Indefinite Integrals: Integration of rational functions

DETAILED DESCRIPTION:
In this chapter we will discuss the problem-solving methods for indefinite integrals of rational functions. We will introduce proper rational functions and improper rational functions, and algebraic methods on how to decompose them into rational fractions or powers and rational fractions. Appropriate integral formulas will be considered. Examples of integration of simple rational functions will be demonstrated.

Students can investigate the examples presented by Symbolab Step-by-Step Calculator; Algebra; Rational Fractions (URL: https://www.symbolab.com/solver/partial-fractions-calculator). With this software it is also possible to check their own solutions by applying the Symbolab calculator for integrals.

AIM: To acquire the technique of integration of rational functions.

Learning Outcomes:

1. Perform the expansion of proper rational function in partial fractions
2. Perform the long division of polynomials to get a polynomial plus a proper rational function
3. Solve the integrals of rational functions

Prior Knowledge: algebraic identities; completing the square; factorising of polynomials; roots of polynomials; basic integration and derivation formulas.

Relationship to real maritime problems: Computations of indefinite integrals are used as a methodology in calculation of definite integrals. Differentiation and integration are widely used to solve many engineering problems. Practical application of integrals is part of navigation theory; for instance, integrals are used in designing the Mercator map. Derivatives and integrals helped to improve understanding of the concept of Earth's curve: the distance ships had to travel around a curve to get to a specific location. Calculus has been used in shipbuilding for many years to determine both the curve of the ship's hull, as well as the area under the hull.

Content

Integration of rational functions

1. Rational functions, proper and improper rational functions

A ***rational function*** has the form

$$\boldsymbol{f(x) = \frac{P_n(x)}{Q_m(x)}},$$

where $P_n(x)$ and $Q_m(x)$ are polynomials

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

$$Q_m(x) = b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \cdots + b_m$$

If the degree n of polynomial $P_n(x)$ is less than the degree m of polynomial $Q_m(x)$ $(n < m)$ then the rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)}$$

is called a ***proper rational function***, otherwise it is called an ***improper rational function***.

For example,

$$f(x) = \frac{2}{x+3}$$

$$g(x) = \frac{2x-1}{x^2-4}$$

$$t(x) = \frac{x+1}{x-3}$$

$$s(x) = \frac{x^2+3x-4}{x+1}$$

functions $f(x)$ and $g(x)$ are proper rational functions. Functions $t(x)$ and $s(x)$ are improper rational functions.

2. Basic integrals for simple cases

Simple cases of rational functions in a general form are the following

$$f(x) = \frac{A}{x+B}$$

$$g(x) = \frac{A}{(x+B)^m}$$

$$s(x) = \frac{2Ax+B}{Ax^2+Bx+D}$$

Let us integrate these functions by applying formulas

$$\int \frac{dx}{x+B} = ln|x+B| + C$$

$$\int \frac{du}{u^m} = \frac{u^{-m+1}}{-m+1} + C$$

Integral of the first function is

$$\int f(x)dx = \int \frac{A}{x+B}dx = A\int \frac{dx}{x+B} = A \cdot ln|x+B| + C$$

We will replace the linear argument of integrand $g(x)$ by the function $u = x + B$

$$\int g(x)\,dx = \int \frac{A}{(x+B)^m}dx = \begin{vmatrix} u = x+B \\ du = dx \end{vmatrix} = A\int \frac{du}{u^m} =$$

$$= A\frac{u^{-m+1}}{-m+1} + C = A\frac{(x+B)^{-m+1}}{-m+1} + C$$

For function $s(x)$ we will use substitution $u = Ax^2 + Bx + D$ and $du = 2Ax + B$

$$\int s(x)dx = \int \frac{2Ax+B}{Ax^2+Bx+D}dx = \int \frac{du}{u} = ln|u| + C = ln|Ax^2+Bx+D| + C$$

To integrate more complicated rational functions, we can use special methods to split them into the sum of simpler terms.

Example 2.1

$$\int \frac{2}{x+3}dx = 2\int \frac{dx}{x+3} = 2ln|x+3| + C$$

Example 2.2

$$\int \frac{dx}{(x-1)^7} = \frac{(x-1)^{-7+1}}{-7+1} + C = \frac{-1}{6(x-1)^6} + C$$

Example 2.3

$$\int \frac{2x+5}{x^2+5x+10}dx = |u = x^2+5x+10 \; and \; du = (2x+5)dx| = ln|x^2+5x+10| + C$$

In other cases, it is necessary to decompose the rational function into partial fractions to simplify the integration.

