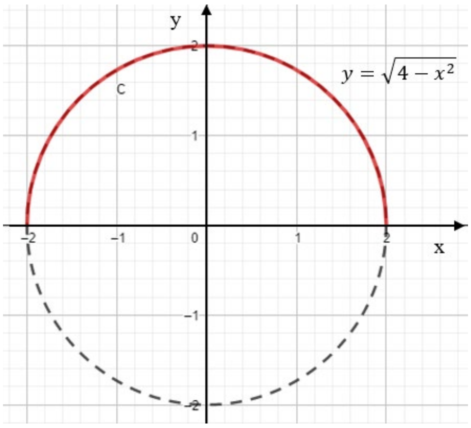


Figure 5.1

Let us choose the upper part of the circle line on the interval where $-2 \le x \le 2$ (see figure 5.1). The equation of this curve is $y = \sqrt{4 - x^2}$

This curve forms the sphere while the arc is rotating around the $x$-axis (see figure 5.2).

We differentiate the function and simplify the expression $1 + y'^2$

$$y' = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

$$1 + y'^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

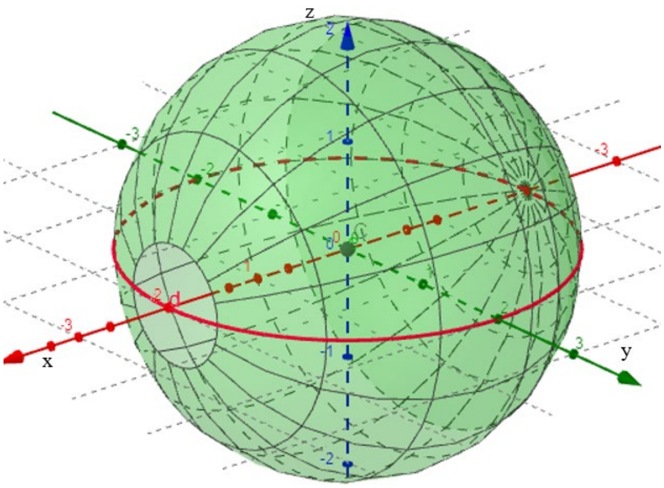


Figure 5.2

For calculation of a surface area of the sphere we apply the formula

$$\boldsymbol{S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

The integral for a given function is

$$S = 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \sqrt{\frac{4}{4 - x^2}}\, dx = 4\pi \int_{-2}^{2} dx = 4\pi x \Big|_{-2}^{2} = 16\pi$$

2. Calculate the area of the surface obtained by the arc of a function $y = e^{-x}$ about the interval $[0, 3]$ rotating around $x$-axis.

Solution

We construct the graph of a given function (see figure 5.3).

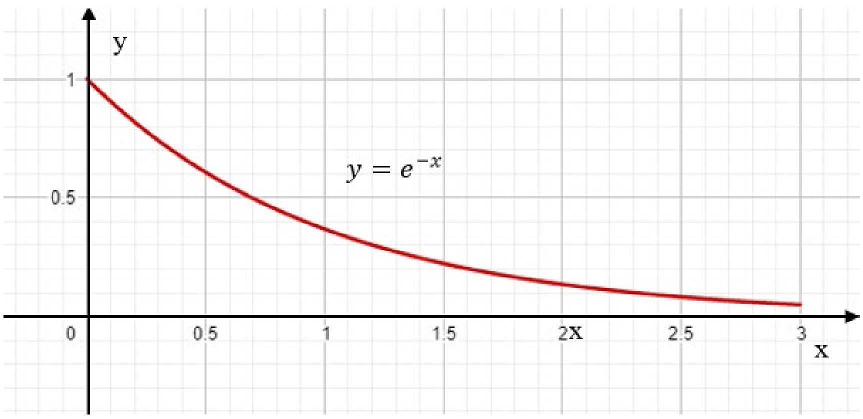


Figure 5.3

The derivative of the function $y = e^{-x}$ is

$$y' = (e^{-x})' = -e^{-x}$$

We compose the integral for calculation of the area of a revolution surface (see figure 5.4)

$$S = 2\pi \int_0^3 e^{-x}\sqrt{1+e^{-2x}}\,dx = \left| \begin{matrix} t = e^{-x} & dt = -e^{-x}dx \\ t_1 = e^0 = 1 & t_2 = e^{-3} \end{matrix} \right| =$$

$$= -2\pi \int_1^{e^{-3}} \sqrt{1+t^2}\,dt = -2\pi \cdot \frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right)\Big|_1^{e^{-3}} =$$

$$= -\pi\left(e^{-3}\sqrt{1+e^{-6}} + \ln\left|e^{-3} + \sqrt{1+e^{-6}}\right| - \sqrt{2} - \ln\left|1+\sqrt{2}\right|\right) \approx 6.9$$

