# Basic Rules of Integration

DETAILED DESCRIPTION:

This section introduces basic formulas of integration of elementary functions and the main properties of indefinite integrals. The section explains how to derive integration formulas from well-known differentiation rules. Several examples with explanations are discussed. Exercises for individual learning of integration are presented. At the end of the section there is an example on how to check the correctness of the solution of an integral.

AIM: To learn basic formulas and properties of integrals; to introduce methods of integration

**Learning Outcomes:**

1. Learning the basic integration formulas
2. Application of the properties of indefinite integrals
3. Computing simple integrals of elementary functions
4. Transformation of integrands if necessary

Prior Knowledge: rules of differentiation; meaning of the term antiderivative; algebraic and trigonometric formulas to transform the integrands.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content



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## 1. Integration formulas

Understanding that integration is the reverse procedure of differentiation, we will write basic formulas of integrals. Any formula

$$\int f(x)dx = F(x) + C$$

can be proved by differentiation – derivative of the function on the right side of the formula must be equal with the integrand:

$$\frac{d(F(x)+C)}{dx}=f(x)$$

For instance, let us prove the formula

$$\int x^n dx=\frac{x^{n+1}}{n+1}+C; \quad n\neq -1$$

The derivative of the right side of the formula

$$\left(\frac{x^{n+1}}{n+1}+C\right)'=\left(\frac{x^{n+1}}{n+1}\right)'+C'=\frac{1}{n+1}\cdot(n+1)x^n+0=x^n$$

The special case $n=-1$ gives another formula

$$\int x^{-1}dx=\int\frac{dx}{x}=lnx+C$$

## 2. List of basic integration formulas

The list of basic differentiation formulas covers all elementary functions. Therefore, the basic list of integration formulas contain the antiderivatives that are elementary functions – power functions, exponent functions, logarithmic functions, trigonometric functions, and cyclometric functions:

1. $\int dx=x+C$

2. $\int x^n dx=\frac{x^{n+1}}{n+1}+C; \quad n\neq -1$

3. $\int x^{-1}dx=\int\frac{dx}{x}=ln|x|+C$

4. $\int e^x dx=e^x+C$

5. $\int a^x dx=\frac{a^x}{lna}+C$

6. $\int sinx\, dx=-cosx+C$

7. $\int cosx\, dx=sinx+C$

8. $\int\frac{dx}{sin^2x}=-cotx+C$

9. $\int\frac{dx}{cos^2x}=tanx+C$

10. $\int\frac{dx}{1+x^2}=arctanx+C$

11. $\int\frac{dx}{\sqrt{1-x^2}}=arcsinx+C$

Usually this list is been supplemented by additional formulas that relate to hyperbolic functions, cyclometric or inverse trigonometric functions, and similar-looking integrals:

12. $\int sinhx\, dx = coshx + C$

13. $\int coshx\, dx = sinhx + C$

14. $\int \frac{dx}{sinh^2 x} = -cothx + C$

15. $\int \frac{dx}{cosh^2 x} = tanhx + C$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} arctan\frac{x}{a} + C$

17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = arcsin\frac{x}{a} + C$

18. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} ln\left|\frac{a+x}{a-x}\right| + C$

19. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

The basic formulas work for any of the mentioned functions whenever the argument of the function is $x$; $t$; $\omega$; $s$, or some other. For instance, the following formulas are true, like formula 6:

$$\int sint\, dt = -cost + C \quad \text{or} \quad \int sin\omega\, d\omega = -cos\omega + C.$$

## 3. Properties of indefinite integrals

Let us look at the most common properties of integrals.

Property 1. The derivative of integral equals to the integrand

$$\left(\int f(x)dx\right)' = f(x)$$

Property 2. The integral of the sum of two function is equal to the sum of two integrals of given functions

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

Property 3. For any arbitrary constant $a$

$$\int af(x)d = a\int f(x)dx$$

Property 4. The integral of the differential of a function is equal to that function plus an arbitrary constant

$$\int d(F(x)) = F(x) + C$$

The first property derives from the definition of the indefinite integral. Other mentioned properties can be proved based on the first property and by the rules of differentials.

