Definite integral

DETAILED DESCRIPTION:

This chapter introduces the definite integral. The description starts with the question: how to calculate the area of a region that is bounded by a curve and straight lines? The method of approximate calculation is discussed. Then this method is generalised and the definition of the definite integral is formulated. Basic properties and the Newton-Leibniz formula are presented for calculations of integrals. For students, we can recommend the "Integral Calculator" https://www.integral-calculator.com/. This software shows the solution of an integral step by step and comes with an interactive graph of integrand and antiderivative. With a help of such software students can check their individual solutions of the task. The graphs can be constructed using free software GeoGebra Classic; DESMOS Graphing Calculator; Microsoft Excel, and others.

The books recommended (in Latvian)

1. K.Šteiners, B.Siliņa. Augstākā matemātika. III, IV daļa. Rīga, Zvaigzne ABC, 1998
2. B.Siliņa, K.Šteiners. Rokasgrāmata augstākajā matemātikā. Rīga, Zvaigzne ABC, 2006
3. E.Kronbergs, P.Rivža, Dz.Bože. Augstākā matemātika. I, II daļa, Rīga, Zvaigzne, 1988
4. V.Liepiņa. Matemātika. Integrālrēķini un diferenciālvienādojumi. Metodiskais līdzeklis un individuālie aprēķinu darbi, II daļa, Rīga, LJA, 2010

AIM: Learn the definition of the definite integral as the limit of a sum.

Learning Outcomes:

1. Understand the meaning of the definite integral
2. Understand and apply the rules for calculating definite integrals

Prior Knowledge: basic rules of integration and differentiation; knowledge of the properties of elementary functions and their graphs; algebra and trigonometry formulas.

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculate the area of different shapes, volumes of solids, and to solve other geometric problems. Definite integrals are used for various calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in theory of cargo transport, in economics, in classical signal theory, and in other specialities.

Content

Definite Integrals

1. Statement of area problem

From the results of Euclidean geometry, we know how to calculate the area of rectangle, triangle, circle and other simple plane figures. If the polygon is given, its area can be found by subdividing the polygon into a finite number of non-overlapping triangles. It is a different case we have if we need to calculate the area of a region enclosed by an arbitrary curve.

The example that we will solve is about calculating the area of a region that is placed in the Cartesian coordinate system.

Example 1.1 Find the area of the region enclosed by the parabola $y = x^2$, two vertical straight lines $x = 0$ and $x = 2.4$, and x-axis $y = 0$.

Solution

We can calculate the area of the region approximately. Let us subdivide the interval $[0, 2.4]$ into parts of equal length. For any such subinterval we construct a rectangle whose height is equal to the value of the given function at the endpoint of the chosen subinterval:

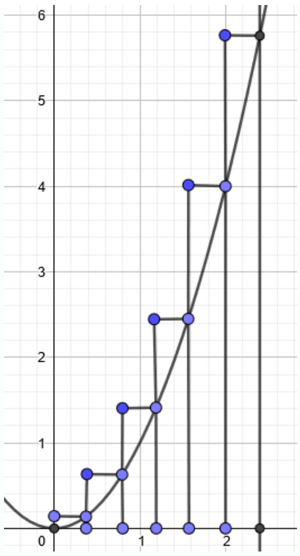

Figure 1.1

The interval $[0;\ 2.4]$ is 2.4 units long (see figure 1.1). Every subinterval is 0.4 units long. We can calculate the area of all rectangles

$$0.4 \cdot 0.4^2 + 0.4 \cdot 0.8^2 + 0.4 \cdot 1.2^2 + 0.4 \cdot 1.6^2 + 0.4 \cdot 2^2 + 0.4 \cdot 2.4^2 = 5.824$$

Figure 1.1 shows that the calculated area of rectangles is larger than the area of the region that we need to find. Subdividing the interval $[0, 2.4]$ into shorter subintervals will make the error of calculations smaller. If the length of the subinterval is 0.01, we get

$$0.01(0.01^2 + 0.02^2 + 0.03^2 + \cdots + 2.4^2) \approx 463.684$$

It is useful to do this very long summation of 261 addends by a software program, for instance, by Microsoft Excel. Thus, we can subdivide the interval $[0, 2.4]$ into more and more detail, thus reducing the error. Anyway, we need to find the correct answer.

