Improper Integrals

DETAILED DESCRIPTION:
In this section, we will extend the concept of the definite integral. Special integrals are investigated over infinite intervals. We will introduce two types of improper integrals: integrals with infinite limits and integrals with an infinite discontinuity in the region of integration. Such integrals are defined using the notion of the limit. The ways of calculation of improper integrals are presented. Examples of convergent and divergent improper integrals are discussed.

The software GeoGebra, DESMOS, Excel are recommended for construction of graphs. To check their solutions and for deeper understanding, students can use *Definite and Improper Integral Calculator* (https://www.emathhelp.net/calculators/calculus-2/definite-integral-calculator/)

AIM: to learn about improper integrals and methods of their evaluation, to understand the concepts of convergence and divergence of integrals.

Learning Outcomes:

1. Acquire the methods of evaluation of improper integrals of type I.
2. Distinguish the improper integrals of type II and acquire the methods of their calculation.

Prior Knowledge: definite integrals; limits; detection of the domain of function; elementary functions and their graphs.

Relationship to real maritime problems: By describing the shape of the hull of a ship mathematically, it is possible to research the ship's wave resistance that can be presented by an improper integral. Improper integrals are used to express the electrical potential of a given field. A probability density function for a continuous random variable can be described by an improper integral.

Content

1. Improper integrals with infinite upper limit
2. Improper integrals with an infinite discontinuity in the region of integration
3. Exercises
4. Solutions

Improper Integrals

1. Improper integrals with infinite upper limit

In the previous sections we got acquainted with the definite integral where integrand is defined on the closed interval. Let us investigate a continuous function over a left-bounded interval $[a, \infty)$ (see figure 1.1).

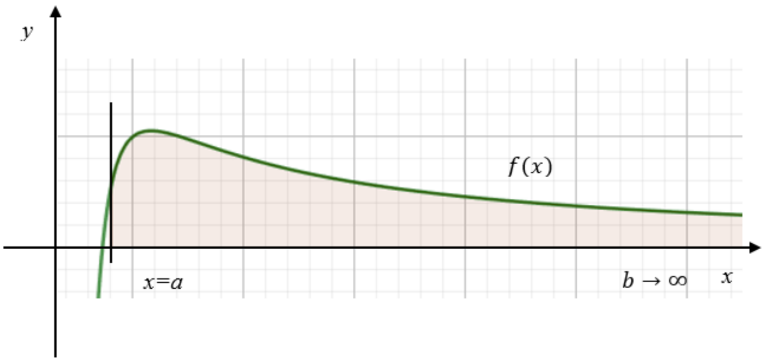

Figure 1.1

Suppose that function $f(x)$ is integrable on the whole interval, so we can calculate the value of any integral with upper limit B∈ $[a,\infty)$

$$\int_a^B f(x)dx$$

By choosing different values of the number B

$$a \leq B_1 < B_2 < B_3 < \cdots,$$

we get a sequence of numbers

$$\int_a^{B_1} f(x)dx\,;\ \int_a^{B_2} f(x)dx\,;\ \int_a^{B_3} f(x)dx;\ \ldots$$

It can be a convergent or a divergent sequence.

Definition 1.1 *The improper integral of type I* (or an integral with an infinite upper limit) is a definite integral with infinite limits of integration evaluated by the limit

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^{b} f(x)dx$$

If the limit is a finite number, we say that the improper integral ***converges***. If the limit does not exist or is infinity (positive or negative), the improper integral ***diverges***.

The evaluation of an improper integral follows the known rules. We find the corresponding antiderivative, apply the Newton-Leibniz formula, and calculate the limit as b tends to infinity

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^{b} f(x)dx = \lim_{b\to\infty} F(x)\Big|_a^b = \lim_{b\to\infty} F(b) - F(a)$$

We will write symbolically

$$\int_a^{\infty} f(x)dx = F(\infty) - F(a),$$

Remembering that $F(\infty)$ means the calculation of a limit.