Example 3.1 Compute the integral

$$\int (sinx + e^x)dx$$

Solution

Combining second property and formulas 4 and 6 we get

$$\int (sinx + e^x)dx = \int sinx\, dx + \int e^x dx =$$

$$= -cosx + e^x + C$$

Example 3.2 Compute the integral

$$\int \frac{5}{9+x^2}dx$$

Solution

Combining third property and formula 16 we get

$$\int \frac{5}{9+x^2}dx = 5\int \frac{dx}{9+x^2} =$$

$$= 5\cdot\frac{1}{3}arctan\frac{x}{3} + C$$

Example 3.3 Compute the integral

$$\int \left(4x^3 - 2\sqrt{x} + \frac{3ln7}{x}\right)dx$$

Solution

$$\int \left(4x^3 - 2\sqrt{x} + \frac{3ln7}{x}\right)dx = 4\int x^3 dx - 2\int x^{\frac{1}{2}}dx + 3ln7\int \frac{dx}{x} =$$

$$= 4\frac{x^3}{4} - 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3ln7\cdot ln|x| + C =$$

$$= x^3 - \frac{4}{3}x\sqrt{x} + 3ln7\cdot ln|x| + C$$

## 4. Alteration of the integrand

In most cases the integrand function is quite complicated, therefore special techniques of integration are needed. However, there are functions that we can alter to use basic formulas. Let us investigate some examples where we will use algebra and trigonometry formulas.

Example 4.1 Compute the integral

$$\int \frac{dx}{\sqrt{81-49x^2}}$$

Solution

We change the expression under the square root to apply basic integration formula 17

$$\int \frac{dx}{\sqrt{81-49x^2}} = \int \frac{dx}{\sqrt{49\left(\frac{81}{49}-x^2\right)}} = \frac{1}{7}\int \frac{dx}{\sqrt{\frac{81}{49}-x^2}} = \frac{1}{7} arcsin\frac{7x}{9} + C$$

Example 4.2 Compute the integral

$$\int (2-x)^2 dx$$

Solution

We will expand the expression

$$\int (2-x)^2 dx = \int (4-4x+x^2)dx = 4x - 2x^2 + \frac{x^3}{3} + C$$

Example 4.3 Compute the integral

$$\int \frac{x^2-5x+6}{x-2} dx$$

Solution

We factorise the nominator and simplify the integrand

$$\int \frac{x^2-5x+6}{x-2} dx = \int \frac{(x-2)(x-3)}{x-2} dx = \int (x-3)dx = \frac{x^2}{2} - 3x + C$$

Example 4.4 Compute the integral

$$\int \frac{x^2 \cdot \sqrt[3]{x}}{x^{-\frac{2}{3}} \cdot \sqrt{x}} dx$$

Solution

Here we use the property of products of power

$$\int \frac{x^2 \cdot \sqrt[3]{x}}{x^{-\frac{2}{3}} \cdot \sqrt{x}} dx = \int x^{2+\frac{1}{3}+\frac{2}{3}-\frac{1}{2}} dx = \int x^{\frac{5}{2}} dx = \frac{2}{7}x^{\frac{7}{2}} + C = \frac{2}{7}x^3 \cdot \sqrt{x} + C$$

Example 4.5 Compute the integral

$$\int cos^2\frac{x}{2} dx$$

Solution

Apply the trigonometric formula of the double angle

$$\int cos^2\frac{x}{2} dx = \int \frac{1+cosx}{2} dx = \frac{1}{2}x + \frac{1}{2} sinx + C$$

For more complex integrals, there are used special integration techniques that we will discuss in the following sections.

## 5. Exercises

Compute the following integrals using basic formulas and algebraic transformations if needed.