2. Definition of the definite integral

An arbitrary continuous function $y = f(x)$ is defined on the closed interval $[a, b]$. The task is to calculate area *S* of the region between the curve determined by the function, x-axis, and two vertical straight lines $x = a, \;\; x = b$ (see figure 2.1).

We choose a set of points inside the interval $[a, b]$, say

$$\{x_1 = a, x_2, x_3, \dots, x_n = b\}, \text{ where}$$

$$x_1 < \; x_2 < x_3 < \cdots < x_n$$

We call this set of points a ***partition of interval*** $[\boldsymbol{a}, \boldsymbol{b}]$ into *n* subintervals $[x_{i-1}, x_i]$, where $i = 1,2,3, \dots, n$. The length of any subinterval is

$$\Delta x_i = x_i - x_{i-1}$$

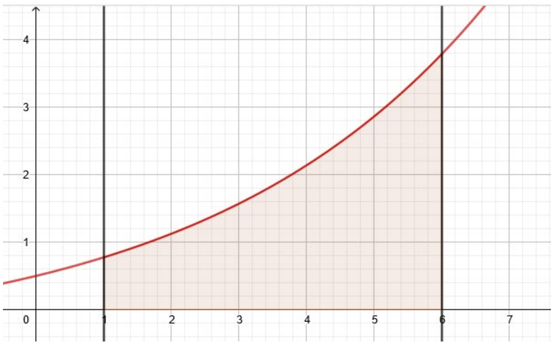

Figure 2.1

We chose an inner point c_i in every interval $c_i \in [x_{i-1}, x_i], where\ i = 1, 2, 3, \dots, n$ and construct the rectangle with side length Δx_i and $f(c_i)$ (see figure 2.2).

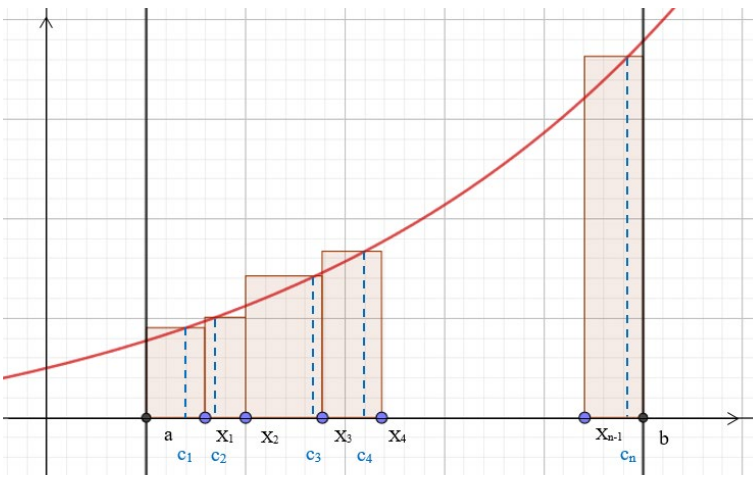

Figure 2.2

Now we compute the area of every rectangle and calculate their sum. The sum of all areas can be written using a sigma notation

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

The sum expresses the approximate value of the area under the curve determined by the function $f(x)$ over the interval $[a, b]$. If we make the partition of this interval into smaller and smaller parts, so that the length of the longest subinterval tends to zero

$$max\Delta x_i \to 0,$$

then the difference between the sum and the area S of the given region will decrease. Taking the limit, we can calculate the precise value of area S

$$S = \lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Definition. If the limit S exists at any partition of the interval $[a, b]$ and any selection of inner points c_i, then we call the limit the ***definite integral*** of the function $f(x)$ on the interval $[a, b]$. The definite integral is denoted by the symbol

$$\boldsymbol{\lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx}$$

If the limit exists, we say that the function $f(x)$ is ***integrable*** on interval $[a, b]$.

The sign ∫ is called the ***integral sign***, it resembles the letter S since it represents the limit of the sum.

Numbers a and b are called the ***limits of integration***, a is the ***lower limit***, and b is the ***upper limit***.

The function $f(x)$ is called ***integrand***; x is the variable of integration.

dx is the ***differential of*** $\boldsymbol{x}$.

The variable x can be replaced with any other variable without changing the value of the integral

$$\boldsymbol{\int_a^b f(x)dx = \int_a^b f(t)dt}$$

Example 2.1 Find the area of the region bounded by $f(x) = 1, x = a, x = b, y = 0$.

Solution

The area of the given region can be calculated by the integral

$$\int_a^b 1 \cdot dx$$

Described region is bounded by straight lines that define a rectangle (see figure 2.3).