3. Decomposition of partial fractions

Let us have a proper rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)}; \quad n < m$$

We can apply the ***method of decomposition of partial fractions*** if the denominator can be factorised into fractions. Here we discuss three cases of decomposition of partial fractions:

Case 1: Denominator can be factorised in all linear multipliers;

Case 2: Denominator contains an irreducible quadratic;

Case 3: Denominator contains the repeated linear factor.

3.1. Case 1. Denominator can be factorised in all linear multipliers

The following example shows that we can integrate the function more easily if it is decomposed into partial fractions with linear denominators

Example 3.1

$$\int \frac{x+8}{x^2+x-2}dx = \int \left(\frac{3}{x-1} - \frac{2}{x+2}\right)dx =$$

$$= \int \frac{3}{x-1}dx - \int \frac{2}{x+2}dx = 3ln|x-1| - 2ln|x+2| + C$$

If the denominator $Q_m(x)$ has real roots $x = -a$ and $x = -b$, it is reducible

$$Q_m(x) = k(x+a)(x+b)$$

We can split the given rational expression into partial fractions

$$\frac{P_n(x)}{Q_m(x)} = \frac{A}{x+a} + \frac{B}{x+b}$$

We know that the coefficients of polynomials and roots of denominator *a* and *b* are definite. To determinate the unknown constants *A* and *B*, we will equalize the denominators of partial fractions, equate the numerators, discard them, and get the equation

$$\frac{P_n(x)}{Q_m(x)} = \frac{kA(x+b)}{(x+a)(x+b)} + \frac{kB(x+a)}{(x+b)(x+a)}$$

$$P_n(x) = kA(x+b) + kB(x+a)$$

Let us plug the values $x = -a$ and then $x = -b$ into the equation to get

$$P_n(-a) = kA(b-a)$$

$$P_n(-b) = kB(a-b)$$

From these equations we can calculate the values of the unknown constants

$$A = \frac{P_n(-a)}{k(b-a)}$$

$$B = \frac{P_n(-b)}{k(a-b)}$$

Example 3.2

Compute the integral

$$\int \frac{4x+7}{x^2+x-6}dx$$

Solution

First part: decomposition of partial fractions

Step 1. Factorise the denominator $x^2+x-6=(x-2)(x+3)$

Step2. Write partial fractions with unknown constants

$$\frac{4x+7}{x^2+x-6}=\frac{4x+7}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$$

Step 3. Equalize the denominators, equate the numerators, and discard them

$$4x+7=A(x+3)+B(x-2)$$

Step 4. Plug in $x=2$ to calculate constant A

$$4\cdot 2+7=A(2+3)+B(2-2)$$

$$8+7=5A$$

$$A=3$$

Step 5. Plug in $x=-3$ to calculate constant B

$$4\cdot(-3)+7=A(-3+3)+B(-3-2)$$

$$-12+7=-5B$$

$$B=1$$

Second part: integration

$$\int \frac{4x+7}{x^2+x-6}dx=\int\left(\frac{3}{x-2}+\frac{1}{x+3}\right)dx=3ln|x-2|+ln|x+3|+C$$

Answer

$$\int \frac{4x+7}{x^2+x-6}dx=3ln|x-2|+ln|x+3|+C$$

In the ***general case***, a denominator can have more than two linear multipliers

$$\frac{P_n(x)}{k(x+a_1)(x+a_2)\cdot\ldots\cdot(x+a_m)}=\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+\ldots+\frac{A_m}{x+a_m}$$

3.2. Case 2. Denominator contains an irreducible quadratic

If the denominator $Q_m(x)$ of a rational function is reducible in the following way

$$Q_m(x)=k(x+a)(x^2+b),$$

we can decompose the given rational expression into partial fractions

$$\frac{P_n(x)}{Q_m(x)}=\frac{A}{x+a}+\frac{Bx+D}{x^2+b}$$

Similarly, we express

$$P_n(x) = kA(x^2 + b) + kBx(x + a) + kD(x + a)$$

or, considering that the degree of polynomial does not exceed 2

$$a_0x^2 + a_1x + a_2 = kA(x^2 + b) + kBx(x + a) + kD(x + a)$$

Two polynomials are equal by definition if they have the same degree and all corresponding coefficients are equal.

