Application of Definite Integrals. Areas of Plane Regions

DETAILED DESCRIPTION:

Definite integrals are used to solve various problems. One of the usual applications is the calculation of the area of a plane region bounded by curves. This chapter presents different types of regions and gives the methods to calculate their areas. Formulas of definite integrals are given for curves expressed analytically, expressed by parametrical equations, as well as for curves given in the polar coordinate system. To construct the curves, the software programs GeoGebra Classic or Desmos Graphing Calculator, or others, can be used. Students can check their solutions with the integral calculator (https://www.integral-calculator.com/) that also constructs graphs of the integrand and the antiderivative.

AIM: **to explore the methods of calculation of the area of plane regions of different types.**

Learning Outcomes:

1. Students understand the geometrical meaning of the definite integral.
2. Students can calculate the area of plane regions enclosed by curves.
3. Students distinguish the cases if a region must be divided into two or more parts.

Prior Knowledge: basic rules of integration and differentiation; Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Relationship to real maritime problems: Calculation of the area of various specific construction parts is one of the core questions in shipbuilding. However, the shapes are so complex that mostly numerical calculations are used. Calculation of the area of a region is part of solving physics problems: for instance, to detect the pressure that is applied to an object it is necessary to calculate the area of the object's surface.

Content

Areas of Plane Regions

1. Area under the graph of a function

The definite integral was introduced as a tool for calculation of the area of a given region. If the region is bounded by the graph of a continuous function $f(x)$ on the interval $[a, b]$, two vertical lines $x = a$ and $x = b$, and x-axis, we can calculate the area S under the graph of the given function

$$S = \int_a^b f(x)dx$$

Example 1.1

Calculate the area of the region bounded by the function $y = cosx$, vertical straight lines $x = -\frac{\pi}{3}$, $x = x = \frac{\pi}{3}$, and x-axis.

Solution Let us construct the graph (see figure 1.1) and let us express the integral

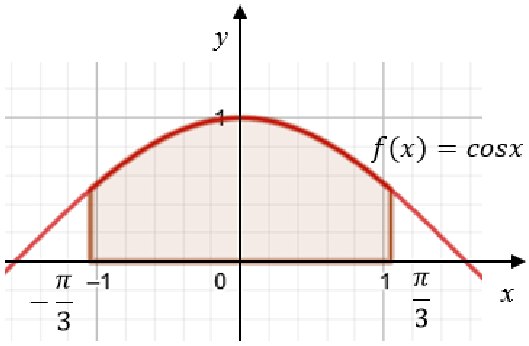

Figure 1.1

$$S = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} cosxdx = 2\int_0^{\frac{\pi}{3}} cosxdx = 2sinx\Big|_0^{\frac{\pi}{3}} = 2sin\frac{\pi}{3} - 0 = \sqrt{3}$$

We notice that this integral has symmetric integration boundaries and cosine function is an even function, so the interval was halved.

Example 1.2 Find the area of the plane region bounded by $y = (x-1)^3 + 1$, $x = 0.5$, $x = 2$, and $y = 0$.

Solution The region is bounded by two vertical lines $= 0.5$, $x = 2$, x-axis, and the cubic parabola (see figure 1.2).

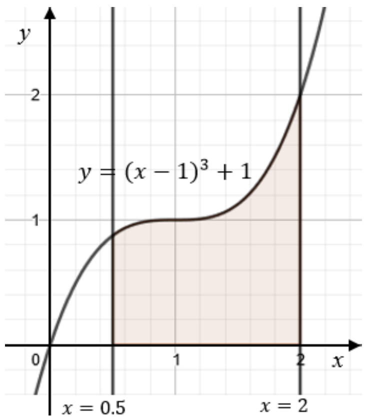

Figure 1.2

Thus, the area of the region is

$$S = \int_{0.5}^{2} ((x-1)^3 + 1)dx = \int_{0.5}^{2} (x-1)^3 d(x-1) + \int_{0.5}^{2} dx =$$

$$= \frac{(x-1)^4}{4} + x \Big|_{0.5}^{2} = \frac{1}{4} + 2 - \frac{1}{64} - 0.5 \approx 1.73$$

If the function has a break point that separates the interval of integration into subintervals where the function has only positive and only negative values, we need to integrate the function separately on every such subinterval, taking the absolute value of the result.

Example 1.3 Calculate the area of the region enclosed by $f(x) = log_2 x,\ x = 0.5,\ x = 2,\ y = 0$.

