Integration Techniques: Integration of Trigonometric Functions

DETAILED DESCRIPTION:

Various oscillation processes can be described by trigonometric functions. The research of such processes requires the calculation of integrals where integrands are composite functions. Trigonometric identities are useful to modify these integrals. In this chapter we will present the application of trigonometric formulas for more common cases and the appropriate substitution for solving integrals. The method of trigonometric substitution will be introduced additionally.

AIM: To learn the use of trigonometric identities and special cases of substitution for trigonometric integrands.

Learning Outcomes:

1. Students will be able integrate the integrals of trigonometric functions applying some trigonometric identities.
2. Students can apply the trigonometric substitution.

Prior Knowledge: rules of integration and differentiation; substitution methods for integrals; algebra and trigonometry formulas.

Relationship to real maritime problems: trigonometric integrals are useful for describing and for solving different problems on sinusoidal processes – for example, to construct an effective shape of a ship propellers' blades, or to calculate wave resistance for steady motion in ship's control equipment.

Content

1. Composite trigonometric functions of a linear argument
2. Product of sines and cosines
3. Powers of trigonometric functions
4. Double-angle trigonometric identity
5. Trigonometric substitution
6. Exercises
7. Solutions

Integration of Trigonometric Functions

1. Composite trigonometric functions of a linear argument

Some of the simplest cases where integrals involve sine and cosine functions are the following

$$\int sinax\,dx\,; \quad \int cosax\,dx\,; \quad \int tanax\,dx$$

For such cases we can use simple substitution, for instance,

Example 1.1 Compute the integral

$$\int sinax\ dx$$

Solution

$$\int sinax\ dx = \begin{vmatrix} let \quad u = ax \\ then\ du = adx \end{vmatrix} = \frac{1}{a}\int sinu\ du = -\frac{1}{a}cosu + C = -\frac{1}{a}cosax + C$$

A slightly more complicated case is the integrand that is a tangent function

Example 1.2 Compute the integral

$$\int tanaxdx$$

Solution

$$\int tanax\ dx = \int \frac{sinax}{cosax}dx = \begin{vmatrix} let \quad u = cosax \\ then\ du = -a\ sinax\ dx \end{vmatrix} =$$

$$= -\frac{1}{a}\int \frac{du}{u} = -\frac{1}{a}ln|u| + C = -\frac{1}{a}ln|cosax| + C$$

As additional special cases we can add these formulas to the list of basic integral formulas:

$$\boldsymbol{\int sinax\ dx = -\frac{1}{a}cosax + C}$$

$$\boldsymbol{\int cosax\ dx = \frac{1}{a}sinax + C}$$

$$\boldsymbol{\int tanax\ dx = -\frac{1}{a}ln|cosax| + C}$$

$$\boldsymbol{\int cotax\ dx = \frac{1}{a}ln|sinax| + C}$$

2. Product of sines and cosines

Here we will look at the integrals of the product of trigonometric functions that have different arguments, for instance,

$$\int sinax \cdot cosbx\ dx$$

For simplifying integrals of this kind we can apply the product-to-sum identities

$$\boldsymbol{sinax \cdot cosbx = \frac{1}{2}(sin(ax + bx) + sin(ax - bx))}$$

$$\boldsymbol{cosax \cdot cosbx = \frac{1}{2}(cos(ax + bx) + cos(ax - bx))}$$

$$\boldsymbol{sinax \cdot sinbx = \frac{1}{2}(cos(ax - bx) - cos(ax + bx))}$$

The substitution takes place after splitting the integral into two parts.

Example 2.1 Compute the integral

$$\int sin5x \cdot sin2x\, dx$$

Solution

$$\int sin5x \cdot sin2x\, dx = \frac{1}{2}\int (cos3x - cos7x)dx =$$

$$= \frac{1}{2}\int cos3x\, dx - \frac{1}{2}\int cos7x\, dx =$$

$$= \frac{1}{6}sin3x - \frac{1}{14}sin7x + C$$

3. Powers of trigonometric functions

In this section we take consider the integrals which include the integer powers of sines and cosines. In general form it is written as

$$\int sin^n x \cdot cos^m x\, dx$$

If at least one of the powers is an odd number, we can apply substitution.

Example 3.1 Compute the integral

$$\int cos^5 x \cdot sin^3 x\, dx$$

Let us notice that both functions have odd powers and the cosine function has bigger power than the sine function. Therefore, we will substitute the cosine, but first we split the power of the sine into multipliers and then use the trigonometric identity

$$\boldsymbol{sin^2 x + cos^2 x = 1}$$

Solution

$$\int cos^5 x \cdot sin^3 x\, dx = \int cos^5 x \cdot sin^2 x \cdot sinx\, dx =$$

$$= \int cos^5 x \cdot (1 - cos^2 x) \cdot sinx\, dx = \begin{vmatrix} let\ \ u = cosx \\ du = -sinx\, dx \end{vmatrix} =$$

$$= -\int u^5(1 - u^2)du = -\int u^5 du + \int u^7 du =$$

$$= -\frac{u^6}{6} + \frac{u^8}{8} + C = -\frac{cos^6 x}{6} + \frac{cos^8 x}{8} + C$$