1. $\int \left(6x^3 + \frac{2}{5x^3} - 12\right) dx$

2. $\int \left(\frac{1}{2\sqrt{x}} - x^{0.5} + \frac{4}{x}\right) dx$

3. $\int \left(3sinx + \frac{2}{sin^2x}\right) dx$

4. $\int \left(12^x - \frac{1}{cos^2x} + e^x\right) dx$

5. $\int \frac{16}{x^2+25} dx$

6. $\int \left(sinht - \frac{1}{\sqrt{t^2-64}}\right) dt$

7. $\int \sqrt[3]{\frac{x^4 \cdot \sqrt{x^3}}{x^{-\frac{1}{2}}}} dx$

8. $\int \frac{x^2-3x+4}{x+1} dx$

## 6. Solutions

1. $\int \left(6x^3 + \frac{2}{5x^3} - 12\right) dx = 6\int x^3 dx + \frac{2}{5}\int x^{-3} dx - 12\int dx =$

$$= \frac{6x^4}{4} - \frac{2}{5} \cdot \frac{x^{-2}}{2} - 12x + C = 1.5x^4 - \frac{1}{5x^2} - 12x + C$$

2. $\int \left(\frac{1}{2\sqrt{x}} - x^{0.5} + \frac{4}{x}\right) dx = \frac{1}{2}\int x^{-0.5} dx - \int x^{0.5} dx + 4\int \frac{dx}{x} =$

$$= \frac{1}{2}\frac{x^{0.5}}{0.5} - \frac{x^{1.5}}{1.5} + 4ln|x| + C = \sqrt{x} - \frac{2}{3}x\sqrt{x} + 4ln|x| + C$$

3. $\int \left(3sinx + \frac{2}{sin^2x}\right) dx = 3\int sinx\, dx + 2\int \frac{dx}{sin^2x} = -3cosx - 2cotx + C$

4. $\int \left(12^x - \frac{1}{cos^2x} + e^x\right) dx = \int 12^x dx - \int \frac{dx}{cos^2x} + \int e^x dx =$

$$= \frac{12^x}{ln12} - tanx + e^x + C$$

5. $\int \frac{16}{x^2+25} dx = 16 \int \frac{dx}{x^2+5^2} = 16 \cdot \frac{1}{5} arctan \frac{x}{5} + C$

6. $\int \left(sinht - \frac{1}{\sqrt{t^2-64}}\right) dt = \int sinht\, dt - \int \frac{dt}{\sqrt{t^2-64}} =$

$$= cosht - ln\left|t + \sqrt{t^2-64}\right| + C$$

7. $\int \sqrt[3]{\frac{x^4 \cdot \sqrt{x^3}}{x^{-\frac{1}{2}}}} dx = \int \left(x^{4+\frac{3}{2}+\frac{1}{2}}\right)^{\frac{1}{3}} dx = \int x^{6 \cdot \frac{1}{3}} dx = \int x^2 dx = \frac{x^3}{3} + C$

8. $\int \frac{x^2+5x+4}{x+1} dx = \int \frac{(x+1)(x+5)}{x+1} dx = \int (x+5)\, dx =$

$$= \int x\, dx + 5 \int dx = \frac{x^2}{2} + 5x + C$$

## 7. Additional note

Every integration result can be checked by differentiation of the antiderivative. For instance, let us check the result of example 3.2 by applying the chain rule of differentiation

$$\left(5 \cdot \frac{1}{3} arctan \frac{x}{3} + C\right)' = \frac{5}{3}\left(arctan \frac{x}{3}\right)' + C' =$$

$$= \frac{5}{3} \cdot \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \left(\frac{x}{3}\right)' = \frac{5}{3} \cdot \frac{1}{1+\frac{x^2}{9}} \cdot \frac{1}{3} =$$

$$= \frac{5}{3} \cdot \frac{9}{9+x^2} \cdot \frac{1}{3} = \frac{5}{9+x^2}$$

Thus, we get the same function as the integrand.