Solution The graph shows that the function has negative values in the interval $[0.5,1]$ and positive values in the interval $[1,2]$ (see figure 1.3). Therefore, we will separate the intervals.

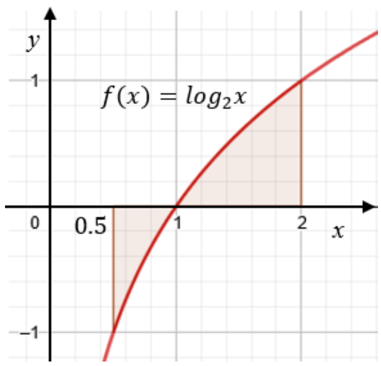

Figure 1.3

We split the integral into two parts to calculate the area of the region

$$S = \int_{0.5}^{2} |log_2 x| dx = \left| \int_{0.5}^{1} log_2 x dx \right| + \int_{1}^{2} log_2 x dx$$

Let us evaluate the corresponding indefinite integral by applying the method of integration by parts

$$\int log_2 x dx = \begin{vmatrix} let\ \ u = log_2 x, & dv = dx \\ du = \dfrac{dx}{x ln2}, & v = x \end{vmatrix} = x log_2 x - \frac{1}{ln2} \int \frac{x dx}{x} = x log_2 x - \frac{x}{ln2} + C$$

Now we use this antiderivative for the calculation of area S according to the Newton-Leibniz formula

$$S = \left| x log_2 x - \frac{x}{ln2} \right| \Big|_{0.5}^{1} + \left(x log_2 x - \frac{x}{ln2} \right) \Big|_{1}^{2} =$$

$$= \left| log_2 1 - \frac{1}{ln2} - 0.5 log_2 0.5 + \frac{1}{2ln2} \right| + 2 log_2 2 - \frac{2}{ln2} - log_2 1 + \frac{1}{ln2} =$$

$$= \left| \frac{1}{2} - \frac{1}{2ln2} \right| + 2 - \frac{1}{ln2} \approx |-0.22| + 0.56 \approx 0.78$$

Example 1.4 At what value of the upper limit b is the integral equal to 4?

$$\int_1^b \frac{dx}{4\sqrt{x}} = 4$$

Solution We calculate the integral

$$\int_1^b \frac{dx}{4\sqrt{x}} = \frac{1}{4}\cdot 2\sqrt{x}\Big|_1^b = \frac{1}{2}\left(\sqrt{b}-1\right)$$

We solve the equation

$$\frac{1}{2}\left(\sqrt{b}-1\right) = 4$$

$$\sqrt{b}-1 = 8$$

$$\sqrt{b} = 9;\quad b = 81$$

Answer The upper limit of the integral should be $b = 81$.

2. Area between two curves

If functions $f(x)$ and $g(x)$ are continuous functions over the interval $[a, b]$ and $f(x) \geq g(x)$ for all arguments $x \in [a, b]$ then area S of the region between the curves $f(x)$ and $g(x)$ in this interval is expressed by the integral

$$\boldsymbol{S = \int_a^b (f(x) - g(x))dx}$$

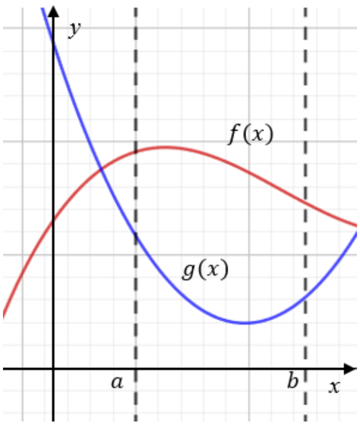

Figure 2.1

Example 2.2 Find the area of a plane region bounded by two curves $y = x^2,\ \ y = x + 2$

Solution We construct the graphs of given functions (see figure 2.2). To detect the integration interval, we need to calculate the coordinates of the projection of the region on the x-axis.