Therefore, we can write a system of equations and calculate the unknown constants A, B, D

$$\begin{cases} a_0 = kA + kB \\ a_1 = kBa + kD \\ a_2 = kAb + kDa \end{cases}$$

Example 3.3

Compute the integral

$$\int \frac{4x^2 + 2x - 3}{(x-2)(x^2+1)} dx$$

Solution

First part: decomposition of partial fractions

Step1. Write partial fractions with unknown constants

$$\frac{4x^2 + 2x - 3}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + D}{x^2 + 1}$$

Step 2. Equalize the denominators, equate the numerators, and discard them

$$4x^2 + 2x - 3 = A(x^2 + 1) + Bx(x - 2) + D(x - 2)$$

Step 3. Create a system of equations

$$\begin{cases} 4 = A + B \\ 2 = -2B + D \\ -3 = A - 2D \end{cases}$$

Step 4. Express *A* from the first equation and plug it into the last equation to get a system of two equations

$$A = 4 - B$$

$$-3 = 4 - B - 2D$$

$$\begin{cases} 2 = -2B + D \\ -7 = -B - 2D \end{cases}$$

Step 5. Multiply the first equation by 2 and add equations

$$-3 = -5B; \quad B = 0.6$$

Step 6. Calculate *A* and *D*

$$A = 4 - 0.6 = 3.4$$

$$D = 2 + 2B = 2 + 1.2 = 3.2$$

Second part: integration

$$\int \frac{4x^2 + 2x - 3}{(x-2)(x^2+1)} dx = \int \left(\frac{3.4}{x-2} + \frac{0.6x + 3.2}{x^2+1} \right) dx =$$

$$= \int \frac{3.4}{x-2} dx + 0.3 \int \frac{2xdx}{x^2+1} + 3.2 \int \frac{dx}{x^2+1} =$$

$$= 3.4ln|x-2| + 0.3ln|x^2+1| + 3.2arctanx + C$$

Answer

$$\int \frac{4x^2 + 2x - 3}{(x-2)(x^2+1)} dx = 3.4ln|x-2| + 0.3ln|x^2+1| + 3.2arctanx + C$$

3.3. Case 3. Denominator contains the repeated linear factor

If the denominator $Q_m(x)$ of a rational function is reducible

$$Q_m(x) = k(x+a)^m = k(x+a) \cdot (x+a) \cdot \ldots \cdot (x+a),$$

we can decompose the given rational expression into partial fractions

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_m}{(x+a)^m}$$

To calculate the unknown constants $A_1, A_2, \ldots, A_m$ we can use the same method as in case 2.

Example 3.4.

Compute

$$\int \frac{x^2 - x - 4}{(x-1)^3} dx$$

Solution

First part: decomposition of partial fractions

Step1. Write partial fractions with unknown constants

$$\frac{x^2 - x - 4}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Step 2. Equalize the denominators, equate the numerators, and discard them

$$x^2 - x - 4 = A(x-1)^2 + B(x-1) + D$$

$$x^2 - x - 4 = Ax^2 - 2Ax + A + Bx - B + D$$

Step 3. Create a system of equations

$$\begin{cases} 1 = A \\ -1 = -2A + B \\ -4 = A - B + D \end{cases}$$

Step 4. Calculate B and D

$$B = -1 + 2 = 1$$

$$D = -4 - 1 + 1 = -4$$

Second part: integration

$$\int \frac{x^2 - x - 4}{(x-1)^3} dx = \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{4}{(x-1)^3} \right) dx =$$

$$= ln|x-1| - (x-1)^{-1} - 4\frac{(x-1)^{-2}}{-2} + C$$

Answer

$$\int \frac{x^2 - x - 4}{(x-1)^3} dx = ln|x-1| - \frac{1}{x-1} + \frac{2}{(x-1)^2} + C$$

4. Computation of improper rational functions

An improper rational function can be expressed as a polynomial plus a proper rational function

$$f(x) = \frac{P_n(x)}{Q_m(x)} = T_{n-m}(x) + \frac{R_k(x)}{Q_m(x)},$$

where polynomial $T_{n-m}(x)$ has degree $n - m$ and $n \geq m;\;\; k < m$. It is necessary to perform *long division of polynomials* to get this result.

Example 4.1.

