Integration Techniques: Substitution

DETAILED DESCRIPTION:

In this chapter we will investigate the methods of integration of composite functions. The Chain Rule of derivation will be presented as an argument for integration of composite functions. The reverse process will be presented that introduces the formula of the Reverse Chain Rule. The method of substitution can help to simplify the notation of an integral. It will be called u-substitution. Several examples are presented in the chapter. There are integrals of composite functions given where the inner function is either linear or non-linear.

AIM: **To master the skills of substitution to solve the integrals of composite functions.**

Learning Outcomes:

1. Students will acquire the method of changing the differential to compute integrals
2. Students will be able to carry out integration by making substitution
3. Students will recognize that the method of substitution is useful with integrals of composite functions

Prior Knowledge: rules of differentiation; basic rules of integration; algebraic formulas; knowledge of elementary mathematics.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

Substitution

1. Integration of composite functions

In the previous sections we solved integrals where the integrands are elementary simple functions of variable x. How we can compute the integral if the integrand is a composite function?

A ***composite function*** is composed of two functions $f(x)$ and $g(x)$ where one of the given functions is the argument of another function $f(g(x))$.

Function $g(x)$ is the inner function and function $f(x)$ is the outer function.

Examples of composite functions

$$1)\ sin3x;\ \ 2)\ cos(x^2);\ \ 3)\ \ e^{tanx};\ \ \ 4)\ (6x+7)^{13}$$

There are given linear inner functions $3x$ and $6x+7$, and non-linear inner functions x^2 and $tanx$. Outer functions are $sinx$; $cosx$; e^x; x^{13}.

2. Reverse Chain Rule

Let us remember the ***Chain Rule*** for differentiation of a composite function

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Suppose that we are trying to detect the function whose derivative is $f'(g(x)) \cdot g'(x)$. We will perform the reverse procedure of differentiation to solve this problem. For instance, can we determine the function whose derivative is $cos(x^2) \cdot 2x$?

Here the composite function is $cosx^2$ whose argument (inner function) is x^2. The derivative of x^2 is

$(x^2)' = 2x$.

Let us check now

$$(sin(x^2))' = cos(x^2) \cdot (x^2)' = cos(x^2) \cdot 2x$$

The performed procedure can be recorded in the notation of integral

$$\int cos(x^2) \cdot 2xdx = sin(x^2) + C$$

Let us simplify the notation by substituting the function

$$let\ u = x^2\ \ then\ \ du = dx^2 = (x^2)'dx = 2xdx$$

$$\int cosu\ du = sinu + C, \qquad where\ u = x^2.$$

Generalising the case, we write the ***formula of Reverse Chain Rule***:

$$\boldsymbol{\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + C}$$

3. Application of Reverse Chain Rule

Based on the definition of the differential of the function $du = u'dx$ for function $u = u(x)$

we can simplify the Reverse Chain Rule

$$\boldsymbol{\int f(u)du = F(u) + C}$$

Therefore, the basic list of integrals given in the section "Basic Rules of Integration" can also be applicable for composite functions. For instance, formulas

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{dx}{cos^2x} = tanx + C$$

can be modified for composite functions whose inner function is the function u

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{cos^2u} = tanu + C$$

Example 3.1 Integrate

$$\int cos^5xd(cosx)$$

Solution

Let us notice

$$\int u^5du = \frac{u^6}{6} + C$$

We can apply this formula for the given integral

$$\int cos^5xd(cosx) = \frac{cos^6x}{6} + C$$

Example 3.2 Integrate

$$\int (8-11x)^5d(8-11x)$$

Solution

Notice that we can apply the same formula as in the example 3.1.

