Integration Techniques: Integration by Parts

DETAILED DESCRIPTION:
The section starts by recalling the Product rule for differentiation of multiplication of two functions. The integration of this formula produces the method of integration by parts. The application of this method is useful if the integrand is a product of two functions of special type. The most popular cases are discussed and are complemented by examples.

AIM: to learn the method of integration by parts and to recognize the types of integrals for which the method is useful.

Learning Outcomes:

1. Recognize the integrals for that the integration by parts is useful.
2. Can apply the method of partial integration to compute the integrals of different type.

Prior Knowledge: rules of differentiation; rules for integration; the method of substitution; algebra and trigonometry formulas.

Relationship to real maritime problems: a well-known application of the method of integration by parts is the calculation of Fourier coefficients of the Fourier series. Fourier series have broad applications in many disciplines. They are used to describe periodical physical phenomena, for instance, in signal processing, to detect and correct sources of vibration in mechanical devices.

Content

1. Formula for integration by parts
2. Special cases
3. Examples
4. Repeated application of the method
5. Exercises
6. Solutions

Integration by Parts

1. Formula for integration by parts

We will discuss the method that is often useful to compute the integral if it's integrand is a product of two functions.

Let us have two differentiable functions $u = u(x)$ and $v = v(x)$. We will calculate the differential of the product of these functions according to the *Product Rule*

$$d(uv) = udv + vdu$$

By integrating both sides of this equation, we obtain

$$\int d(uv) = \int udv + \int vdu.$$

By transposing terms and applying the property of integrals $\int d(uv) = uv + C$, we get

$$\int \boldsymbol{udv} = \boldsymbol{uv} - \int \boldsymbol{vdu}.$$

This formula expresses the ***method of integration by parts*** or ***partial integration***. The method is recommended if the integral on the right side of the formula is not more complicated than the given integral.

Example 1.1 Find the integral

$$\int xe^x dx$$

Solution

$$\int xe^x dx = \left| \begin{matrix} let & u = x; \; dv = e^x dx \\ then & du = dx; \; v = \int e^x dx = e^x \end{matrix} \right| =$$

$$= xe^x - \int e^x dx = xe^x - e^x + C$$

Comment. Let us note that here the formula for calculation of the differential of the function $v(x)$ is applied

$$dv = v'dx$$

that we have to integrate to find the function $v(x)$.

2. Special cases

There are some special forms of integrals for which we can apply the method described above. They include polynomials whose degree is decreasing at derivation. On the other hand, there are functions that we cannot simplify by derivation. For instance, the exponential function e^x does not change by derivation. These observations point us to some standard situations for selection of the function $u = u(x)$.

Case 1. The integrand is a product of a polynomial and a trigonometric function $sinx$ or $cosx$

$\int P_n(x) sinx\, dx;$ we choose $u = P_n(x),$ then the rest of the expression is the differential $dv = sinxdx$

Case 2. The integrand is a product of a polynomial and an exponential function e^x or a^x

$$\int P_n(x)a^x dx;\ \text{we choose } u = P_n(x),\ \text{then the differential is } dv = a^x dx$$

Case 3. The integrand is a product of a polynomial and a logarithmic function lnx or $log_a x$

$$\int P_n(x)log_a x dx;\ \text{we choose } u = log_a x,\ \text{then the differential is } dv = P_n(x)dx$$

Comment. Instead of the polynomial there can be given an arbitrary power function

$$\int x^k lnx dx$$

Case 4. The integrand is a cyclometric function $arcsinx$ or $arctanx$

$$\int arcsinx\, dx;\ \text{we choose } u = arcsinx,\ \text{then the differential is } dv = dx$$

