Application of the Definite Integral. Volume of a Solid of Revolution

DETAILED DESCRIPTION:

This chapter introduces the main principles for calculation of the volume of solids of revolution. Different examples are discussed where solids are generated by elementary curves and lines. Some composite constructions are explained. The case of the parametrically given curve is included to describe the solid of revolution. The content is supplemented with examples of graphs and surfaces constructed with GeoGebra tools.

AIM: to show the methods of calculation of the volume of solids of revolution.

Learning Outcomes:

1. Students understand the application of the definite integral in solving geometry tasks.
2. Students can construct regions of a revolution and understand what surfaces they form.
3. Students can calculate the volume of solids of revolution.

Prior Knowledge: basic rules of integration and differentiation; the Newton-Leibniz formula; properties of functions; the construction of graphs of functions; algebra and trigonometry formulas.

Relationship to real maritime problems: Volume is a very important concept if we are speaking about the capacity of cargo holds, the capacity of fuel oil tanks or ballast water tanks, tanks of lubricating oil, or others. It is important to know the amount of material required for producing a specific part with a definite volume. Calculations of the volume of containers, cauldrons, and tanks are among the necessary premises for designing a ship's engineering equipment.

Contents

Application of the Definite Integral. Volume of a Solid of Revolution

1. Volume of a solid of revolution obtained by rotating an area about x-axis

Let us recall the concept of the solid of revolution.

Definition. The *solid of revolution* is a solid figure obtained by rotating a plane curve around a straight line (the axis of revolution) that lies in the same plane.

Let f be a continuous function on the interval $[a, b]$.

Consider the solid formed by rotating (revolving) the region bounded by the curve $y = f(x)$, straight lines $x = a$, $x = b$, and x- axis **about the x-axes** (see figure 1.1). This solid is called a **solid of revolution**.

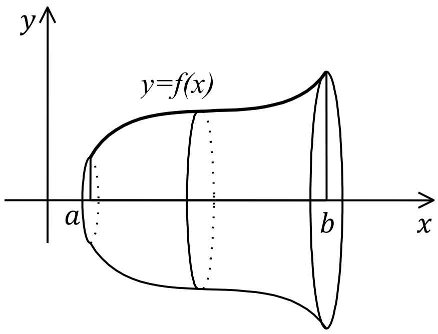

Figure 1.1

The volume of this solid can be calculated by the formula:

$$V_x = \pi \int_a^b (f(x))^2 dx$$

Example 1.1

Let us find the volume of solid of revolution obtained by revolving the area bounded by the curve $y = x^3$ and x-axis between $x = 0$ and $x = 2$ about x-axis. The region is given in figure 1.2. Figure 1.3 presents the solid of revolution.