$$\int (8-11x)^5d(8-11x) = \frac{(8-11x)^6}{6} + C$$

Example 3.3 Integrate

$$\int \frac{d(4^x)}{cos^2(4^x)}$$

Solution

$$\int \frac{d(4^x)}{cos^2(4^x)} = tan(4^x) + C$$

4. The change of differential

The examples in the previous chapter demonstrate the integration method where the expression under the integral has the differential of a function as a variable of integration. We can create such differentials in simple cases. Especially if the argument of the composite function is linear

$$\int f(ax+b)dx$$

Let us compute the differential of a linear function

$$d(ax+b)=(ax+b)'dx=adx$$

Calculation shows that differentials dx and $d(ax+b)$ differ only by a constant number a. Therefore, we can easy change the integral

$$\boldsymbol{\int f(ax+b)dx=\frac{1}{a}\int f(ax+b)d(ax+b)=\frac{1}{a}F(ax+b)+C}$$

Example 4.1

$$\int sin7xdx=\frac{1}{7}\int sin7x\,d(7x)=-\frac{1}{7}cos7x+C$$

Example 4.2

$$\int\frac{dx}{\sqrt{2x+1}}=\frac{1}{2}\int\frac{d(2x+1)}{\sqrt{2x+1}}=\sqrt{2x+1}+C$$

Example 4.3

$$\int sin^5x\cdot cosx\,dx=\int sin^5x\,d(sinx)=\frac{sin^6x}{6}+C$$

5. Method of substitution

In a more general case, we can try to simplify the integral of a composite function by substitution if we can construct the derivative of the inner function. For example, if the given integral is the following

$$\int x^3(0.5x^4+21)^{10}dx$$

we note the connection between the inner function $0.5x^4+21$ and the multiplier x^3. The multiplier is part of derivative of the inner function

$$(0.5x^4+21\ )'=2x^3$$

Therefore, we can change the integral substituting the inner function by u

$$\int x^3(0.5x^4+21)^{10}dx=\left|\begin{matrix}let\ \ u=0.5x^4+21\\ then\ du=2x^3dx\end{matrix}\right|=$$

$$=\frac{1}{2}\int 2x^3(0.5x^4+21)^{10}dx=\frac{1}{2}\int u^{10}du=\frac{1}{2}\cdot\frac{u^{11}}{11}+C=$$

$$=\frac{(0.5x^4+21)^{11}}{22}+C$$

In a more general case we can integrate the composite function $f(u)$ with respect to the function $u=u(x)$ if the integrand contains the derivative of the argument function

$$\int f(u)u'dx = \int f(u)du$$

We can call this method of substitution ***u-substitution***.

6. Examples

Example 6.1 Compute

$$\int \frac{dx}{5x+2}$$

Solution

$$\int \frac{dx}{5x+2} = \left| \begin{matrix} let \quad u = 5x+2 \\ then \;\; du = (5x+2)'dx = 5dx \end{matrix} \right| =$$

$$= \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\cdot ln|u| + C = \frac{1}{5}\cdot ln|5x+2| + C$$

Example 6.2 Compute

$$\int \frac{dx}{1+16x^2}$$

Solution

$$\int \frac{dx}{1+16x^2} = \left| \begin{matrix} let \;\; u = 4x \\ then \; du = 4dx \end{matrix} \right| = \frac{1}{4}\int \frac{du}{1+u^2} =$$

$$= \frac{1}{4}arctan(4x) + C$$

Example 6.3 Compute

$$\int \frac{e^{tanx}}{cos^2x}dx$$

Solution

$$\int \frac{e^{tanx}}{cos^2x}dx = \left| \begin{matrix} let \; u = tanx \\ then \; du = \dfrac{1}{cos^2x}dx \end{matrix} \right| =$$

$$= \int e^u du = e^u + C = e^{tanx} + C$$

Example 6.4 Compute

$$\int \frac{tdt}{(t+1)^3}$$

Solution

$$\int \frac{tdt}{(t+1)^3} = \left| \begin{matrix} let \; u = t+1; \;\; t = u-1 \\ than \; du = dt \end{matrix} \right| =$$

$$= \int \frac{u-1}{u^3}du = \int u^{-2}du - \int u^{-3}du =$$

$$= \frac{u^{-1}}{-1} - \frac{u^{-2}}{-2} + C = -\frac{1}{t+1} + \frac{2}{(t+1)^2} + C$$