Similarly, the improper integral can have an infinite lower limit, or both

$$\int_{\infty}^{b} f(x)dx\,; \quad \int_{\infty}^{\infty} f(x)dx$$

Example 1.1 Evaluate the integral

$$\int_{0.5}^{\infty} \frac{dx}{x^2}$$

Solution

Let us construct the graph

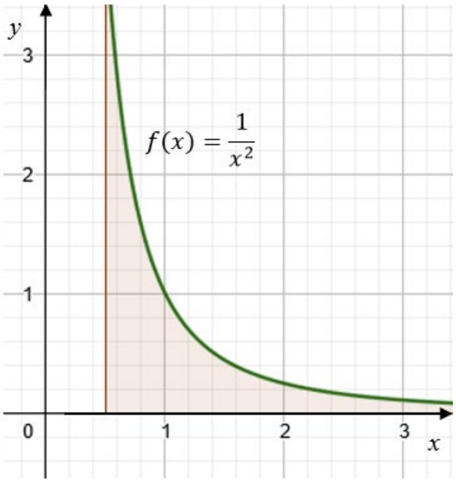

Figure 1.2

Figure 1.2 shows that the graph of the given integrand approaches the x-axis asymptotically. Let us evaluate the integral

$$\int_{0.5}^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_{0.5}^{b} x^{-2}\,dx = \lim_{b\to\infty} \frac{x^{-1}}{-1}\Big|_{0.5}^{b} = -\lim_{b\to\infty} \left(\frac{1}{x}\right)\Big|_{0.5}^{b} = -\left(\lim_{b\to\infty} \frac{1}{b} - 2\right) = 2$$

Answer The given integral converges to 2.

Example 1.2 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Solution

The graph of the integrand is

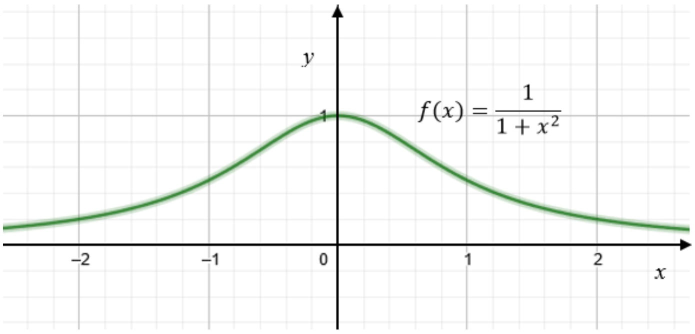

Figure 1.3

Figure 1.3 demonstrates the graph of an even function; its graph is symmetric with respect to the y-axis. To evaluate this integral we will split it into two integrals with half-bounded integration limits taking the intermediate value $x = 0$. We will also follow the principle of symmetry.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} = 2\lim_{b\to\infty} arctanx \Big|_{0}^{b} = 2\lim_{b\to\infty} arctanb - arctan0 = 2\frac{\pi}{2} = \pi$$

The graph of function $f(x) = arctanx$ has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ (see figure 1.4)

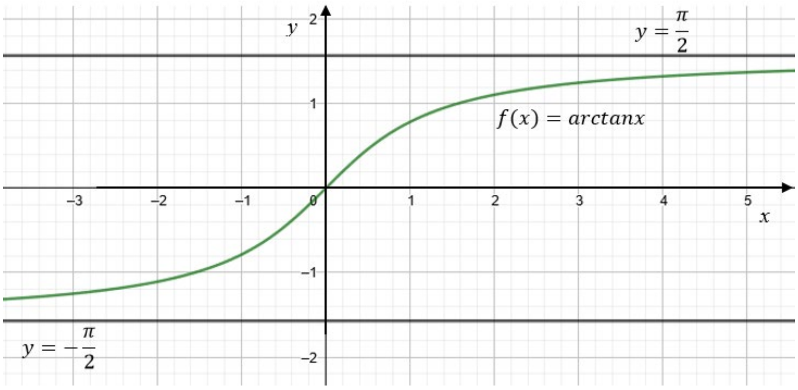

Figure 1.4

Answer The given integral converges to π.

Example 1.3 Evaluate the integral

$$\int_{0.2}^{\infty} \frac{dx}{x}$$

Solution

$$\int_{0.2}^{\infty} \frac{dx}{x} = \lim_{b\to\infty} lnx \Big|_{0.2}^{b} = \lim_{b\to\infty} lnb - ln0.2 = \infty$$

The graphs of integrand $f(x)$ and of antiderivative $F(x)$are presented in figure 1.5. Function $F(x)$ increases indefinitely.

