Some Methods for Calculation of the Definite Integral

DETAILED DESCRIPTION:
The basic rules for calculation of the definite integral were discussed in the previous section. In this section we present methods of integration of composite functions and the method of integration by parts for the definite integral.

AIM: to introduce certain methods of calculation of the definite integral if the integrand is non-trivial.

Learning Outcomes:
Students can evaluate different definite integrals using various integration methods

Prior Knowledge: basic rules of integration and differentiation; methods of integration of indefinite integrals; the Newton-Leibniz formula.

Relationship to real maritime problems: Definite integrals have a wide range of applications. With the help of definite integrals, it is possible to calculated the area of various shapes; the volumes of solids, and to solve other geometric problems. Definite integrals are used for different calculations of constructions in shipbuilding. Integrals are applied in the theory of stability, in electrical engineering, in the theory of cargo transport, in economics, in classical signal theory, and in other specialities.

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2. Substitution method for the definite integral
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Some Methods for Calculation the Definite Integral

1. Integration by parts

Suppose that we have an integral that can be integrated by parts. We can find the antiderivative part by part according the formula for indefinite integrals. Let us recall it

$$\int udv = uv - \int vdu$$

Following the method for integration of the definite integral, we need to find the antiderivative and to apply the Newton-Leibniz formula as follows

$$\int_a^b udv = \left(uv - \int vdu\right)\Big|_a^b$$

or

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

Example 1.1 Find the integral

$$\int_0^2 xe^x dx$$

Solution

$$\int_0^2 xe^x dx = \left| \begin{matrix} let\ \ u = x, & dv = e^x dx \\ du = dx, & v = e^x \end{matrix} \right| = xe^x\Big|_0^2 - \int_0^2 e^x dx =$$

$$= 2 \cdot e^2 - 0 - e^x\Big|_0^2 = 2e^2 - e^2 + e^0 = e^2 + 1$$

Example 1.2 Find the integral

$$\int_4^{e^2} \frac{lnx}{\sqrt{x}} dx$$

Solution

$$\int_4^{e^2} \frac{lnx}{\sqrt{x}} dx = \left| \begin{matrix} let\ \ u = lnx, & dv = \dfrac{dx}{\sqrt{x}} \\ du = \dfrac{1}{x} dx, & v = 2\sqrt{x} \end{matrix} \right| = 2\sqrt{x} lnx\Big|_4^{e^2} - 2\int_4^{e^2} \frac{\sqrt{x}}{x} dx =$$

$$= 2\sqrt{e^2} ln(e^2) - 2 \cdot 2ln4 - 2 \cdot 2\sqrt{x}\Big|_4^{e^2} = 4e - 4ln4 - 4\sqrt{e^2} + 8 = 8 - 4ln4$$

2. Substitution method for the definite integral

Let us recall the substitution method for indefinite integrals. If the integrand contains a composite function multiplied by the derivative of its argument, we can simplify the notation of the integral by introducing a new argument

$$\int f(g(x))g'(x)dx = \int f(u)du$$

The indefinite integral represents a set of antiderivatives $F(x) + C$. The value of a definite integral is numerical. Since the definite integral has limits of integration defined by the interval $[a, b]$ with respect to the argument x, the introduced argument u belongs to a new interval $[\alpha, \beta]$. The case is expressed precisely by the following theorem:

Theorem 2.1. Suppose that the function $g(x)$ is a differentiable function on the interval $[a,b]$, and satisfies $g(a)=\alpha$ and $g(b)=\beta$. Suppose that the function $f(x)$ is continuous on the range of $g(x)$. Then by performing the substitution $u=g(x)$ it follows

$$\int_a^b f(g(x))g'(x)dx=\int_\alpha^\beta f(u)du$$

Example 2.1 Find the integral

$$\int_0^{\frac{\pi}{2}} cosx e^{sinx}\, dx$$

Solution

$$\int_0^{\frac{\pi}{2}} cosx e^{sinx}\, dx=\left|\begin{array}{c} let\ \ u=sinx,\ \ du=cosx\ dx \\ u_1=sin0=0,\qquad u_2=sin\frac{\pi}{2}=1\end{array}\right|=$$

$$=\int_0^1 e^u du=e^u\Big|_0^1=e-1$$

Example 2.2 Find the integral

$$\int_{\frac{1}{2}}^{1}(2-8x^5)^2x^4dx$$

Solution

$$\int_{\frac{1}{2}}^{1}(2-8x^5)^2x^4dx=\left|\begin{array}{c} let\ \ u=2-8x^5,\ \ du=-40x^4dx \\ u_1=1.75,\ \ u_2=-6\end{array}\right|=$$

$$=-\frac{1}{40}\int_{1.75}^{-6}u^2du=\frac{1}{40}\int_{-6}^{1.75}u^2du=\frac{1}{40}\cdot\frac{u^3}{3}\Big|_{-6}^{1.75}=\frac{1}{120}\left(\frac{343}{64}-\left(\frac{-1}{216}\right)\right)\approx 0.045$$

Solving this integral we reversed the limits of integration.