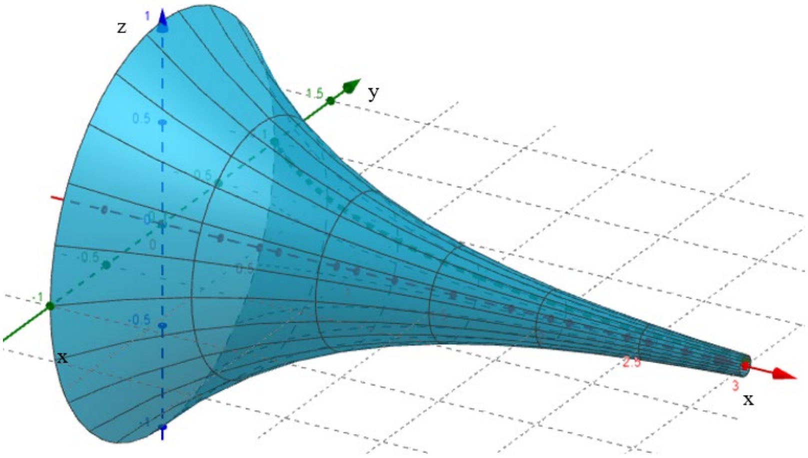


Figure 5.4

<u>Comment.</u> The integral

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right) + C$$

can be solved using integration by parts (see Appendix at the end of this chapter).

3. Find the area of the surface obtained by rotating the curve $y = \arccos x$ on the interval $[-1, 1]$ around the $y$–axis.

## Solution

The curve is given on the segment $[-1, 1]$ on $x$-axis (see figure 5.5). Let us find the projection of the curve onto the $y$-axis – it is the segment $AC = [0, \pi]$. We express the inverse function $x$ with respect to the argument $y$ and find the derivative

$$x = x(y) = \cos y$$

$$x' = -\sin y$$

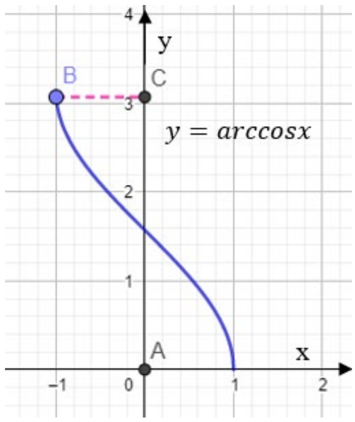


Figure 5.5

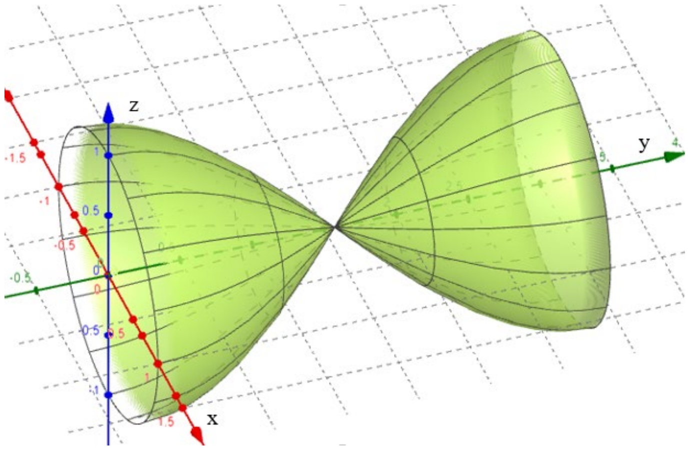


Figure 5.6

The curve rotating about the $y$-axis create symmetrical two-part surface (see figure 5.6). Therefore, we split the surface into two equal parts and calculate the surface area of one part, where the argument $y$ changes from 0 to $\frac{\pi}{2}$. For calculation the surface area we use the formula

$$\boldsymbol{S = 2\pi \int_{c}^{d} g(y)\sqrt{1 + (g'(y))^2}\, dy}$$

The integral for calculation of the surface area of one part is

$$S_1 = 2\pi \int_0^{\frac{\pi}{2}} \cos y\sqrt{1 + \sin^2 y}\, dy = \left| \begin{matrix} t = \sin y; & dt = \cos y dy \\ t_1 = \sin 0 = 0; & t_2 = \sin\frac{\pi}{2} = 1 \end{matrix} \right| =$$

$$= 2\pi \int_0^1 \sqrt{1 + t^2} dt = \pi \left( t\sqrt{1 + t^2} + \ln\left|t + \sqrt{1 + t^2}\right| \right) \Bigg|_0^1 =$$

$$= \pi\left(\sqrt{2} + \ln\left|1 + \sqrt{2}\right|\right) \approx 10.34$$

The total area of a given surface of revolution is

$$S = 2\pi\left(\sqrt{2} + \ln\left|1 + \sqrt{2}\right|\right) \approx 20.68$$

4. The arc of the cycloid rotates around the $x$-axis. Find the area of this surface if the parametric equations of the cycloid are

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

## Solution

An arc of the cycloid is given if the range of a parameter $t \in [0, 2\pi]$ (see the arc AB in the figure 5.7).