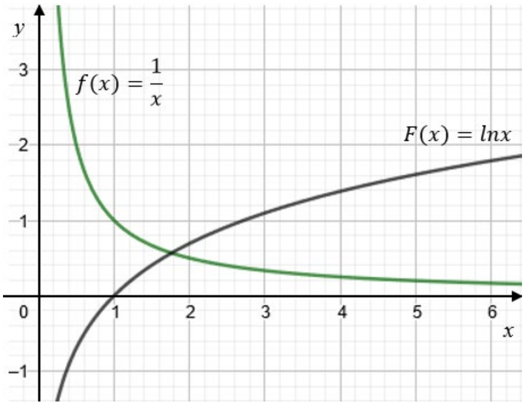

Figure 1.5

Answer The given improper integral diverges.

2. Improper integrals with an infinite discontinuity in the region of integration

Let us investigate the case when the function $f(x)$ becomes unbounded as its argument x approaches one or both endpoints of the interval $[a, b]$. Figure 2.1 presents the function whose value tends to infinity when the argument x approaches endpoint b of the interval. To integrate the function over such an interval we will evaluate the one-sided limit.

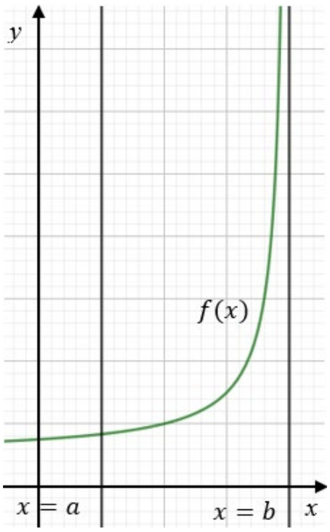

Figure 2.1

Definition 2.1 Let function $f(x)$ be continuous on the interval $[a, b)$ and be discontinuous at endpoint b of the interval, *the improper integral of type II* is defined in the following way

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx$$

If the limit exists, we say that the improper integral *converges*. If the limit does not exist or it is infinity, the improper integral *diverges*.

Similarly, if the function is discontinuous at the left endpoint of the interval (see figure 2.2), we have

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)dx$$

It is necessary to separate the integral into two parts if the function is discontinuous at the inner point of the interval (see figure 2.3). The improper integral converges only in the case if both its parts converge.

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^b f(x)dx$$

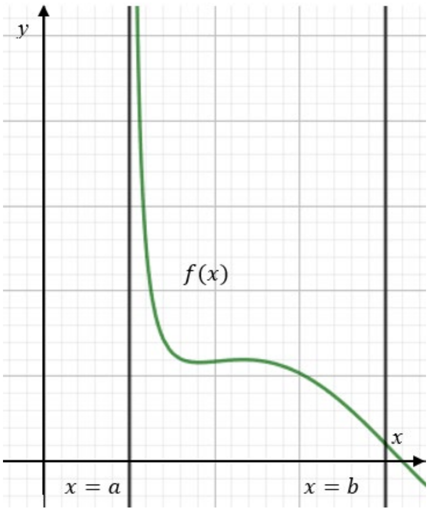

Figure 2.2

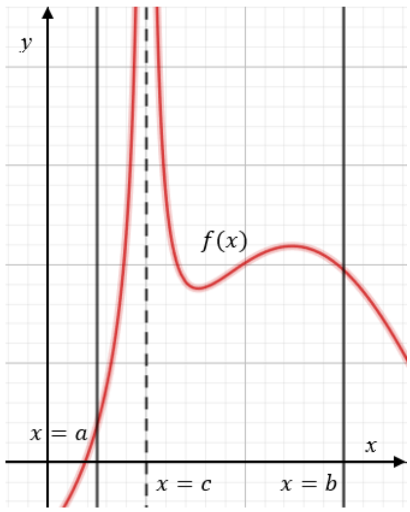

Figure 2.3

Example 2.1 Evaluate the integral

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x}}$$

Solution

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x}} = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} (1-x)^{-1/3} dx = \begin{vmatrix} let\ \ u = 1-x, \ \ du = -dx \\ u_1 = 1, \ \ u_2 = 0 \end{vmatrix} =$$

$$= \lim_{\varepsilon \to 0+} - \int_1^{0+\varepsilon} u^{-1/3} du = - \lim_{\varepsilon \to 0+} \frac{3}{2} u^{\frac{2}{3}} \Big|_1^{0+\varepsilon} = -\frac{3}{2}(0-1) = \frac{3}{2}$$

Answer This integral converges to $\frac{3}{2}$.