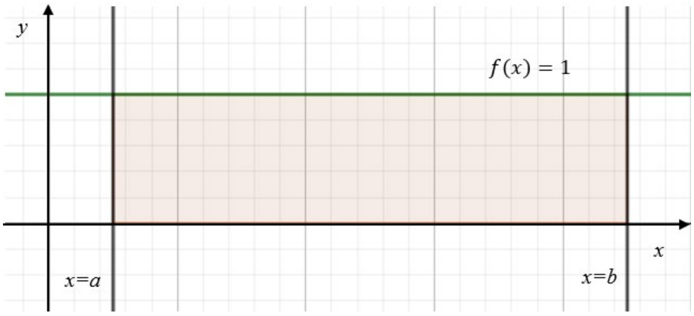

Figure 2.3

The area of this rectangle is equal to the value of the integral

$$\int_a^b dx = (b-a)\cdot 1 = b-a$$

3. Properties of the definite integral

Some of the most important properties of definite integrals are included in the following list. Most of the properties listed below can be directly deduced from the definition of the definite integral.

Let functions $f(x)$ and $g(x)$ be continuous and differentiable on the interval $[a,b]$, then

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c is a particular constant.

4. $\int_a^b (cf(x)+kg(x))\,dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx$, where c and k are particular constants.

5. If $f(x) \le g(x)$ for all arguments $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

6. For all arguments $x \in [a,b]$ is true $\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$

7. If $f(x)$ is an odd function ($f(-x) = -f(x)$), then $\int_{-a}^a f(x)\,dx = 0$

8. If $f(x)$ is an even function ($f(-x) = f(x)$), then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

Other properties present methods of evaluation of the definite integral.

9. $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$, where m is the minimum value of the function $f(x)$ in the interval $[a,b]$, M is the maximum value in the interval $[a,b]$.

The 10^{th} property is called ***the Mean value theorem***.

10. $\int_a^b f(x)dx = f(c)(b-a)$, where $c \in [a,b]$.

By applying the Mean value theorem, we can calculate the ***average value*** of an integrable function $f(x)$ on the interval $[a,b]$:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

4. Calculation of the definite integral

The Newton–Leibniz formula. If the function $f(x)$ is continuous on the interval $[a,b]$ and the function $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

How to apply this formula? First, we need to compute the corresponding indefinite integral

$$\int f(x)dx = F(x) + C,$$

and then calculate the values of the antiderivative at the upper limit of integral and at its lower limit, and subtract them

$$F(b) + C - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$$

Calculation shows that we can omit the constant *C* of integration. Therefore, we will expand the Newton – Leibniz formula with an evaluation symbol (vertical line segment)

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Now we can precisely calculate the value of the region defined in example 1.1. We calculate the integral

$$\int_0^{2.4} x^2 dx = \frac{x^3}{3}\Big|_0^{2.4} = \frac{2.4^3}{3} - 0 = 4.608$$

Example 4.1

Application of the Newton – Leibniz formula

$$\int_1^3 2x dx = x^2\Big|_1^3 = 3^2 - 1 = 8$$

Example 4.2 Compute the integral

$$\int_1^4 (3+\sqrt{x})\,dx$$

Solution

Here we use property 4 of definite integrals in the solution

$$\int_1^4 (3+\sqrt{x})\,dx = \int_1^4 3\,dx + \int_1^4 \sqrt{x}\,dx = \left(3x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^4 =$$

$$= 3(4-1) + \frac{2}{3}\left(\sqrt{4^3}-1\right) = 9 + \frac{2}{3}\cdot 7 = 13\frac{2}{3}$$

Example 4.3 Compute the integral

$$\int_0^{\frac{\pi}{2}} \frac{1-cos2\varphi}{2}\,d\varphi$$

Solution

To solve this integral we change the differential

$$\int_0^{\frac{\pi}{2}} \frac{1-cos2\varphi}{2}\,d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi - \frac{1}{4}\int_0^{\frac{\pi}{2}} cos2\varphi\; d(2\varphi) = \frac{\varphi}{2} - \frac{1}{4}sin2\varphi\Bigg|_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{4} - \frac{1}{4}sin\pi - (0 - sin0) = \frac{\pi}{4}$$

Example 4.4 Compute the integral

$$\int_{-1}^1 x^2\,dx + \int_1^2 \sqrt[3]{x^4}\cdot x^{\frac{2}{3}}\,dx$$

Solution

We apply property 3 to simplify the task

$$\int_{-1}^1 x^2\,dx + \int_1^2 \sqrt[3]{x^4}\cdot x^{\frac{2}{3}}\,dx = \int_{-1}^1 x^2\,dx + \int_1^2 x^2\,dx =$$

$$= \int_{-1}^2 x^2\,dx = \frac{x^3}{3}\Bigg|_{-1}^2 = \frac{8}{3} + \frac{1}{3} = 3$$

Example 4.5

$$\int_{-\pi}^{\pi} sin^5 2x\,dx = 0,$$

while the integrand is an odd function (see property 7).