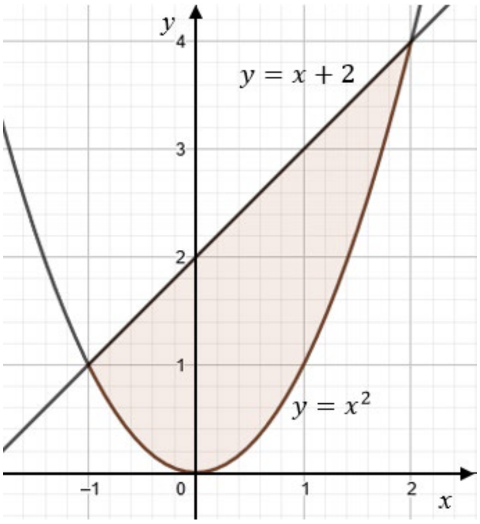

Figure 2.2

$$\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$$

$$x^2 - x - 2 = 0$$

The equation has two roots

$$x_1 = -1,\ \ x_2 = 2$$

Considering that the parabola is the lower curve, the area is

$$S = \int_{-1}^{2} (x + 2 - x^2)\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Bigg|_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5$$

3. The problem of the compound region

We will investigate the case of the region bounded by more than two curves. Let it be bounded by curves $f(x),\ g(x),\ z(x)$ (see figure 3.1).

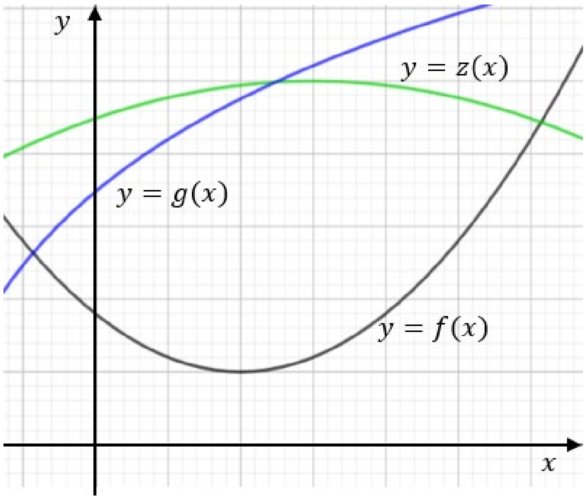

Figure 3.1

There are given two upper functions $g(x)\ and\ z(x)$ and one lower function $f(x)$. We determine the intersection points of graphs that define two separate regions with different intervals of projection $[a,b]$ and $[b,c]$ (see figure 3.2).

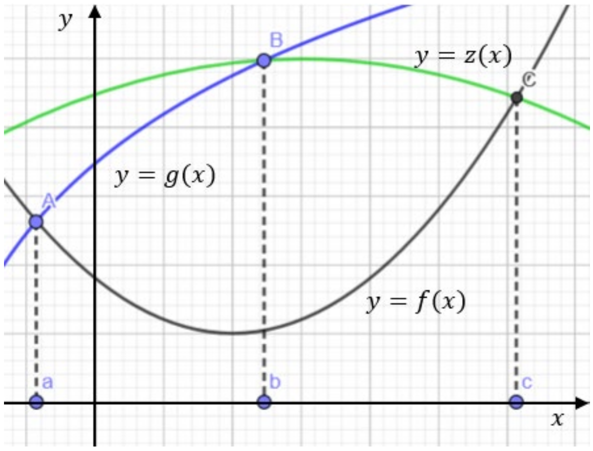

Figure 3.2

We compose two integrals to solve the problem of the area

$$S = \int_a^b (g(x) - f(x))\,dx + \int_b^c (z(x) - f(x))\,dx$$

Example 3.1 Calculate the area of a region enclosed by the curve $y = x^2 - 2x + 1$ and two lines $x + y = 3,\ y = 0$.

Solution

The figure 3.3 shows several closed regions. We find the region that is enclosed by the curve and exactly two lines, where one of the lines is the x-axis (see the coloured region).

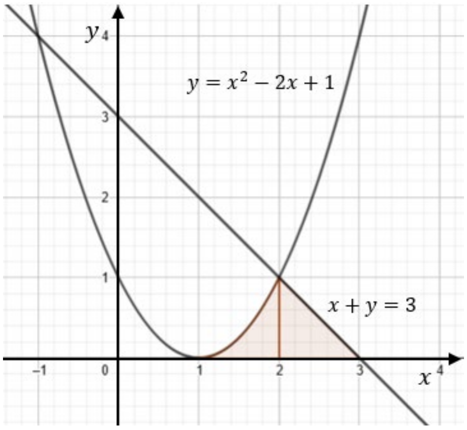

Figure 3.3

Now our solution has the following steps:

Step 1. Determine the boundaries of integration

$$x^2 - 2x + 1 = 0; \quad x = 1$$

$$x + y - 3 = 0; \quad x = 3$$

The boundaries are given by the interval $[1, 3]$.

Step 2.