Example 2.3 Find the integral

$$\int_1^2\frac{2t^3}{t^2+1}dt$$

Solution

$$\int_1^2\frac{2t^3}{t^2+1}dt=\int_1^2\frac{2t\cdot t^2}{t^2+1}dt=\left|\begin{array}{c} let\ \ u=t^2+1,\ \ du=2tdt \\ u_1=1+1=2,\ \ u_2=2^2+1=5 \\ t^2=u-1\end{array}\right|=$$

$$= \int_{2}^{5} \frac{u-1}{u} du = \int_{2}^{5} du - \int_{2}^{5} \frac{du}{u} = (u - lnu)\Big|_{2}^{5} =$$

$$= 5 - ln5 - 2 + ln2 = 3 - ln2.5$$

3. Exercises

Find the integrals

1. $\int_{0}^{2} (x^2 + 1)x\, dx$

2. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cot\varphi\, d\varphi$

3. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5^{arcsinx} dx}{\sqrt{1-x^2}}$

4. $\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75 + 2t^4} dx$

5. $\int_{-1}^{3} \frac{4dx}{\sqrt{x}(x+1)}$

4. Solutions

1. $\int_{0}^{2} (x^2 + 1)x\, dx$

Solution

$$\int_{0}^{2} (x^2 + 1)x\, dx = \left| \begin{matrix} let\ \ u = x^2 + 1, & du = 2xdx \\ u_1 = 1, & u_2 = 5 \end{matrix} \right| = \frac{1}{2} \int_{1}^{5} udu = \frac{u^2}{4}\Big|_{1}^{5} = \frac{25}{4} - \frac{1}{4} = 6$$

2. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cot\varphi\, d\varphi$

Solution

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cot\varphi d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{cos\varphi}{sin\varphi} d\varphi = \left| \begin{array}{c} let\ \ u = sin\varphi,\ \ du = cos\varphi d\varphi \\ u_1 = \frac{\sqrt{2}}{2},\ \ u_2 = \frac{\sqrt{3}}{2} \end{array} \right| =$$

$$= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{du}{u} = lnu \Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = ln\frac{\sqrt{3}}{2} - ln\frac{\sqrt{2}}{2} = 0.5ln1.5$$

3. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5^{arcsinx} dx}{\sqrt{1-x^2}}$

Solution

$$\int_{-\frac{1}{2}}^{0} \frac{5^{arcsinx} dx}{\sqrt{1-x^2}} = \left| \begin{array}{c} let\ \ u = arcsinx,\ \ du = \frac{dx}{\sqrt{1-x^2}} \\ u_1 = arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6},\ \ \ u_2 = 0 \end{array} \right| = \int_{-\frac{\pi}{6}}^{0} 5^u du =$$

$$= \frac{5^u}{ln5}\Big|_{-\frac{\pi}{6}}^{0} = \frac{1}{ln5} - \frac{5^{-\frac{\pi}{6}}}{ln5}$$

4. $\displaystyle\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75+2t^4} dx$

Solution

$$\int_{0}^{\frac{1}{2}} \frac{t^3}{0.75-16\,t^4} dt = \left| \begin{array}{c} let\ \ u = 0.75-16t^4,\ \ du = -64t^3 dt \\ u_1 = 0.75,\ \ u_2 = -0.25 \end{array} \right| = -\frac{1}{64} \int_{0.75}^{-0.25} \frac{du}{u} =$$

$$= \frac{1}{64} \int_{-0.25}^{0.75} \frac{du}{u} = \frac{1}{64} lnu \Big|_{-0.25}^{0.75} = \frac{1}{64}(ln0.75 - ln|-0.25|) =$$

$$= \frac{1}{64}(ln0.75 - ln0.25) = \frac{1}{64} ln\frac{0.75}{0.25} = \frac{ln3}{64}$$

5. $\displaystyle\int_{-1}^{3} \frac{4dx}{\sqrt{x}(x+1)}$

Solution

$$\int_1^3 \frac{4dx}{\sqrt{x}(x+1)} = \left| \begin{array}{c} let \quad u = \sqrt{x}, \quad du = \dfrac{dx}{2\sqrt{x}} \\ u_1 = 1, \;\; u_2 = \sqrt{3} \\ x = u^2 \end{array} \right| = 4 \cdot 2 \int_1^{\sqrt{3}} \frac{du}{1+u^2} = 8 arctan\sqrt{x} \Big|_1^{\sqrt{3}} =$$

$$= 8\left(arctan\sqrt{3} - arctan1\right) = 8\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{2\pi}{3}$$