Example 2.2 Evaluate the integral

$$\int_0^3 \frac{2dx}{(x-1)^2}$$

Solution The integrand is not defined at the inner point $x = 1$ of the integration interval. It is necessary to split the interval into two parts.

$$\int_0^3 \frac{2dx}{(x-1)^2} = \int_0^1 \frac{2dx}{(x-1)^2} + \int_1^3 \frac{2dx}{(x-1)^2} =$$

$$= \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{2dx}{(x-1)^2} + \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^3 \frac{2dx}{(x-1)^2} =$$

Let us solve these limits separately

$$\lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{2dx}{(x-1)^2} = 2 \lim_{\varepsilon \to 0+} \frac{-1}{x-1}\bigg|_0^{1-\varepsilon} =$$

$$= -2 \lim_{\varepsilon \to 0+} \frac{1}{-\varepsilon} + 1 = \infty$$

Similarly, the second improper integral tends to infinity. It is not necessary to calculate it, as if at least one of the addends tends to infinity, the given integral diverges.

3. Exercises

Evaluate the integral and draw the graph of its integrand and of its antiderivative.

1. $\int_2^\infty \frac{dx}{xlnx}$
2. $\int_{-\infty}^0 xe^{-x^2} dx$
3. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$
4. $\int_{-1}^2 \frac{x}{\sqrt{x+1}} dx$
5. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} tanx\, dx$

4. Solutions

1. $\int_2^\infty \frac{dx}{xlnx}$

Solution The graph of the integrand in figure 4.1 demonstrates a decreasing function

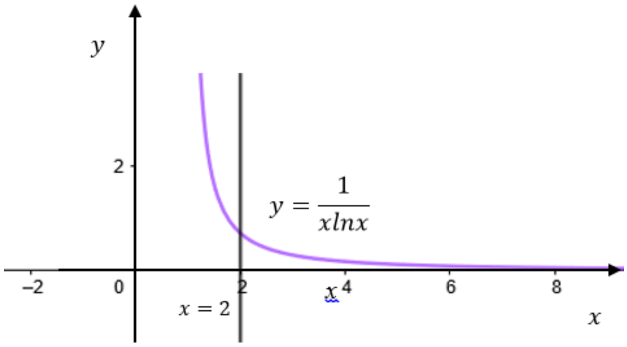

Figure 4.1

The upper limit of the given integral is infinity. Therefore, the given integral is an improper integral of type 1. After the primitive function or antiderivative lnu is found, we calculate the limit. While the logarithmic function is an increasing function, its limit is infinite as x tends to infinity (see figure 4.2).

$$\int_{2}^{\infty} \frac{dx}{xlnx} = \left| \begin{matrix} let \;\; u = lnx \quad then\; du = \dfrac{dx}{x} \\ u_1 = ln2, \;\; u_2 = \infty \end{matrix} \right| =$$

$$= \int_{ln2}^{\infty} \frac{du}{u} = ln|u| \Big|_{ln2}^{\infty} = \lim_{u \to \infty} lnu - lnln2 = \infty$$

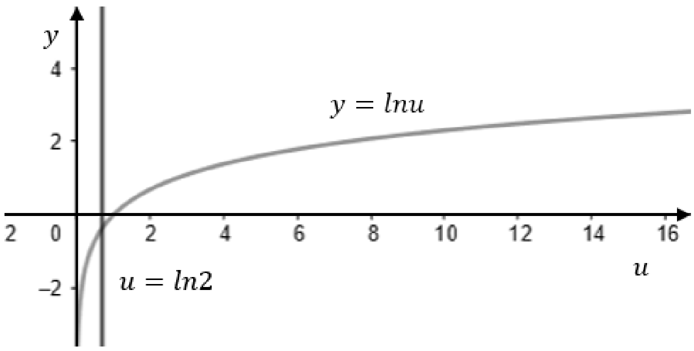

Figure 4.2

The given improper integral diverges.

2. $\displaystyle\int_{-\infty}^{0} xe^{-x^2} dx$

Solution

Figure 4.3 shows the graph of the integrand (red curve) and of the antiderivative (green curve). Both graphs are symmetric with respect to the origin of coordinate system, which means both functions are odd.