Calculate the point of intersection of the curve and the line $x + y = 3$.

$$\begin{cases} y = x^2 - 2x + 1 \\ y = 3 - x \end{cases}$$

$$3 - x = x^2 - 2x + 1$$

$$x^2 - x - 2 = 0$$

The equation has two roots $x = -1; \quad x = 2$. The point $x = 2$ belongs to the interval $[1, 3]$.

Step 3. To calculate the area of the region it is necessary to break up the interval of boundaries into two parts

$$[1, 3] = [1, 2] + [2, 3]$$

and set up two integrals of two different upper functions

$$S = \int_1^2 (x^2 - 2x + 1)\, dx + \int_2^3 (3 - x)\, dx$$

Step 4. Calculate the area

$$S = \int_1^2 (x^2 - 2x + 1)\, dx + \int_2^3 (3 - x)\, dx =$$

$$= \frac{x^3}{3} - x^2 + x \Big|_1^2 + 3x - \frac{x^2}{2} \Big|_2^3 = \frac{8}{3} - 4 + 2 - \frac{1}{3} + 1 - 1 + 9 - \frac{9}{2} - 6 + 2 = \frac{5}{6}$$

Example 3.2 Calculate the area of a region enclosed by $y = \sqrt{x},\ \ 3x - 5y - 12 = 0,\ \ y = 0$.

We will calculate the area of the given region in two different ways.

Solution 1

Step 1. Construct the given region.

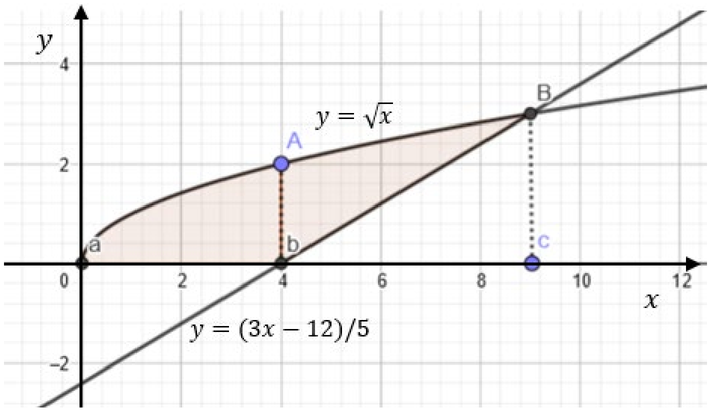

Figure 3.4

Step 2. Detect the boundaries of the integrals. Figure 3.4 presents the compound region whose area will be calculated as the sum of two integrals. The first integral is defined in the interval $[0, 4]$ because the point $x = 4$ is the x-intercept of the straight line. The boundaries of the second integral are $[4, 9]$. We can find the upper bound x=9 by solving the system of equations

$$\begin{cases} y = \sqrt{x} \\ y = \dfrac{3x - 12}{5} \end{cases}$$

$$5\sqrt{x} = 3x - 12$$

$$25x = 9x^2 - 72x + 144$$

$$9x^2 - 97x + 144 = 0$$

$$x = 9;\ \ x = \frac{16}{9}$$

Point B has coordinates $B(9, 3)$

Step 3. Set up integrals

$$S = \int_0^4 \sqrt{x}\, dx + \int_4^9 \left(\sqrt{x} - \frac{3x - 12}{5}\right) dx = \frac{x^{3/2}}{3/2}\bigg|_0^4 + \frac{x^{3/2}}{3/2}\bigg|_4^9 - \frac{3x^2}{10} + \frac{12x}{5}\bigg|_4^9 =$$

$$= \frac{2}{3} \cdot 27 - \frac{3}{10} \cdot 65 + \frac{12}{5} \cdot 5 = 10.5$$

Solution 2

We solved the problem by the calculation of two integrals. If we turn the construction with x-axis up, we can express the given functions as functions with respect to the argument y

$$x = y^2; \quad x = \frac{5y + 12}{3}$$

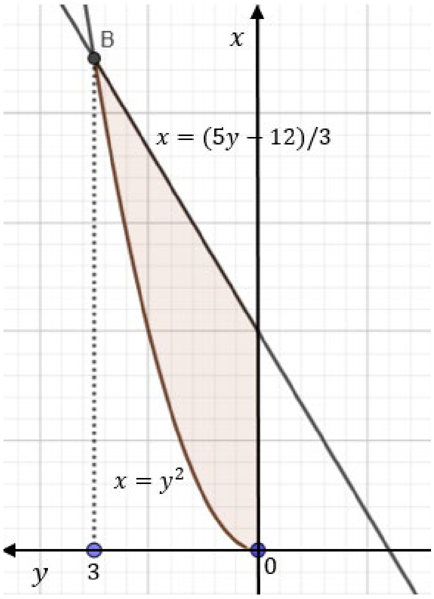

Figure 3.5

Point B has coordinates $(9, 0)$ (see figure 3.5). The boundaries on the y-axis are $[0, 3]$. Therefore, we can set up a simpler integral

