Application of the Definite Integral. Arc length

DETAILED DESCRIPTION:

Definite integrals can be applied to calculate the length of various curves. This chapter explains the creation of the formula for calculation of arc length. The formula can be transformed for curves that are given as parametric equations or in polar form. The content is supplemented with examples of graphs constructed with GeoGebra and Desmos.

AIM: to demonstrate the calculation of the arc length for curves given in the Cartesian coordinate system and for curves given in the polar coordinate system.

Learning Outcomes:

1. Students understand the application of definite integral to solve geometry tasks.
2. Students can calculate the arc length of given curves.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of a functions; the construction of the graph of a function; algebra and trigonometry formulas.

Relationship to real maritime problems: With the help of definite integrals it is possible to calculate the lengths of different objects that can be described by functions. For instance, it is possible to calculate the length of a rope hanging between two supports by integration.

Content

Arc length

1. The formula for calculation of the length of an arc

Let the function $y = f(x)$ be given over the interval $[a, b]$. We will calculate the length of the arc $\widetilde{AB}$ of this curve (see figure 1.1).

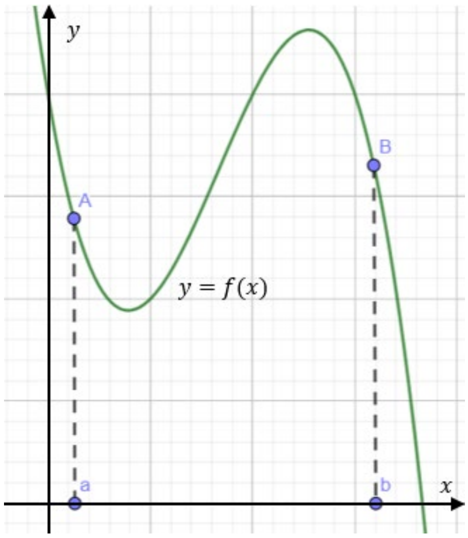

Figure 1.1

We will form a polygonal line by choosing points $A = P_0, P_1, P_2, P_3, \ldots,$ and $P_n = B$ (see figure 1.2).

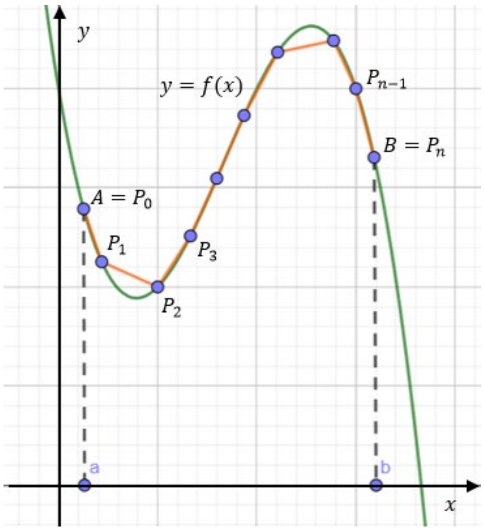

Figure 1.2

Now we can calculate the length of every segment $P_{i-1}P_i$ for every index i (see figure 1.3):

$$\Delta l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

where

$$\Delta x_i = x_i - x_{i-1}; \;\; \Delta y_i = f(x_i) - f(x_{i-1})$$

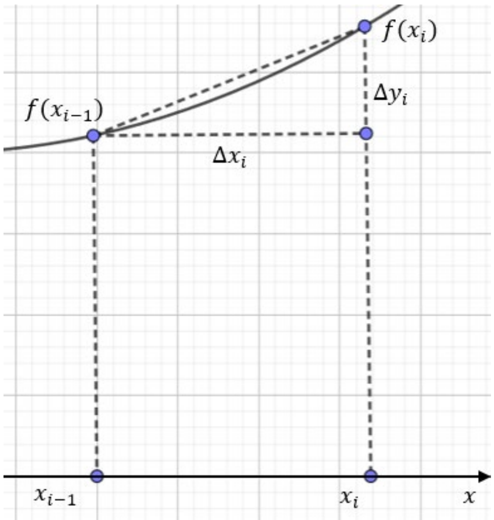

Figure 1.3

We change the expression

$$\Delta l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

The approximate value of the arc length is the sum of the lengths of all segments Δl_i

$$\overset{\frown}{AB} \approx \sum_{i=1}^{n} \Delta l_i = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

The result will be better if we divide the arc into smaller and smaller parts. Taking the limit when the maximum length of the interval Δx_i tends to zero, we get the real length of the arc

$$\overset{\frown}{AB} = \lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

Here we can recall the definition of derivative of the function $y = f(x)$ with respect to the argument x

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'$$

Thus, we express the limit as an integral in the following way

$$\overset{\frown}{AB} = \int_a^b \sqrt{1 + (y')^2}\, dx$$

Example 1.1 Calculate the length of the line segment given by equation $y = 3x - 2$ from $a = -2$ and $b = 3$.