3. Examples

Example 3.1 Find the integral

$$\int 2xcosx\, dx$$

Solution

$$\int 2xcosx\, dx = \left| \begin{matrix} let \quad u = 2x; \ dv = cosx\, dx \\ then \ du = 2dx; \ v = \int cosx\, dx = sinx \end{matrix} \right| =$$

$$= 2xsinx - \int 2sinx\, dx = 2xsinx + 2cosx + C$$

Example 3.2 Find the integral

$$\int x^3 lnx\, dx$$

Solution

$$\int x^3 lnx\, dx = \left| \begin{matrix} let \quad u = lnx; \ dv = x^3 dx \\ then \ du = \frac{1}{x}dx; \ v = \int x^3 dx = \frac{x^4}{4} \end{matrix} \right| =$$

$$= \frac{x^4}{4}lnx - \int \frac{x^4}{4}\cdot\frac{1}{x}dx = \frac{x^4}{4}lnx - \frac{1}{4}\int x^3 dx = \frac{x^4}{4}lnx - \frac{x^4}{16} + C$$

Example 3.3 Find the integral

$$\int arctanx\, dx$$

Solution

$$\int arctanx\, dx = \begin{vmatrix} let \quad u = arctanx; \; dv = dx \\ then \; du = \dfrac{dx}{1+x^2}; \quad v = \int dx = x \end{vmatrix} =$$

$$= xarctanx - \int \frac{x}{1+x^2} dx = xarctanx - \frac{1}{2}\int \frac{2x}{1+x^2} dx =$$

$$= xarctanx - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = = xarctanx - \frac{1}{2} ln|1+x^2| + C$$

In more general cases the functions can be composite, for instance, $sinax$; $arctan(ax)$.

Example 3.4 Find the integral

$$\int (x+1) sin4x\, dx$$

Solution

$$\int (x+1) sin4x\, dx = \begin{vmatrix} let \quad u = x+1; \; dv = sin4x\, dx \\ then \; du = dx; \; v = \int sin4x\, dx = -\dfrac{1}{4} cos4x \end{vmatrix} =$$

$$= -\frac{1}{4}(x+1)cos4x + \frac{1}{4}\int cos4x\, dx = -\frac{1}{4}(x+1)cos4x + \frac{1}{16} sin4x + C$$

4. Repeated application of method

If the polynomial factor of integrand is not linear, we can apply partial integration repeatedly. If the polynomial has degree n, we apply the method n times repeatedly to eliminate the degree of the polynomial.

Example 4.1 Find the integral

$$\int x^3 sinx\, dx$$

Solution

For the given integral we will use the method of integration by parts three times because the degree of the given polynomial is three.

$$\int x^3 sinx\,dx = \left| \begin{matrix} let \quad u = x^3; \;\; dv = sinx\,dx \\ then \;\; du = 3x^2dx; \;\; v = \int sinx\,dx = -cosx \end{matrix} \right| =$$

$$= -3x^3cosx + 3\int x^2\,cosx\,dx = \left| \begin{matrix} let \quad u = x^2; \;\; dv = cosx\,dx \\ then \;\; du = 2xdx; \;\; v = \int cosx\,dx = sinx \end{matrix} \right| =$$

$$= -3x^3cosx + 3\left(x^2sinx - 2\int xsinx\,dx\right) = \left| \begin{matrix} let \;\; u = x; \;\; dv = sinx\,dx \\ then \;\; du = dx; \;\; v = -cosx \end{matrix} \right| =$$

$$= -3x^3cosx + 3x^2sinx - 6\left(-xcosx + \int cosx\,dx\right) =$$

$$= -3x^3cosx + 3x^2sinx + 6xcosx - 6sinx + C$$

Example 4.2 Find the integral

$$\int (x^2 - 3x + 7)2^x dx$$

Solution

$$\int (x^2 - 3x + 7)2^x dx = \left| \begin{matrix} let \quad u = x^2 - 3x + 7; \;\; dv = 2^x dx \\ then \;\; du = (2x-3)dx; \;\; v = \int 2^x dx = \dfrac{2^x}{ln2} \end{matrix} \right| =$$