$$S = \int_0^3 \left(\frac{5y + 12}{3} - y^2\right) dy = \frac{5y^2}{6} + 4y - \frac{y^3}{3}\Bigg|_0^3 = 5 \cdot \frac{9}{6} + 12 - 9 = 10.5$$

4. Area under a parametric curve

Parametric equations are used to describe many different types of curves. Circle, ellipse, cycloid, and hypocycloid are some of the best-known curves that can be expressed parametrically. To calculate the area under the curve, we modify the area formula by substitution.

The area of a region S enclosed by function $f(x)$, two vertical lines $x = a$ and $x = b$ and x-axis can be calculated by the formula

$$S = \int_a^b f(x) dx$$

If the function is described by $x = x(t)$ and $y = y(t)$ and the parameter t runs between t_1 and t_2 where

$$a = x(t_1); \quad b = x(t_2)$$

We substitute

$$\boldsymbol{S = \int_{t_1}^{t_2} y(t) d(x(t)) = \int_{t_1}^{t_2} y(t) x'(t) dt}$$

Example 4.1 Calculate the area of an ellipse.

Solution

The ellipse is symmetric with respect to its axes. Therefore, we calculate the area of the fourth part of the ellipse (see figure 4.1)

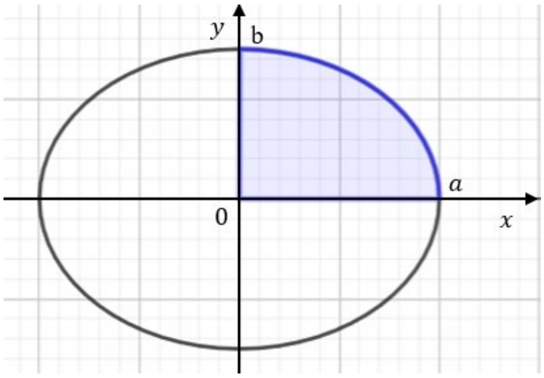

Figure 4.1

Parametric equations of the ellipse are

$$\begin{cases} x = acost \\ y = bsint \end{cases}$$

We calculate

$$S = 4\int_0^a f(x)dx = \left| \begin{matrix} let\ \ x = a\,cost, & then\ \ dx = -asint\ dt \\ x_1 = 0,\ \ then\ t_1 = \frac{\pi}{2}; & x_2 = a,\ \ t_2 = 0 \end{matrix} \right| =$$

$$= -4\int_{\frac{\pi}{2}}^{0} bsint\ asint\ dt = -4ab\int_{\frac{\pi}{2}}^{0} sin^2 t\ dt = 4ab\int_0^{\frac{\pi}{2}} \frac{1 - cos2t}{2}\ dt =$$

$$= 2ab\int_0^{\frac{\pi}{2}} dt\ - ab\int_0^{\frac{\pi}{2}} cos2t\ d2t = 2ab\Big|_0^{\frac{\pi}{2}} - absin2t\Big|_0^{\frac{\pi}{2}} = 2ab\frac{\pi}{2} = \pi ab$$

5. Curve in a polar coordinate system

The curvilinear sector is given by the function $r = r(\varphi)$ and two rays $\varphi = \alpha;\ \ \varphi = \beta$. To calculate the area of this sector we apply the formula

$$\boldsymbol{S = \frac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi}$$

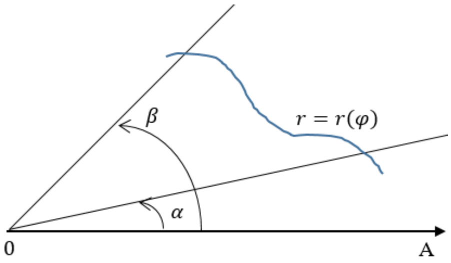

Figure 5.1

Example 5.1 Find the area inside the cardioid $r = 2 + 2cos\varphi$.