Solution

It is necessary to differentiate the given function to apply the formula

$$y' = (3x - 2)' = 3$$

Now

$$L=\int_{-2}^{3}\sqrt{1+(3)^2}\,dx=\sqrt{10}\int_{-2}^{3}dx=\sqrt{10}x\Big|_{-2}^{3}=\sqrt{10}(3+2)=5\sqrt{10}\approx 15.8$$

2. The length of an arc given by parametric equations

If the arc $\overset{\frown}{AB}=L$ is described by parametric equations on the interval $[a,b]$ with respect to the argument x

$$\begin{cases}x=x(t)\\y=y(t)\end{cases}$$

we can apply substitution with respect to the argument t

$$L=\int_{a}^{b}\sqrt{1+(y')^2}\,dx=\left|\begin{matrix}x=x(t),\ \ dx=d(x(t))=(x'\,(t))_t dt=\dot{x}dt\\ y'\ =\dfrac{\dot{y}}{\dot{x}},\ \ x(t_1)=a,\ \ x(t_2)=b\end{matrix}\right|=$$

$$=\int_{t_1}^{t_2}\sqrt{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2}\cdot\dot{x}\,dt=\int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2}\,dx$$

We can calculate the arc length for a parametrically given function by the formula

$$\boldsymbol{L=\int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2}\,dx}$$

Example 2.1

Calculate the arc length of an astroid

$$\begin{cases}x=4cos^3t\\y=4sin^3t\end{cases}$$

Solution

An astroid is the locus of a point on a circle as it rolls inside a fixed circle with four times the radius (see figure 2.1). Parameter t expresses the angle. The curve can be constructed if the parameter t changes from 0 to 2π.

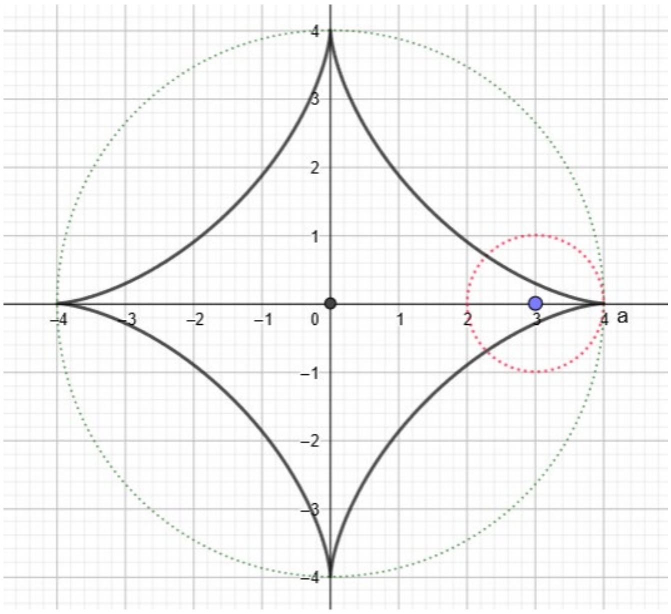

Figure 2.1

The curve is centrally symmetric with respect to the origin. Therefore, we can calculate one-fourth of the astroid where parameter t changes from 0 to $\pi/2$.

To create an integral, it is necessary to differentiate the parametric functions

$$\begin{cases} \dot{x} = 4 \cdot 3cos^2t(-sint) \\ \dot{y} = 4 \cdot 3sin^2t \cdot cost \end{cases}$$

The length of the given astroid is

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{(4 \cdot 3cos^2t(-sint))^2 + (4 \cdot 3sin^2t \cdot cost)^2}dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{144cos^4tsin^2t + 144sin^4tcos^2t}\,dt =$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{144cos^2tsin^2t(cos^2t + sin^2t)}\,dt =$$

$$= 4\int_0^{\frac{\pi}{2}} 12sint \cdot cost\,dt = 48\int_0^{\frac{\pi}{2}} sint\,d(sint) \; = 48\frac{sin^2t}{2}\Bigg|_0^{\frac{\pi}{2}} = 24(1-0) = 24$$

3. The arc length of a polar curve

The arc length of a polar curve $r = r(\varphi)$ between the rays α and β is given by the integral

$$\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\varphi}$$

Example 3.1

Calculate the length of the arc of the circle $r = 6$ between $o \leq \varphi \leq \frac{2\pi}{3}$