$$= (x^2 - 3x + 7)\frac{2^x}{ln2} - \frac{1}{ln2}\int (2x-3)2^x dx = \left| \begin{matrix} let\, u = 2x - 3; \;\; dv = 2^x dx \\ then\, u = 2dx; \;\; v = \int 2^x dx = \dfrac{2^x}{ln2} \end{matrix} \right| =$$

$$= (x^2 - 3x + 7)\frac{2^x}{ln2} - \frac{1}{ln2}\left((2x-3)\frac{2^x}{ln2} - \frac{2}{ln2}\int 2^x dx\right) =$$

$$= (x^2 - 3x + 7)\frac{2^x}{ln2} - (2x-3)\frac{2^x}{(ln2)^2} + \frac{2 \cdot 2^x}{(ln2)^3} + C$$

5. Exercises

Find the integrals

1. $\int \frac{x}{8} sinx\,dx$

2. $\int 5x \cdot 5^x dx$

3. $\int \frac{lnx}{x^2} dx$

4. $\int (x^2+1)cos2x\,dx$

5. $\int arcsinx\,dx$

6. Solutions

1. $\int \frac{x}{8} sinx\,dx$

Solution

$$\int \frac{x}{8} sinx\,dx = \left| \begin{matrix} let\ \ u=\frac{x}{8}, & dv = sinx\,dx \\ du=\frac{1}{8}dx, & v=\int sinx\,dx = -cosx \end{matrix} \right| =$$

$$= -\frac{x}{8}cosx + \frac{1}{8}\int cosx\,dx = -\frac{x}{8}cosx + \frac{1}{8}sinx + C$$

2. $\int 5x \cdot 5^x dx$

Solution

$$\int 5x \cdot 5^x dx = \left| \begin{matrix} let\ u = 5x, & dv = 5^x dx \\ du = 5dx, v = & \int 5^x dx = \frac{5^x}{ln5} \end{matrix} \right| =$$

$$= 5x\frac{5^x}{ln5} - \frac{5}{ln5}\int 5^x dx = 5x\frac{5^x}{ln5} - \frac{5 \cdot 5^x}{ln^2 5} + C$$

3. $\int \frac{lnx}{x^2} dx$

Solution

$$\int \frac{lnx}{x^2} dx = \left| \begin{matrix} let\ \ u = lnx, & dv = \frac{dx}{x^2} \\ du = \frac{1}{x}dx, & v = \int x^{-2}dx = -x^{-1} \end{matrix} \right| =$$

$$= -\frac{lnx}{x} + \int x^{-2}dx = -\frac{lnx}{x} - \frac{1}{x} + C$$

4. $\int (x^2+1)cos2x\, dx$

Solution

$$\int (x^2+1)cos2x\, dx = \left| \begin{matrix} let\ \ u = x^2+1, \ \ dv = cos2x\, dx \\ du = 2xdx;\ v = \int cos2x\, dx = \frac{1}{2}sin2x \end{matrix} \right| =$$

$$= \frac{x^2+1}{2}sin2x - \frac{1}{2}\int sin2x\, dx = \frac{x^2+1}{2}sin2x + \frac{1}{4}cos2x + C$$

5. $\int arcsinx\, dx$

Solution

$$\int arcsinx\, dx = \left| \begin{matrix} let\ \ u = arcsinx, \ \ \ dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}dx, \ \ v = x \end{matrix} \right| =$$

$$= xarcsinx - \int \frac{x}{\sqrt{1-x^2}}dx = \left| \begin{matrix} let\ \ u = 1-x^2 \\ du = -2xdx \end{matrix} \right| =$$

$$= xarcsinx + \frac{1}{2}\int \frac{du}{\sqrt{u}} = xarcsinx + \sqrt{u} + C = \ xarcsinx + \sqrt{1-x^2} + C$$

Here we used the substitution for the second integral to simplify the integrand.