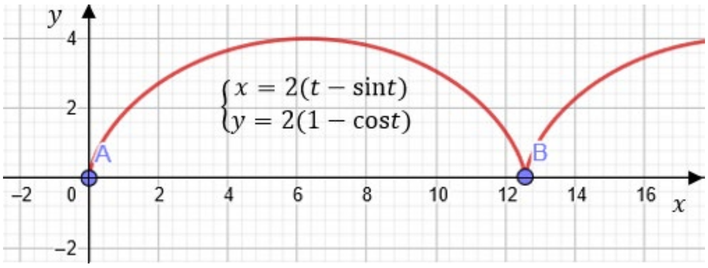


Figure 5.7

To calculate the surface of revolution (see figure 5.8) we apply the formula

$$\boldsymbol{S = 2\pi \int_{\alpha}^{\beta} y(t)\sqrt{\dot{x}^2 + \dot{y}^2}\, dt}$$

At first we will calculate the derivatives

$$\dot{x} = 2(1 - \cos t)$$
$$\dot{y} = 2\sin t$$

Now we will simplify the expression applying algebra and trigonometry formulas

$$\dot{x}^2 + \dot{y}^2 = 4(1 - \cos t)^2 + 4\sin^2 t = 4(1 - 2\cos t + \cos^2 t + \sin^2 t) =$$

$$= 4(2 - 2\cos t) = 16\sin^2\frac{t}{2}$$

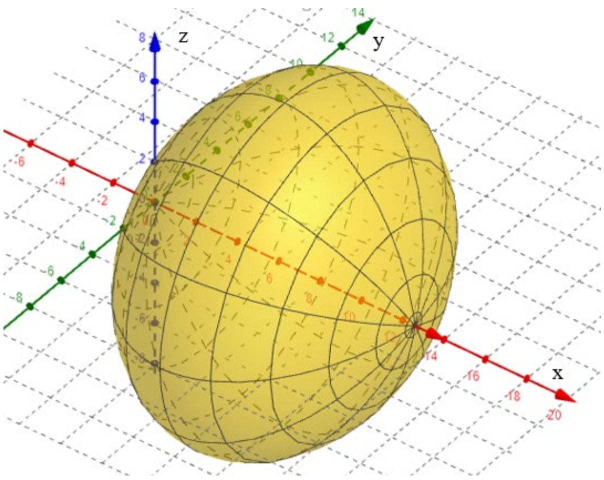


Figure 5.8

The surface area formed by the first arc of cycloid revolving around the $x$-axis is

$$S = 2\pi \int_0^{2\pi} 2(1-\cos t)\sqrt{16\sin^2\frac{t}{2}}\,dt =$$

$$= 2\pi \int_0^{2\pi} 2\sin^2\frac{t}{2}\cdot 4\sin\frac{t}{2}\,dt = \left| \begin{matrix} u = \cos\frac{t}{2}; & du = -\frac{1}{2}\sin\frac{t}{2}\,dt \\ u_1 = \cos 0 = 1 & u_2 = \cos\pi = -1 \end{matrix} \right| =$$

$$= -\frac{16\pi}{2}\int_1^{-1}(1-u^2)\,du = 8\pi\left(\frac{u^3}{3} - u\right)\Big|_1^{-1} = 8\pi\cdot\frac{4}{3} \approx 33.5$$

5. Find the area of the arc of the astroid revolving around the $y$-axis

$$\begin{cases} x = 3\cos^3 t \\ y = 3\sin^3 t \end{cases}; \quad 0 \le t \le \frac{\pi}{4}$$