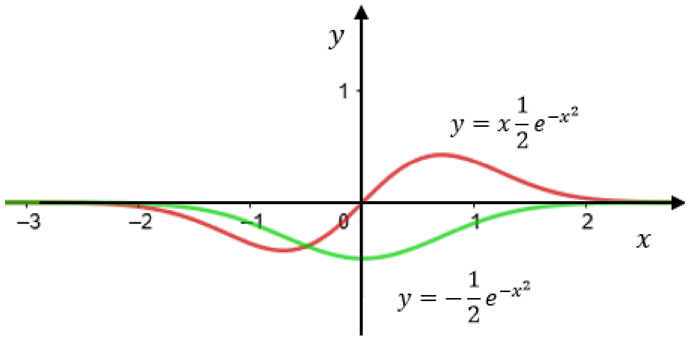

Figure 4.3

Since there is infinity in the lower bound, this is an improper integral of type 1. We evaluate the integral by changing the differential, and find the limit.

$$\int_{-\infty}^{0} xe^{-x^2}dx = -\frac{1}{2}\int_{-\infty}^{0} e^{-x^2}d(x^2) = -\frac{1}{2}e^{-x^2}\Big|_{-\infty}^{0} =$$

$$= -\frac{1}{2}\left(e^0 - \lim_{x\to-\infty} e^{-x^2}\right) = -\frac{1}{2}(1-0) = -\frac{1}{2}$$

The given integral converges.

3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Solution

The integral with both infinite limits is an improper integral of type I. We split it in two parts

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2}$$

Both integrals have the same integrand. Therefore, we compute the corresponding indefinite integral by completing the full square

$$\int \frac{dx}{x^2+2x+2} = \left|\begin{matrix} x^2+2x+2 = x^2+2x+1+1 = \\ = (x+1)^2+1 \end{matrix}\right| =$$

$$= \int \frac{d(x+1)}{(x+1)^2+1} = arctan(x+1) + C$$

We use this antiderivative for evaluation of improper integrals, and we apply the odd property of $arctanx$ function

$$\int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2} = arctan(x+1)\Big|_{-\infty}^{0} + arctan(x+1)\Big|_{0}^{\infty} =$$

$$= arctan0 - arctan(-\infty) + arctan\infty - arctan0 = 2\lim_{x\to\infty} arctanx = 2\frac{\pi}{2} = \pi$$

The integral converges. The graphs of both functions are presented in figure 4.4

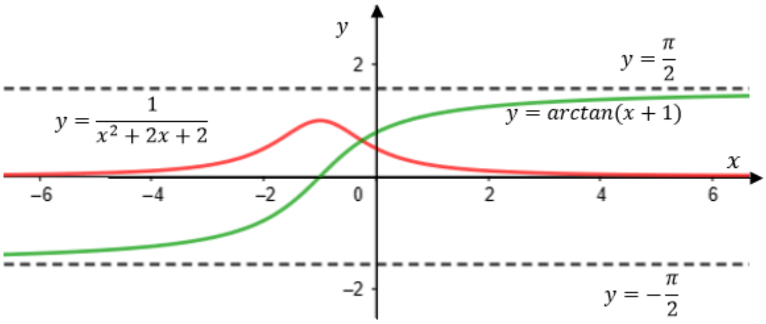

Figure 4.4

4. $\int_{-1}^{2} \frac{x}{\sqrt{x+1}} dx$

Solution

We detect the domain of the integrand

$$x > -1$$

The function is not defined at the lower point of the integration region; therefore, the given integral is an improper integral of type II. The graph of this function is sketched in figure 4.5

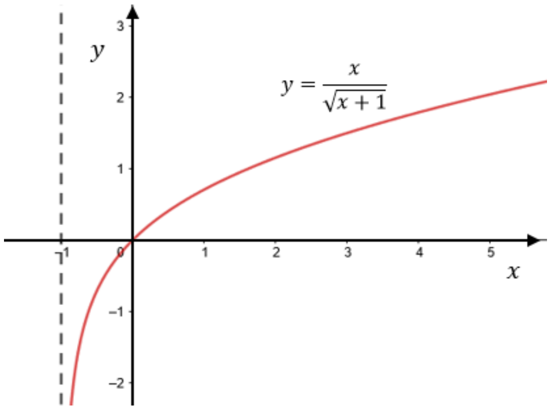

Figure 4.5

We compute the corresponding indefinite integral by applying substitution

$$\int \frac{x}{\sqrt{x+1}} dx = \left| \begin{matrix} let \ \ x+1 = u^2 \\ dx = 2udu \end{matrix} \right| = \int \frac{u^2-1}{u} 2udu =$$

$$= 2\int (u^2-1)du = 2\left(\frac{u^3}{3} - u\right) + C = 2\left(\frac{\sqrt{x+1}^3}{3} - \sqrt{x+1}\right) + C$$