Solution

We will calculate the length of the arc that is one-third of the circle with radius 6 (see figure 3.1)

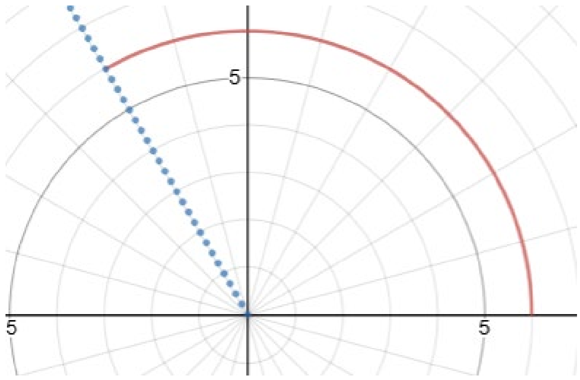

Figure 3.1

Provided that the value of derivative $r' = 0$, the length of the arc is

$$L = \int_{0}^{\frac{2\pi}{3}} \sqrt{36 + 0}\, d\varphi = \int_{0}^{\frac{2\pi}{3}} 6\, d\varphi = 6\varphi \Big|_{0}^{\frac{2\pi}{3}} = 4\pi$$

4. Exercises

1. Calculate the arc length of the curve $y = \frac{4}{3}\sqrt{x^3}$ from 1 to 2.

2. Calculate the arc length of the curve $y = lnx$ from 1 to $\sqrt{2}$.

3. Find the length of one arc of the cycloid given by parametric equations

$$\begin{cases} x = 2(1 - sint) \\ y = 2(t - cost) \end{cases}$$

4. Calculate the arc length of the circle $r = 4cos\varphi$ included between the polar rays $\varphi = -\frac{\pi}{3}$ and $\varphi = \frac{\pi}{5}$.

5. Solutions

1. Calculate the arc length of the curve $y = \frac{4}{3}\sqrt{x^3}$ from 1 to 2.

Solution

Construct the graph

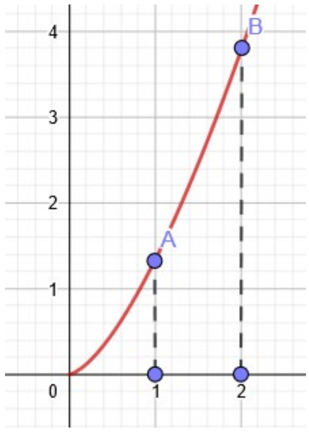

Figure 5.1

We calculate the arc length between the points A and B. The derivative of the function

$$y' = \left(\frac{4}{3}x^{\frac{3}{2}}\right)' = 2\sqrt{x}$$

We use the arc length formula

$$L = \int_1^2 \sqrt{1 + \left(2\sqrt{x}\right)^2}\,dx = \int_1^2 \sqrt{1+4x}\,dx = \left| \begin{matrix} u = 1 + 4x, \quad du = 4dx \\ x_1 = 1, \ \ u_1 = 5, x_2 = 2, \ \ u_2 = 9 \end{matrix} \right| =$$

$$= \frac{1}{4}\int_5^9 u^{\frac{1}{2}}\,dx = \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_5^9 = \frac{1}{6}\left(3^3 - \sqrt{5^3}\right) = \frac{1}{6}\left(27 - 5\sqrt{5}\right) \approx 2.64$$

2. Calculate the arc length of the curve $y = lnx$ from 1 to $\sqrt{2}$.

Solution

Construct the graph of the given function

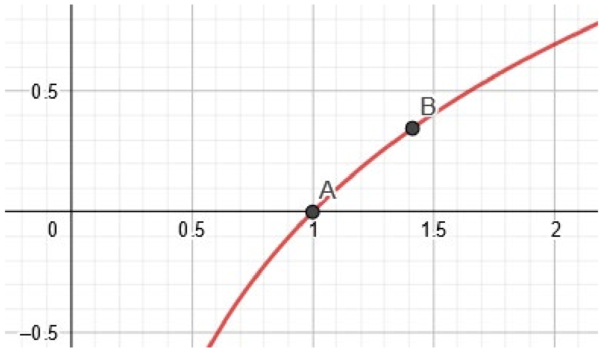

Figure 5.2

The derivative of the function $y = lnx$ is

$$y' = (lnx)' = \frac{1}{x}$$

To calculate the integral, we use an algebraic transformation of the integrand and apply substitution

$$L = \int_1^{\sqrt{2}} \sqrt{1 + \frac{1}{x^2}}\,dx = \int_1^{\sqrt{2}} \sqrt{\frac{x^2+1}{x^2}}\,dx =$$