Solution

The shape of the given cardioid is represented in figure 5.2

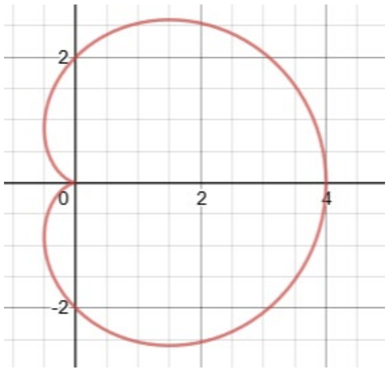

Figure 5.2

Polar axis is the symmetry line of the cardioid. We create the integral for half of the region where the angle changes from 0 to 180°. The area of this region is

$$S = 2 \cdot \frac{1}{2}\int_0^{\pi}(2 + 2cos\varphi)^2 d\varphi = \int_0^{\pi}(4 + 4cos\varphi + cos^2\varphi)d\varphi =$$

$$= 4\int_0^{\pi} d\varphi + 4\int_0^{\pi} cos\varphi\, d\varphi + \frac{1}{2}\int_0^{\pi}(1 + cos2\varphi)d\varphi =$$

$$= \left(4\varphi + 4sin\varphi + \frac{1}{2}\varphi + \frac{1}{4}sin2\varphi\right)\Big|_0^{\pi} = 4.5\pi$$

6. Exercises

1. Calculate the area of the region between the curve $y = sinx$ and x-axis in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{4}\right]$.
2. Calculate the area of a region enclosed by straight lines $y = x$ and $x + 2y - 6 = 0$, and x-axis.
3. Calculate the area between two curves $y = (x + 2)^2$ and $y = 4 - x^2$.
4. Calculate the area enclosed by $y = 0.5^x$, $y = 0.5x\sqrt{1 + x^2}$, $x = -2$ and y-axis.
5. Calculate the area under one arc of the cycloid

$$\begin{cases} x = 2(t - sint) \\ y = 2(1 - cost) \end{cases}$$

6. Calculate the area of one petal of the polar rose $r = 4cos3\varphi$.

7. Solutions

1. Calculate the area of the region between the curve $y = sinx$ and x-axis in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{4}\right]$.

Solution

We construct the curve and vertical lines (see figure 7.1).

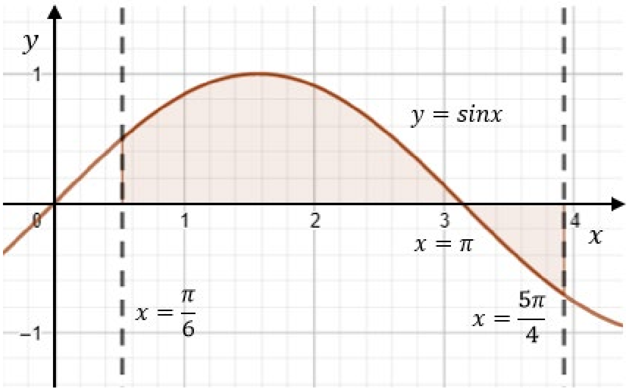

Figure 7.1

The function $y = sinx$ has positive and negative values over the given interval. To calculate the area of the region it is necessary to divide the interval into two parts. The x-intercept of the function is $x = \pi$. We compose two integrals to calculate the area

$$S = S_1 + S_2 = \int_{\frac{\pi}{6}}^{\pi} sinx\,dx + \left|\int_{\pi}^{\frac{5\pi}{4}} sinx\,dx\right| = -cosx\Big|_{\frac{\pi}{6}}^{\pi} + |-cosx|\Big|_{\pi}^{\frac{5\pi}{4}} =$$

$$= -\left(cos\pi - cos\frac{\pi}{6}\right) + cos\frac{5\pi}{4} - cos\pi = 1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} + 1 = \frac{4 + \sqrt{3} - \sqrt{2}}{2} \approx 2.16$$

2. Calculate the area of a region enclosed by straight lines $y = x$ and $x + 2y - 6 = 0$, and x-axis.

Solution

We construct the straight lines and choose the projection of the region to the y-axis.