Return to the improper integral

$$\int_{-1}^{2} \frac{x}{\sqrt{x+1}} dx = \lim_{\varepsilon \to 0+} \int_{-1+\varepsilon}^{2} \frac{x}{\sqrt{x+1}} dx =$$

$$= \lim_{\varepsilon \to 0+} 2\left(\frac{\sqrt{x+1}^3}{3} - \sqrt{x+1}\right)\Bigg|_{-1+\varepsilon}^{2} =$$

$$= 2\frac{\sqrt{3}^3}{3} - 2\sqrt{3} - 2\lim_{\varepsilon \to 0+}\left(\frac{\sqrt{-1+\varepsilon+1}^3}{3} - \sqrt{-1+\varepsilon+1}\right) = 2\frac{\sqrt{3}^3}{3} - 2\sqrt{3} = 0$$

The given improper integral converges as the limit is finite. The graph of the antiderivative is presented in figure 4.6. The value of this function at the point $x = 2$ is 0, the graph crosses the x-axis at this point. The extreme value of the antiderivative in the given interval is at point $x = 0$.

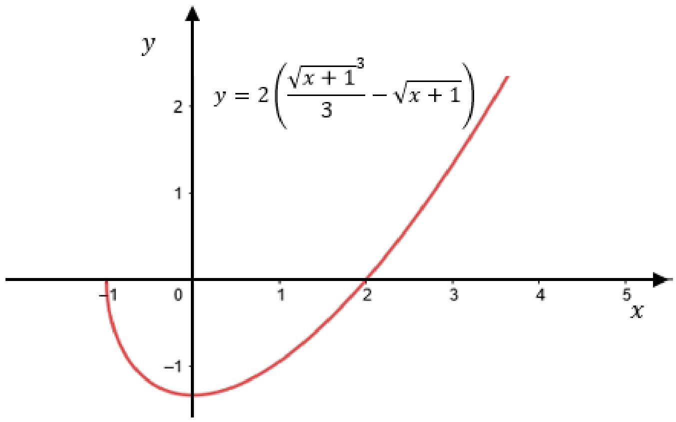

Figure 4.6

5. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} tanx\, dx$

Solution

The tangent function is not defined at the point $x = \frac{\pi}{2}$. We solve the improper integral of type II. The graphs of the integrand and the corresponding antiderivative are given in figure 4.7.

We solve the integral by the change of differential

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} tanxdx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{sinx}{cosx} dx = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d(cosx)}{cosx} =$$

$$= -ln|cosx|\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -ln\left(cos\frac{\pi}{3}\right) + \lim_{\varepsilon \to 0+} ln\left(cos\left(\frac{\pi}{2} - \varepsilon\right)\right) = -\infty$$

Since the value of the integral is not finite, it is divergent.

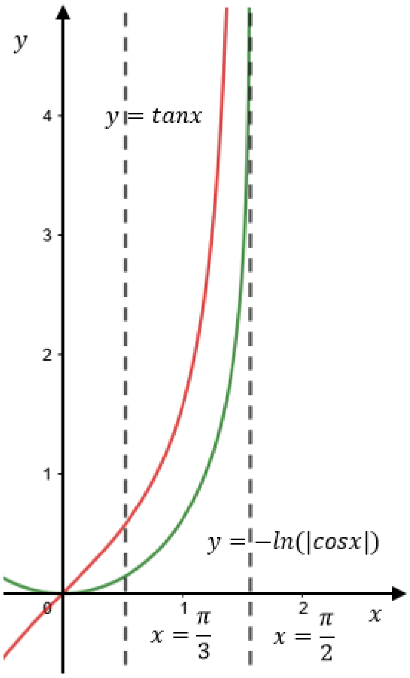

Figure 4.7