Compute

$$\int \frac{4x^3 - 12x + 16}{x-2} dx$$

Solution

Step 1. Perform long division of polynomials

$$\begin{array}{r} 4x^2 + 8x + 4 \\ x - 2\overline{)4x^3 - 12x + 16} \\ \underline{-(4x^3 - 8x^2)} \\ 8x^2 - 12x + 16 \\ \underline{-(8x^2 - 16x)} \\ 4x + 16 \\ \underline{-(4x - 8)} \\ 24 \end{array}$$

Step 2. Integrate step by step

$$\int \frac{4x^3 - 12x + 16}{x-2} dx = \int \left(4x^2 + 8x + 4 + \frac{24}{x-2} \right) dx =$$

$$\int 4x^2 dx + \int 8x dx + \int 4 dx + \int \frac{24}{x-2} dx =$$

$$= \frac{4x^3}{3} + \frac{8x^2}{2} + 4x + 24ln|x-2| + C$$

Answer

$$\int \frac{4x^3 - 12x + 16}{x - 2} dx = \frac{4x^3}{3} + 4x^2 + 4x + 24ln|x - 2| + C$$

5. Summary

To evaluate the integral of a rational function, the following steps are recommended

Starting step. Evaluate the given rational function $\int \frac{P_n(x)}{Q_m(x)} dx$

Case 1. The given function is an improper rational function ($n \geq m$)

Step 1.1. Perform long division of polynomials

Step 1.2. Rewrite the integral as an integral of a polynomial plus a proper rational part

Step 1.3. Integrate the polynomial

Step 1.4. For the integral of the proper rational part complete case 2 if necessary or integrate

Case 2. The given function is a proper rational function ($n < m$)

Step 2.1. Factorise the denominator if necessary

Step 2.2. Complete the decomposition of the partial fraction

Step 2.3. Integrate simple rational fractions

Comment. There are two methods how to complete Step 2.2. We can plug in useful values of x into the polynomial equation (see example 3.2.), or we can equate the coefficients of two equal polynomials (see examples 3.3 and 3.4), or we can combine both methods.

Example 5.1.

Compute

$$\int \frac{x^4 + 12x - 6}{x^2(x - 1)} dx$$

Solution

Let us follow the action plan above. We have **case 1** (given function is an improper rational function) and we complete **step 1.1**

$$\begin{array}{r l}
 & \underline{\;x + 1\qquad\qquad} \\
x^3 - x^2 \big) & x^4 + 12x - 6 \\
 & \underline{-(x^4 - x^3)} \\
 & \qquad x^3 + 12x - 6 \\
 & \qquad \underline{-(x^3 - x^2)} \\
 & \qquad\qquad x^2 + 12x - 6
\end{array}$$

Step 1.2.

$$\int \frac{x^4 + 12x - 6}{x^2(x - 1)} dx = \int \left(x + 1 + \frac{x^2 + 12x - 6}{x^2(x - 1)} \right) dx$$

Step 1.3.

$$\int \left(x+1+\frac{x^2+12x-6}{x^2(x-1)}\right)dx=\frac{x^2}{2}+x+\int \frac{x^2+12x-6}{x^2(x-1)}dx$$

Last integral is that of a proper rational function. We complete **step 2.2**.

$$\frac{x^2+12x-6}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

Let us apply the "plug-in" method. Get rid of all the denominators and write the equation

$$x^2+12x-6=Ax(x-1)+B(x-1)+Cx^2$$

We can use the roots of denominator $x=0$ and $x=1$. We choose the constant $x=-1$ additionally.

If $x=0$ we get the equation

$$-6=-B;\quad B=6$$

If $x=1$ we get the equation

$$1+12-6=C;\quad C=7$$

If $x=-1$ we get the equation

$$1-12-6=A(-1)(-2)+B(-2)+C$$

$$A=-6$$

We get simple rational fractions and can integrate these

$$\frac{x^2+12x-6}{x^2(x-1)}=\frac{-6}{x}+\frac{6}{x^2}+\frac{7}{x-1}$$

$$\int \left(\frac{-6}{x}+\frac{6}{x^2}+\frac{7}{x-1}\right)dx=-6ln\,|x|+6\frac{x^{-1}}{-1}+7ln|x-1|+C$$

Answer

$$\int \frac{x^4+12x-6}{x^2(x-1)}dx=\frac{x^2}{2}+x-6ln|x|-\frac{6}{x}+7ln|x-1|+C$$

6. Exercises

Evaluate the following indefinite integrals of rational functions

1. $\int \frac{dx}{1+7x}$
2. $\int \frac{2}{(x-1)(x+2)}dx$
3. $\int \frac{x-1}{x(x-2)(x-3)}dx$