Solution

Let us construct the astroid and show the given arc (see figure 5.9). We calculate the derivatives and simplify the expression to be written under the square root

$$\dot{x} = -9\cos^2 t\cdot\sin t$$
$$\dot{y} = 9\sin^2 t\cdot\cos t$$

$$\dot{x}^2 + \dot{y}^2 = 81\cos^4 t\sin^2 t + 81\sin^4 t\cos^2 t =$$

$$= 81\cos^2 t\sin^2 t(\sin^2 t + \cos^2 t) = 81\sin^2 t\cos^2 t$$

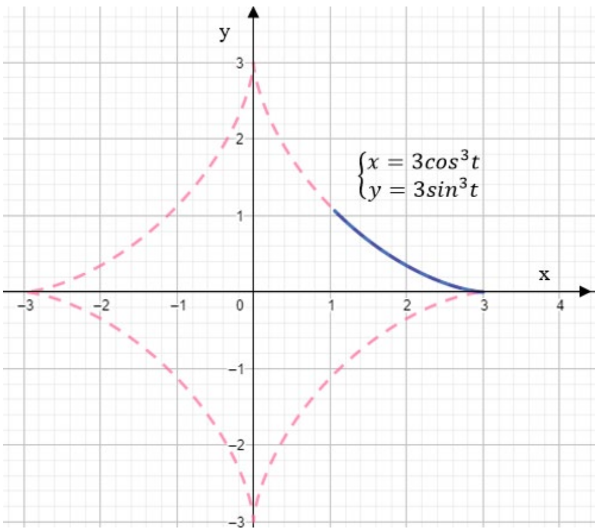


Figure 5.9

We create an integral to calculate the area of the surface (see figure 5.10)

$$S = 2\pi \int_0^{\frac{\pi}{4}} 3\cos^3 t \sqrt{81\sin^2 t\cos^2 t}\, dt =$$

$$= 2\pi \cdot 27 \int_0^{\frac{\pi}{4}} \cos^3 t \,\mathrm{sin}t\, \mathrm{cos}t\, dt = \begin{vmatrix} u = \mathrm{cos}t & du = -\mathrm{sin}t dt \\ u_1 = \cos 0 = 1 & u_2 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{vmatrix} =$$

$$= -54\pi \int_1^{\frac{\sqrt{2}}{2}} u^4\, du = -54\pi \frac{u^5}{5}\Bigg|_1^{\frac{\sqrt{2}}{2}} = -\frac{54\pi}{5}\left(\frac{\sqrt{2}}{8} - 1\right) \approx 14.05$$

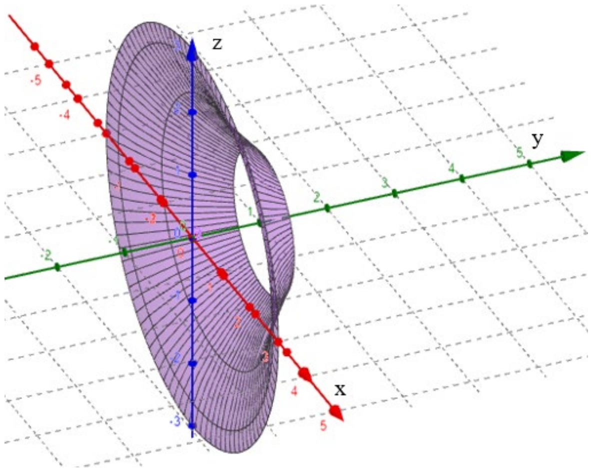


Figure 5.10

Appendix: calculation of integral

In the solution of exercises 2 and 3 we used a special formula

$$\int \sqrt{1+t^2}\, dt = \frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right) + C$$

The integral can be evaluated by the method of integration by parts. Let us denote the given integral by $Int$ and apply the method

$$Int = \int \sqrt{1+t^2}\, dt = \begin{vmatrix} u = \sqrt{1+t^2} & du = \dfrac{t}{\sqrt{1+t^2}} dt \\ dv = dt & v = t \end{vmatrix} =$$

$$= t\sqrt{1+t^2} - \int \frac{t \cdot t}{\sqrt{1+t^2}} dt = t\sqrt{1+t^2} - \int \frac{t^2 + 1 - 1}{\sqrt{1+t^2}} dt =$$

$$= t\sqrt{1+t^2} - \int \frac{t^2+1}{\sqrt{1+t^2}} dt + \frac{1}{\sqrt{1+t^2}} dt =$$

$$= t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + \ln\left|t+\sqrt{1+t^2}\right|$$

Looking at the given expression, its beginning and end, we have obtained the equation

$$Int = t\sqrt{1+t^2} - \int \sqrt{1+t^2}\,dt + \ln\left|t+\sqrt{1+t^2}\right|$$

or

$$Int = t\sqrt{1+t^2} - Int + \ln\left|t+\sqrt{1+t^2}\right|$$

We can express unknown $Int$ from the equation

$$2Int = t\sqrt{1+t^2} + \ln\left|t+\sqrt{1+t^2}\right|$$

$$Int = \frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t+\sqrt{1+t^2}\right|\right)$$

or

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}\left(t\sqrt{1+t^2} + \ln\left|t+\sqrt{1+t^2}\right|\right) + C$$