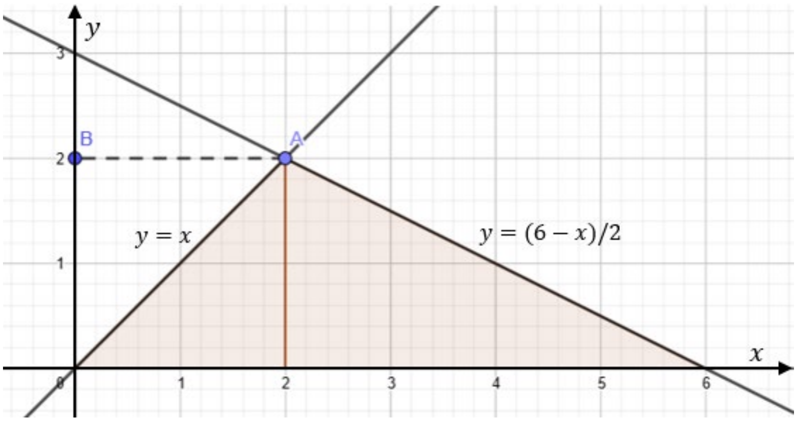

Figure 7.2

Let us calculate the coordinates of the intersection point A

$$\begin{cases} y = x \\ y = (6 - x)/2 \end{cases}$$

$$x = (6 - x)/2$$

$$2x = 6 - x$$

$$3x = 6; \quad x = 2$$

The intersection point A has the coordinates A (2,2). The boundaries of the integral are [0,2] with respect to variable y. The integral is

$$S = \int_0^2 (6 - 2y - y)\, dy = 6y - \frac{3y^2}{2}\Bigg|_0^2 = 12 - 3 \cdot 2 = 6$$

3. Calculate the area between two curves $y = (x + 2)^2$ and $y = 4 - x^2$.

Solution

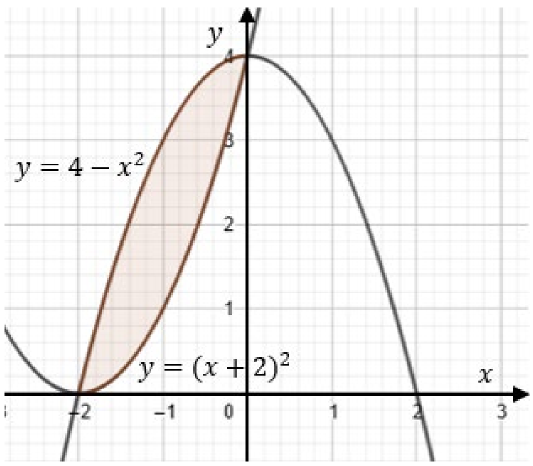

Figure 7.3

The region is defined in the interval $[-2,0]$. We find its area

$$S=\int_{-2}^{0}(4-x^2-(x+2)^2)\,dx=4x-\frac{x^3}{3}-\frac{(x+2)^3}{3}\Bigg|_{-2}^{0}=$$

$$=0-\frac{8}{3}+8-\frac{8}{3}-0=\frac{8}{3}$$

4. Calculate the area enclosed by $y=0.5^x$, $y=0.5x\sqrt{1+x^2}$, $x=-2$ and y-axis.

Solution

Construct the curves and the vertical line (see figure 7.4)

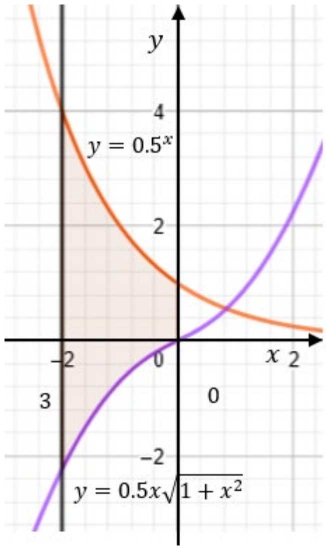

Figure 7.4

Set up the integral

$$S=\int_{-2}^{0}\left(0.5^x-0.5x\sqrt{1+x^2}\right)dx=\int_{-2}^{0}0.5^x\,dx-\int_{-2}^{0}0.5x\sqrt{1+x^2}\,dx$$

Let us solve the second integral separately

$$\int_{-2}^{0}0.5x\sqrt{1+x^2}\,dx=\left|\begin{matrix}let\ \ u=1+x^2, then\ \ du=2xdx\\ u_1=5,\ \ u_2=1\end{matrix}\right|=\frac{1}{4}\int_{5}^{1}\sqrt{u}\,du=$$

$$=\frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{5}^{1}=\frac{1}{6}(1-5\sqrt{5})$$

Now

$$S=\frac{0.5^x}{ln0.5}\Bigg|_{-2}^{0}-\frac{1-5\sqrt{5}}{6}=\frac{1}{ln0.5}(1-0.5^{-2})+\frac{5\sqrt{5}-1}{6}\approx 6.03$$

5. Calculate the area under one arc of the cycloid

$$\begin{cases} x = 2(t - sint) \\ y = 2(1 - cost) \end{cases}$$

Solution

The cycloid is the locus of a point on the rim of a circle of radius R rolling along a straight line. We can see the way of construction of the cycloid on the webpage:

Weisstein, Eric W. "Cycloid." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/Cycloid.html

The radius of the given cycloid is $R = 2$. Then the area under the first arc is over the interval $[0, 2\pi R] = [0, 4\pi]$ (see figure 7.5).