4. $\int \frac{x}{4-x^2}dx$

5. $\int \frac{x+6}{x^2-8x}dx$

6. $\int \frac{x+5}{x^2-4x-12}dx$

7. $\int \frac{2x}{x^2-4x+20}dx$

8. $\int \frac{2x+1}{x^2(x+2)}dx$

9. $\int \frac{3}{x(1+x^2)}dx$

10. $\int \frac{3x}{x+1}dx$

11. $\int \frac{x^3}{x(x+3)}dx$

7. Solution of the exercises

1. $\int \frac{dx}{1+7x}$

Solution

$$\int \frac{dx}{1+7x} = \frac{1}{7}\int \frac{d(7x)}{1+7x} = \frac{1}{7} ln|1+7x| + C$$

2. $\int \frac{2}{(x-1)(x+2)}dx$

Solution

Let us find partial fractions

$$\frac{2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$2 = A(x+2) + B(x-1)$$

$$x = 1; \quad 2 = A \cdot 3; \quad A = \frac{2}{3}$$

$$x = -2; \; 2 = B \cdot (-3); \; B = \frac{-2}{3}$$

$$\frac{2}{(x-1)(x+2)} = \frac{2}{3} \cdot \frac{1}{x-1} - \frac{2}{3} \cdot \frac{1}{x+2}$$

Now we can change the integral

$$\int \frac{2}{(x-1)(x+2)}dx = \frac{2}{3}\int \frac{dx}{x-1} - \frac{2}{3}\int \frac{dx}{x+2} =$$

$$= \frac{2}{3}ln|x-1| - \frac{2}{3}ln|x+2| + C$$

3. $\int \frac{x-1}{x(x-2)(x-3)}dx$

Solution

Let us find partial fractions

$$\frac{x-1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x-1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2)$$

$$x=0;\ \ -1 = A(-2)(-3);\ \ A = -\frac{1}{6}$$

$$x=2;\ \ 1 = B\cdot 2\cdot(-1);\ \ B = -\frac{1}{2}$$

$$x=3;\ \ 2 = C\cdot 3;\ \ C = \frac{2}{3}$$

Let us compute the integral

$$\int \frac{x-1}{x(x-2)(x-3)}dx = -\frac{1}{6}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x-3} =$$

$$= -\frac{1}{6}ln|x| - \frac{1}{2}ln|x-2| + \frac{2}{3}ln|x-3| + C$$

4. $\int \frac{x}{4-x^2}dx$

Solution

Factorise the denominator

$$\frac{x}{4-x^2} = \frac{x}{(2-x)(2+x)}$$

Find partial fractions

$$\frac{x}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$x = A(2+x) + B(2-x)$$

$$x=2;\ \ 2 = A\cdot 4;\ \ A = \frac{1}{2}$$

$$x=-2;\ \ -2 = B\cdot 4;\ \ B = -\frac{1}{2}$$

Compute the integral

$$\int \frac{x}{4-x^2}dx = \frac{1}{2}\int \frac{dx}{2-x} - \frac{1}{2}\int \frac{dx}{2+x} = -\frac{1}{2}ln|2-x| - \frac{1}{2}ln|2+x| + C$$

Comment

$$\int \frac{dx}{2-x} = -\int \frac{dx}{x-2} = -ln|x-2| + C = -ln|2-x| + C$$

5. $\int \frac{x+6}{x^2-8x}dx$

Solution

Factorise the denominator

$$\frac{x+6}{x^2-8x} = \frac{x+6}{x(x-8)}$$

Find partial fractions

$$\frac{x+6}{x(x-8)} = \frac{A}{x} + \frac{B}{x-8}$$

$$x+6 = A(x-8) + Bx$$

$$x = 0;\ \ 6 = A\cdot(-8);\ \ A = -\frac{3}{4}$$

$$x = 8;\ \ 14 = B\cdot 8;\ \ B = \frac{7}{4}$$

Compute the integral

$$\int \frac{x+6}{x^2-8x}dx = -\frac{3}{4}\int \frac{dx}{x} + \frac{7}{4}\int \frac{dx}{x-8} = -\frac{3}{4}ln|x| + \frac{7}{4}ln|x-8| + C$$