Example 3.2 Compute the integral

$$\int \frac{cotx}{sin^3x}dx$$

Solution

$$\int \frac{cotx}{sin^3x}dx = \int \frac{cosx}{sinx \cdot sin^3x}dx = \int \frac{cosx}{sin^4x}dx =$$

$$= \begin{vmatrix} let\ \ u = sinx \\ du = cosxdx \end{vmatrix} = \int \frac{du}{u^4} = \int u^{-4}du =$$

$$= \frac{u^{-3}}{-3} + C = \frac{sin^{-3}x}{-3} + C = -\frac{1}{3sin^3x} + C$$

4. Double-angle trigonometric identities

In the previous section we discussed the methods of integration of sine and cosine functions on the integer powers where at least one of the powers is an odd integer. For even cases we can apply double-angle identities to eliminate the powers

$$\boldsymbol{sin^2x = \frac{1 - cos2x}{2}}$$

$$\boldsymbol{cos^2x = \frac{1 + cos2x}{2}}$$

Example 4.1 Compute the integral

$$\int cos^2x\,dx$$

Solution

$$\int cos^2x\,dx = \int \frac{1 + cos2x}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int cos2x\,dx =$$

$$= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}sin2x + C$$

Example 4.2 Compute the integral

$$\int sin^4x\,dx$$

Solution

$$\int sin^4x\,dx = \int (sin^2x)^2dx = \int \left(\frac{1 - cos2x}{2}\right)^2 dx =$$

$$= \frac{1}{4}\int (1 - 2cos2x + cos^2 2x)dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int 2cos2x\,dx + \frac{1}{4}\int \frac{1 + cos4x}{2}dx =$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int cos2x\ d2x + \frac{1}{4}\cdot\frac{1}{2}\int dx + \frac{1}{4}\cdot\frac{1}{2}\int cos4x\ dx =$$

$$= \frac{1}{4}x - \frac{1}{4}sin2x + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}sin4x + C =$$

$$= \frac{3}{8}x - \frac{1}{4}sin2x + \frac{1}{32}sin4x + C$$

Notice that in the solution of example 4.2 the double-angle formula is applied repeatedly.

5. Trigonometric substitution

Here we shall look at more complex integrals where an integrand contains the square root of a quadratic expression, for instance,

$$\int x\sqrt{4-x^2}dx; \quad \int \frac{\sqrt{x^2-25}}{x^3}dx$$

Trigonometric substitution is useful to simplify the integrands. The substitution method is based on the trigonometric identity

$$\boldsymbol{sin^2x + cos^2x = 1}$$

Dividing the identity by cos^2x we derive a special case

$$\boldsymbol{tan^2x + 1 = \frac{1}{cos^2x}}$$

We apply the-above mentioned identities for the following cases

Case 1. For $\sqrt{a^2-x^2}$ we substitute $x = asinu \quad or \quad x = acosu$

Case 2. For $\sqrt{a^2+x^2}$ we substitute $x = atanu$

Case 3. For $\sqrt{x^2-a^2}$ we substitute $x = \frac{a}{cosu}$

Example 5.1 Compute the integral

$$\int x\sqrt{4-x^2}dx$$

Solution

To solve the integral, we use trigonometric substitution and then we change the differential to get the integral of the power function.

$$\int x\sqrt{4-x^2}dx = \left| \begin{matrix} let\ \ x = 2sinu,\ \ then\ \ dx = 2cosu\ du \\ 4-x^2 = 4-4sin^2u = 4cos^2u \end{matrix} \right| =$$

$$= \int 2sinu\sqrt{4cos^2u}\ 2cosu\ \ du = 8\int cos^2u\ \ sinu\ du =$$

$$= -8\int cos^2u\, d(cosu) = -8\frac{cos^3u}{3} + C$$

Now we return to the function with respect to the argument x

$$cos^3u = cos^2u\cdot cosu = (1-sin^2u)\sqrt{1-sin^2u} =$$

$$=\left| as\ \ x=2sinu\ follows\ \ 1-sin^2u=1-\frac{x^2}{4} \right| =$$

$$=\left(1-\frac{x^2}{4}\right)\sqrt{1-\frac{x^2}{4}}$$

The solution of the integral is

$$=-8\frac{cos^3u}{3}+C=-\frac{8}{3}\left(\sqrt{1-\frac{x^2}{4}}\right)^3+C=-\frac{\left(\sqrt{4-x^2}\right)^3}{3}+C$$