$$= \int_1^{\sqrt{2}} \frac{\sqrt{x^2+1}}{x}\,dx = \int_1^{\sqrt{2}} \frac{\sqrt{x^2+1}}{x^2}\,xdx = \left| \begin{matrix} u^2 = x^2+1, & 2udu = 2xdx \\ u_1 = \sqrt{2}, & u_2 = \sqrt{3} \end{matrix} \right| =$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} \frac{u}{u^2-1}\,udu = \int_{\sqrt{2}}^{\sqrt{3}} \frac{u^2 - 1 + 1}{u^2-1}\,du =$$

$$= \int_{\sqrt{2}}^{\sqrt{3}} du + \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{u^2-1}\,du = u + \frac{1}{2} ln\left|\frac{1+u}{1-u}\right| \Bigg|_{\sqrt{2}}^{\sqrt{3}} =$$

$$= \sqrt{3} - \sqrt{2} + \frac{1}{2}\left(ln\left|\frac{\sqrt{3}-1}{\sqrt{3}+1}\right| - ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| \right) \approx 0.54$$

3. Find the length of one arc of the cycloid given by parametric equations

$$\begin{cases} x = 2(t - sint) \\ y = 2(1 - cost) \end{cases}$$

Solution

We can calculate the length of one arc of the cycloid (see figure 5.3) if the range of the parameter $t \in [0,2\pi]$.

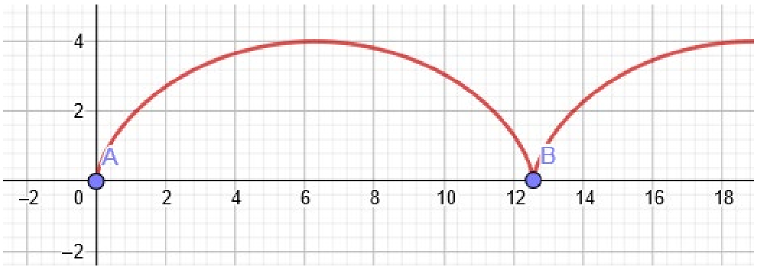

Figure 5.3

First we will calculate the derivatives

$$\dot{x} = 2(1 - cost)$$
$$\dot{y} = 2sint$$

Now we will simplify the expression by applying algebra and trigonometry formulas

$$\dot{x}^2 + \dot{y}^2 = 4(1 - cost)^2 + 4sin^2t = 4(1 - 2cost + cos^2t + sin^2t) =$$

$$= 4(2 - 2cost) = 8 \cdot 2sin^2\frac{t}{2} = 16sin^2\frac{t}{2}$$

We use trigonometry formulas

$$\boldsymbol{cos^2t + sin^2t = 1}$$

$$\boldsymbol{1 - cost = 2sin^2\frac{t}{2}}$$

The length of the first arc of cycloid is

$$L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^{2\pi} \sqrt{16sin^2\frac{t}{2}}\, dt =$$

$$= \int_0^{2\pi} 4sin\frac{t}{2}dt = -8cos\frac{t}{2}\Bigg|_0^{2\pi} = -8(cos\pi - cos0) = -8 \cdot (-2) = 16$$

4. Calculate the arc length of the circle $r = 4cos\varphi$ included between the polar rays $\varphi = -\frac{\pi}{3}$ and $\varphi = \frac{\pi}{5}$.

Solution

Construct the curve in the polar coordinate system (see figure 5.4)

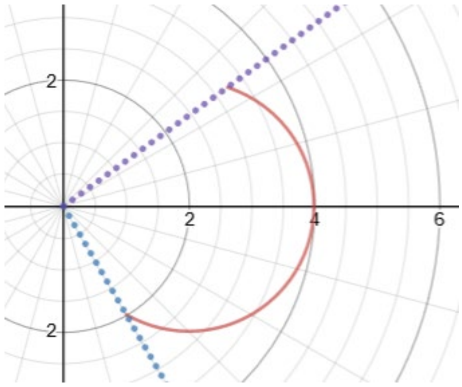

Figure 5.4

Calculate the derivative and transform the trigonometric expression

$$r' = (4cos\varphi)' = -4sin\varphi$$

$$r^2 + r'^2 = 16cos^2\varphi + 16sin^2\varphi = 16$$

Create an integral

$$L = \int_{-\frac{\pi}{3}}^{\frac{\pi}{5}} \sqrt{r^2 + r'^2}\, d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{5}} \sqrt{16}\, d\varphi = 4\varphi \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{5}} = 4\left(\frac{\pi}{5} + \frac{\pi}{3}\right) = \frac{32\pi}{15}$$