Example 4.5 Calculate the average value of the function $f(x) = tan^2 x$ over the interval $\left[0, \frac{\pi}{4}\right]$.

Solution

We apply the mean value theorem

$$\frac{1}{b-a}\int_a^b f(x)dx$$

The length of interval is

$$b - a = \frac{\pi}{4}$$

We calculate the average value of the function

$$\frac{1}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} tan^2 x\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{sin^2 x}{cos^2 x}dx = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{1 - cos^2 x}{cos^2 x}dx =$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \frac{1}{cos^2 x}dx - \frac{4}{\pi}\int_0^{\frac{\pi}{4}} 1\,dx = \frac{4}{\pi}(tanx - x)\Big|_0^{\frac{\pi}{4}} = \frac{4}{\pi}\left(1 - \frac{\pi}{4}\right) = \frac{4}{\pi} - 1$$

5. Exercises

Compute the integrals

1. $\int_0^2 (x^4 - x^3)\,dx$

2. $\int_{-\frac{\pi}{4}}^{0} sinx\,dx$

3. $\int_{-1}^{1} \frac{dt}{\sqrt{4 - t^2}}$

4. $\int_{0}^{\frac{1}{3}} \frac{2^{3x}}{7} dx$

5. $\int_{-2}^{7} \frac{2dx}{x+3}$

6. Solutions

1. $\int_{0}^{2} (x^4 - x^3)\, dx$

Solution

$$\int_{0}^{2} (x^4 - x^3)\, dx = \frac{x^5}{5} - \frac{x^4}{4}\Big|_{0}^{2} = \frac{32}{5} - \frac{16}{4} = 2.4$$

2. $\int_{-\frac{\pi}{4}}^{0} sinx\, dx$

Solution

$$\int_{-\frac{\pi}{4}}^{0} sinxdx = -cosx\Big|_{-\frac{\pi}{4}}^{0} = -cos0 + cos\left(-\frac{\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2}$$

3. $\int_{-1}^{1} \frac{dt}{\sqrt{4-t^2}}$

Solution

$$\int_{-1}^{1} \frac{dt}{\sqrt{4-t^2}} = arcsin\frac{t}{2}\Big|_{-1}^{1} = arcsin\frac{1}{2} - arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

We calculate the value of the integral according to the odd property of sine function.

4. $\int_{0}^{\frac{1}{3}} \frac{2^{3x}}{7} dx$

Solution

$$\int_{0}^{\frac{1}{3}} \frac{2^{3x}}{7} dx = \frac{1}{7} \cdot \frac{1}{3} \int_{0}^{\frac{1}{3}} 2^{3x} d(3x) = \frac{1}{27} \cdot \frac{2^{3x}}{ln2}\Big|_{0}^{\frac{1}{3}} = \frac{2}{27ln2} - \frac{1}{27ln2} = \frac{1}{27ln2}$$

Here is another calculation method using the properties of powers and logarithms

$$\int_0^{\frac{1}{3}} \frac{2^{3x}}{7} dx = \frac{1}{7}\int_0^{\frac{1}{3}} 8^x dx = \frac{1}{7}\cdot\frac{8^x}{ln8}\Bigg|_0^{\frac{1}{3}} = \frac{1}{7}\cdot\frac{\sqrt[3]{8}}{ln8} - \frac{1}{7}\cdot\frac{1}{ln8} = \frac{2}{7\cdot 3ln2} - \frac{1}{7\cdot 3ln2} = \frac{1}{27ln2}$$

5. $\int_{-2}^{7} \frac{2dx}{x+3}$

Solution

$$\int_{-2}^{7} \frac{2dx}{x+3} = 2\int_{-2}^{7} \frac{d(x+3)}{x+3} = 2ln|x+3|\Bigg|_{-2}^{7} = 2ln10 - 2ln1 = 2ln10$$