Example 5.2 Compute the integral

$$\int\frac{\sqrt{x^2-25}}{x^3}dx$$

Solution

$$\int\frac{\sqrt{x^2-25}}{x^3}dx=\left| \begin{matrix} let\ \ x=\dfrac{5}{cosu}\ \ then\ \ dx=\dfrac{2sinu}{cos^2u}du \\ 25-x^2=25tan^2u \end{matrix} \right| =$$

$$=\int\frac{\sqrt{25tan^2u}\cdot cos^3u}{125}\frac{2sinu}{cos^2u}du\ \ =\frac{2}{25}\int tanu\cdot cosu\cdot sinu\ du=$$

$$=\frac{2}{25}\int sin^2u\ du=\frac{1}{25}\int(1-cos2u)du=\frac{1}{25}\left(u-\frac{1}{2}sin2u\right)+C=$$

$$=\left| as\ \ cosu=\frac{5}{x}\ \ we\ get\ sin2u=2\cdot\frac{5}{x}\sqrt{1-\frac{25}{x^2}}=\frac{10}{x^2}\sqrt{x^2-25} \right| =$$

$$=\frac{1}{25}\left(arccos\frac{5}{x}-\frac{5}{x^2}\sqrt{x^2-25}\right)+C$$

The following formula is applied to expand the function $sin2u$

$$\boldsymbol{sin2x=2sinxcosx}$$

6. Exercises

Compute the integrals

1. $\int sin12x\ dx$
2. $\int cot\frac{3x}{4}dx$
3. $\int sin5x\cdot cos4.5x\ dx$
4. $\int sin^{11}x\cdot cosx\ dx$

5. $\int \frac{cos^3x}{sin^5x}dx$

6. $\int 8(1-cos^2x)dx$

7. Apply trigonometric substitution

$$\int \sqrt{1-x^2}\,dx$$

8. Solutions

1. $\int sin12x\,dx$

Solution

$$\int sin12x\,dx = \frac{1}{12}\int sin12x\,d12x = -\frac{1}{12}cos12x + C$$

2. $\int cot\frac{3x}{4}dx$

Solution

$$\int cot\frac{3x}{4}dx = \int \frac{cos\frac{3x}{4}}{sin\frac{3x}{4}}dx = \left| \begin{matrix} u = sin\frac{3x}{4} \\ du = \frac{3}{4}cos\frac{3x}{4}dx \end{matrix} \right| =$$

$$= \frac{4}{3}\int \frac{du}{u} = \frac{4}{3}ln|u| + C = \frac{4}{3}ln\left|sin\frac{3x}{4}\right| + C$$

3. $\int sin5x \cdot cos4.5x\,dx$

Solution

$$\int sin5x \cdot cos4.5x\,dx = \frac{1}{2}\int (sin9.5x + sin0.5x)dx =$$

$$= \frac{1}{2}\cdot\frac{2}{19}\int sin9.5x\,d9.5x + \frac{1}{2}\cdot 2\int sin0.5x\,d0.5x =$$

$$= -\frac{1}{19}cos9.5x - cos0.5x + C$$

4. $\int sin^{11}x \cdot cosx\,dx$

Solution

$$\int sin^{11}x \cdot cosx\,dx = \left| \begin{matrix} u = sinx \\ du = cosx\,dx \end{matrix} \right| = \int u^{11}du =$$

$$= \frac{u^{12}}{12} + C = \frac{sin^{12}x}{12} + C$$

5. $\int \frac{cos^3x}{sin^5x}dx$

Solution

$$\int \frac{cos^3x}{sin^5x}dx = \int \frac{(1-sin^2x)cosx}{sin^5x}dx = \begin{vmatrix} u = sinx \\ du = cosx\,dx \end{vmatrix} =$$

$$= \int \frac{1-u^2}{u^5}du = \int u^{-5}\,du - \int u^{-3}\,du =$$

$$= \frac{u^{-4}}{-4} - \frac{u^{-2}}{-2} + C = -\frac{1}{4sin^4x} + \frac{1}{2sin^2x} + C$$

6. $\int 8(1-cos^2x)dx$

Solution

$$\int 8(1-cos^2x)dx = 8\int sin^2x\;dx = 4\int (1-cos2x)\,dx =$$

$$= 4\int dx - 2\int cos2x\,d2x = 4x - 2sin2x + C$$

7. Apply trigonometric substitution

$$\int \sqrt{1-x^2}\,dx$$

Solution

$$\int \sqrt{1-x^2}\,dx = \begin{vmatrix} let\;\; x = sinu \\ dx = cosu\,du \end{vmatrix} = \int \sqrt{1-sin^2u}\,cosu\,du =$$

$$= \int cosu \cdot cosu\,du = \int \frac{1+cos2u}{2}\,du = \frac{1}{2}\int du + \frac{1}{4}\int cos2u\,d2u =$$

$$= \frac{1}{2}u + \frac{1}{4}sin2u + C = \begin{vmatrix} u = arcsinx \\ sin2u = 2sinucosu = 2x\sqrt{1-x^2} \end{vmatrix} =$$

$$= \frac{1}{2}arcsinx + \frac{1}{2}x\sqrt{1-x^2} + C$$