7. Exercises

Find the integrals by suitable u-substitution

1. $\int cos(8\pi x)dx$
2. $\int \frac{dx}{(3-2x)^4}$
3. $\int \frac{dx}{\sqrt{1-(x+6)^2}}$
4. $\int \frac{ln^5 x}{x}dx$
5. $\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right)dx$
6. $\int e^{\theta}\sqrt{12+e^{\theta}}d\theta$
7. $\int cot\, dt$
8. $\int \frac{arctan^7 x}{1+x^2}dx$

8. Solutions

1. $\int cos(8\pi x)dx$

Solution

$$\int cos(8\pi x)dx = \left| \begin{matrix} let\ u = 8\pi x \\ du = 8\pi dx \end{matrix} \right| = \frac{1}{8\pi}\int cosu\, du =$$

$$= \frac{1}{8\pi}sinu + C = \frac{1}{8\pi}sin(8\pi x) + C$$

2. $\int \frac{dx}{(3-2x)^4}$

Solution

$$\int \frac{dx}{(3-2x)^4} = \left| \begin{matrix} let\ u = 3-2x \\ du = -2dx \end{matrix} \right| = -\frac{1}{2}\int \frac{du}{u^4} =$$

$$= -\frac{1}{2}\cdot\frac{u^{-3}}{-3} + C = \frac{1}{6(3-2x)^3} + C$$

3. $\int \frac{dx}{\sqrt{1-(x+6)^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-(x+6)^2}} = \left| \begin{matrix} let\ u = x+6 \\ du = dx \end{matrix} \right| = \int \frac{du}{\sqrt{1-u^2}} =$$

$$= arcsinu + C = arcsin(x+6) + C$$

4. $\int \frac{ln^5 x}{x} dx$

Solution

$$\int \frac{ln^5 x}{x} dx = \left| \begin{matrix} let\ u = lnx \\ du = \frac{1}{x} dx \end{matrix} \right| = \int u^5 du =$$

$$= \frac{u^6}{6} + C = \frac{ln^6 x}{6} + C$$

5. $\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right) dx$

Solution

$$\int \sqrt{x}\left(3+8x^{\frac{3}{2}}\right) dx = \left| \begin{matrix} let\ u = 3+8x^{\frac{3}{2}} \\ du = 12\sqrt{x} dx \end{matrix} \right| = \frac{1}{12}\int u\, du =$$

$$= \frac{1}{12}\frac{u^2}{2} + C = \frac{\left(3+8x^{\frac{3}{2}}\right)^2}{24} + C$$

6. $\int e^{\theta}\sqrt{12+e^{\theta}} d\theta$

Solution

$$\int e^{\theta}\sqrt{12+e^{\theta}} d\theta = \left| \begin{matrix} let\ u = 12+e^{\theta} \\ du = e^{\theta} d\theta \end{matrix} \right| = \int \sqrt{u}\, du =$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{12+e^{\theta}}^3 + C$$

7. $\int cot\ dt$

Solution

$$\int cot\,dt = \int \frac{cost}{sint}dt = \left| \begin{matrix} let\ u = sint \\ du = cost\ dt \end{matrix} \right| =$$

$$= \int \frac{du}{u} = ln|u| + C = ln|sint| + C$$

8. $\int \frac{arctan^7 x}{1+x^2}dx$

Solution

$$\int \frac{arctan^7 x}{1+x^2}dx = \left| \begin{matrix} let\ u = arctanx \\ du = \frac{dx}{1+x^2} \end{matrix} \right| = \int u^7 du =$$

$$= \frac{u^8}{8} + C = \frac{arctan^8 x}{8} + C$$