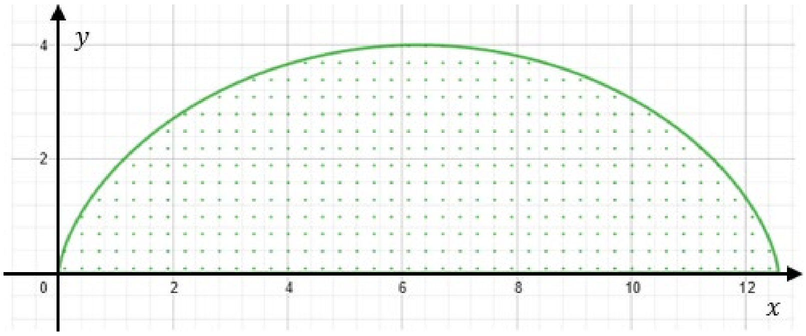

Figure 7.5

We have to calculate the integral of the function given in parametric form. We calculate the boundaries with respect to the argument t in the following way:

We have $0 \leq x \leq 4\pi$.

For the lower bound $x = 0$ then $0 = 2(t - sint)$. We calculate $t_1 = 0$

For the upper bound $x = 4\pi$ then $4\pi = 2(t - sint)$; $2\pi = t - sint$. We calculate $t_2 = 2\pi$.

According to the formula given in chapter 4, we differentiate the function x with respect to variable t

$$x' = 2(1 - cost)$$

The area is

$$S = \int_0^{2\pi} 2(1 - cost)2(1 - cost)\, dt = 4\int_0^{2\pi} (1 - 2cost + cos^2 t)\, dt =$$

$$= 4\int_0^{2\pi} (1 - 2cost)dt + 2\int_0^{2\pi} (1 + cos2t)\, dt = 4t - 8sint + 2t + sin2t \Big|_0^{2\pi} =$$

$$= 8\pi + 4\pi = 12\pi$$

6. Calculate the area of one petal of the polar rose $r = 4cos3\varphi$.

Solution

The given polar rose has three petals:

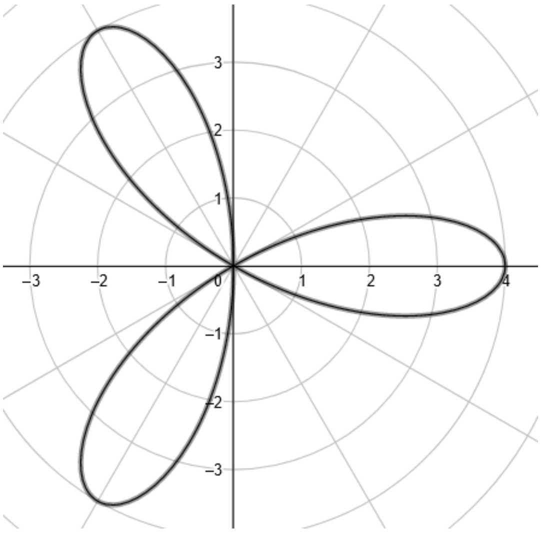

Figure 7.6

We calculate the area of the petal whose line of symmetry is the polar axis. Therefore, we can calculate half of the petal's area and double the integral. First, we need to detect the upper bound of the integral. It appears when the distance of a point on the ray is zero

$$0 = 4cos3\varphi;\ 3\varphi = \frac{\pi}{2};\ \ \varphi_2 = \frac{\pi}{6}$$

We set up the integral to calculate the area of one petal

$$S = 2 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} (4cos3\varphi)^2\, d\varphi =$$

$$= \frac{16}{2} \int_0^{\frac{\pi}{6}} (1 + cos6\varphi)\, d\varphi = 8\left(\varphi + \frac{1}{6} sin6\varphi\right)\Bigg|_0^{\frac{\pi}{6}} = \frac{4\pi}{3}$$