6. $\int \frac{x+5}{x^2-4x-12}dx$

Solution

Factorise the denominator

$$\frac{x+5}{x^2-4x-12} = \frac{x+5}{(x-6)(x+2)}$$

Find partial fractions

$$\frac{x+5}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2}$$

$$x+5 = A(x+2) + B(x-6)$$

$$x = 6;\ \ 11 = A\cdot 8;\ \ A = \frac{11}{8}$$

$$x = -2;\;\; 3 = B\cdot(-8);\;\; B = -\frac{3}{8}$$

Compute the integral

$$\int \frac{x+5}{x^2-4x-12}dx = \frac{11}{8}\int\frac{dx}{x-6} - \frac{3}{8}\int\frac{dx}{x+2} =$$

$$= \frac{11}{8}ln|x-6| - \frac{3}{8}ln|x+2| + C$$

7. $\int \frac{2x}{x^2-4x+20}dx$

Solution

Here we cannot factorise the dominator. We will use another approach: substitution

Let us construct two partial fractions

$$\frac{2x}{x^2-4x+20} = \frac{2x-4+4}{x^2-4x+20} = \frac{2x-4}{x^2-4x+20} + \frac{4}{x^2-4x+20}$$

Now we will integrate two integrals

$$\int \frac{2x}{x^2-4x+20}dx = \int \frac{2x-4}{x^2-4x+20}dx + 4\int\frac{dx}{x^2-4x+20} =$$

$$= \left| \begin{matrix} for\ first\ integral & for\ second\ integral \\ let\ \ u = x^2-4x+20; & let\ u = x-2;\ \ x^2-4x+4+16 = u^2+16 \\ then\ du = (2x-4)dx & then\ \ du = dx \end{matrix} \right| =$$

$$= \int\frac{du}{u} + 4\int\frac{du}{u^2+16} = ln|u| + 4\cdot\frac{1}{4}arctan\frac{u}{4} + C =$$

$$= ln|x^2-4x+20| + arctan\frac{x-2}{4} + C$$

8. $\int \frac{2x+1}{x^2(x+2)}dx$

Solution

Let us find partial fractions of integrand

$$\frac{2x+1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$2x+1 = Ax(x+2) + B(x+2) + Cx^2$$

$$x = 0;\qquad 1 = 2B;\quad B = 0.5$$

$$x = -2;\quad -4+1 = 4C;\;\; C = -0.75$$

$$x = -1;\quad -2+1 = -A+B+C;$$

$$-1 = -A - 0.25;\quad A = 0.75$$

We can split the integral in separate parts

$$\int\frac{2x+1}{x^2(x+2)}dx = 0.75\int\frac{dx}{x} + 0.5\int\frac{dx}{x^2} - 0.75\int\frac{dx}{x+2} =$$

$$= 0.75ln|x| - 0.5\frac{1}{x} - 0.75ln|x+2| + C$$

9. $\int \frac{3}{x(1+x^2)}dx$

Solution

We find partial fractions

$$\frac{3}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$3 = A(1+x^2) + Bx^2 + Cx$$

$$x = 0;\;\; 3 = A$$

For other coefficients we create a system of equations

$$\begin{cases} A + B = 0 \\ C = 0 \end{cases};\;\; B = -3$$

We integrate

$$\int \frac{3}{x(1+x^2)}dx = 3\int \frac{dx}{x} - 3\int \frac{xdx}{1+x^2} =$$

$$= 3ln|x| - 1.5\int \frac{2xdx}{1+x^2} = 3ln|x| - 1.5ln(1+x^2) + C$$

10. $\int \frac{3x}{x+1}dx$

Solution

The integrand is an improper rational part. We will transform this rational in the following way

$$\frac{3x}{x+1} = 3\frac{x+1-1}{x+1} = 3 - \frac{3}{x+1}$$

We integrate

$$\int \frac{3x}{x+1}dx = 3\int dx - 3\int \frac{dx}{x+1} = 3x - 3ln|x+1| + C$$

11. $\int \frac{x^3}{x(x+3)}dx$

Solution

We simplify the expression and then perform a long division

$$\frac{x^3}{x(x+3)} = \frac{x^2}{x+3}$$

$$\begin{array}{r} x-3 \\ x+3\overline{)x^2} \\ \underline{-(x^2+3x)} \\ -3x \\ \underline{-(-3x-9)} \\ 9 \end{array}$$

$$\frac{x^2}{x+3} = x - 3 + \frac{9}{x+3}$$

We integrate

$$\int \frac{x^3}{x(x+3)}dx = \int \frac{x^2}{x+3}dx = \int xdx - 3\int dx + 9\int \frac{dx}{x+3}$$

$$= \frac{x^2}{2} - 3x + 9ln|x+3| + C$$