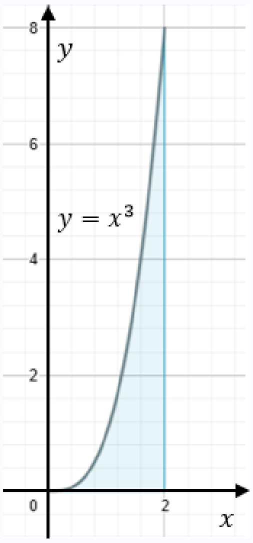

Figure 1.2

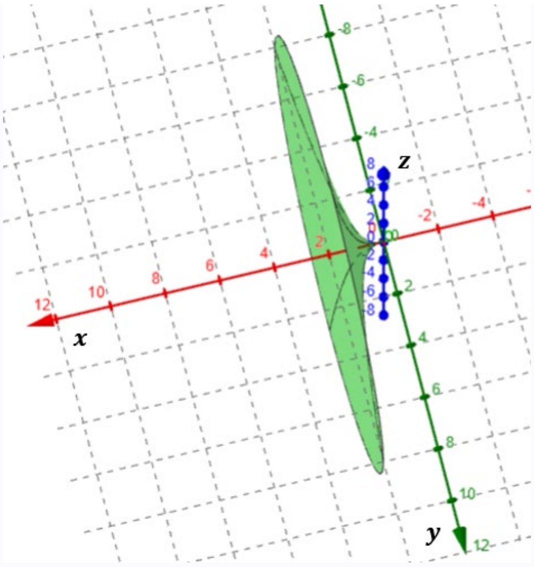

Figure 1.3

The Volume is

$$V_x = \pi \int_0^2 (x^3)^2 dx = \pi \int_0^2 x^6 dx = \pi \frac{x^7}{7}\Big|_0^2 = \frac{2^7 \cdot \pi}{7} = \frac{128\pi}{7} \approx 57.45$$

2. Volume of a solid of revolution generated by two curves

Let us consider two functions $f_1 \leq f_2$ on the interval $[a, b]$ (see figure 2.1).

The volume of the solid formed by rotating the area bounded by two curves $y = f_2(x)$ and $y = f_1(x)$ between $x = a$ and $x = b$ **about the x-axis** (see figure 2.2) is defined as:

$$V_x = \pi \int_a^b [(f_2(x))^2 - (f_1(x))^2] dx$$

Note that due to $f_1(x) \leq f_2(x)$ the curve $y = f_2(x)$ bounds the area on the top and curve $y = f_1(x)$ bounds the area on the bottom.

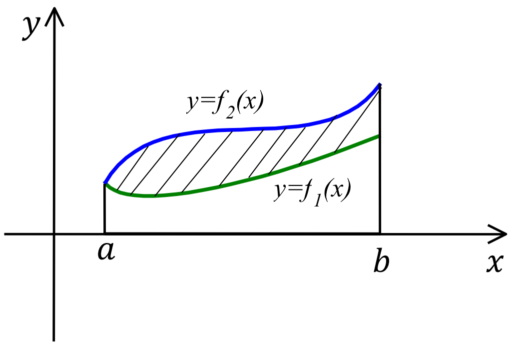

Figure 2.1

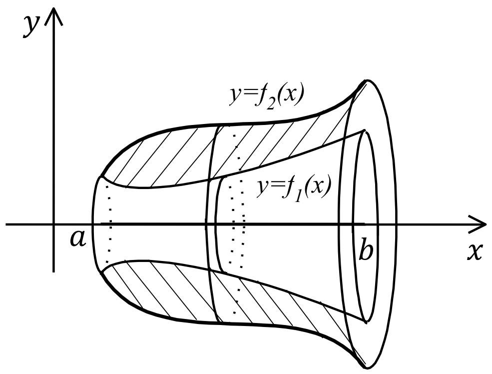

Figure 2.2

Example 2.1

Find the volume of a solid of revolution obtained by rotating the area bounded by the curve $y = x^3$ and the straight line $y = 4x$ about the x-axis.

We sketch the region of the revolution (see figure 2.3) and the solid of revolution (see figure 2.4).

The intersection points of the two lines are $(0,0)$ and $(2,8)$, therefore the area is between $x = 0$, $x = 2$. In this region $x^3 < 4x$. It means that the region is bounded by the line $y = 4x$ on the top and by the curve $y = x^3$ on the bottom.

We use the formula to calculate the volume of a solid:

$$V_x = \pi \int_a^b [(f_2(x))^2 - (f_1(x))^2]dx$$

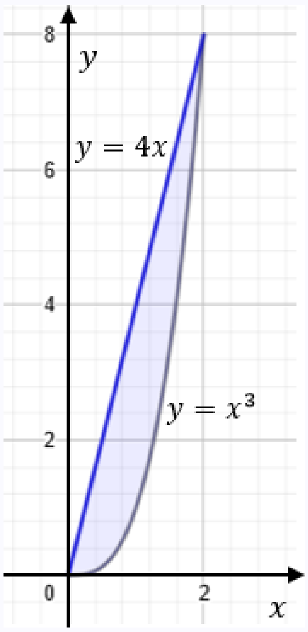

Figure 2.3

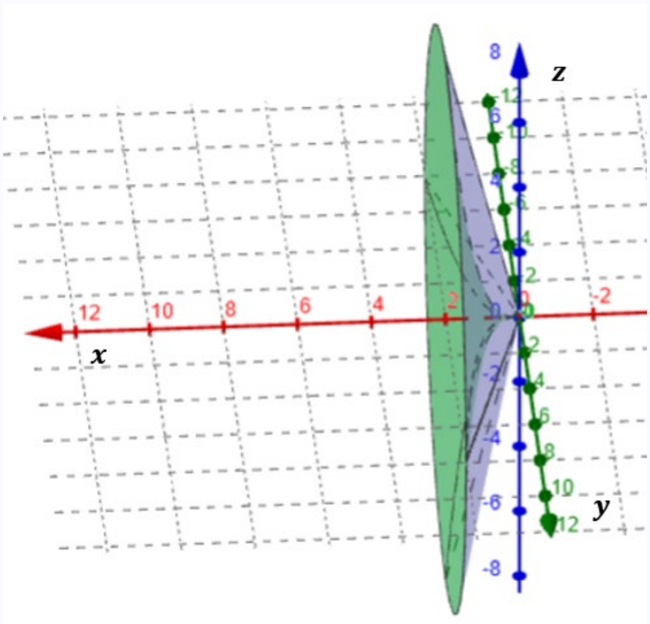

Figure 2.4

For the given case

$$V_x = \pi \int_0^2 [(4x)^2 - (x^3)^2]dx = \pi \int_0^2 [16x^2 - x^6]dx = \pi \left(\frac{16x^3}{3} - \frac{x^7}{7} \right) \Bigg|_0^2$$
$$= \pi \left(\frac{128}{3} - \frac{128}{7} \right) = \frac{512\pi}{21} \approx 76.59$$

3. Rotation about the y-axis

The volume of the solid, when the region bounded by the curve $x = x(y)$ and y-axis between $y = c$ and $y = d$, revolves **about the y-axis** (see figure 3.1) can be found by using the formula:

$$V_y = \pi \int_c^d \left(x(y)\right)^2 dy$$

Here the function $x = x(y)$ is continuous on the interval $[c, d]$.

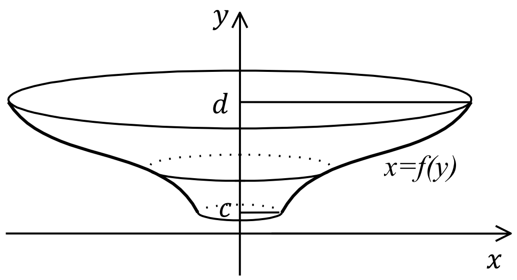

Figure 3.1

In the case when the area located between $y = c$ and $y = d$ and bounded by the curve $x = x_2(y)$ on the right side and by the curve $x = x_1(y)$ on the left side of the area, revolves **about the y-axis**, the volume of the obtained solid of revolution is calculated by the formula

$$V_y = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2 \right] dy$$

The functions $x = x_1(y)$ and $x = x_2(y)$ are continuous and non-negative on the interval $[c, d]$ and $x_1(y) \leq x_2(y)$ on the interval $y \in [c, d]$.

Example 3.1

We find the volume of the solid obtained by revolving about the **y-axis** the area bounded by the curve $y = x^3$, the line $y = 8$, and the y-axis.

The region is sketched in figure 3.2. The solid is sketched in figure 3.3.

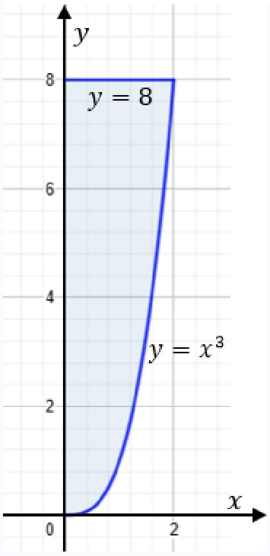

Figure 3.2

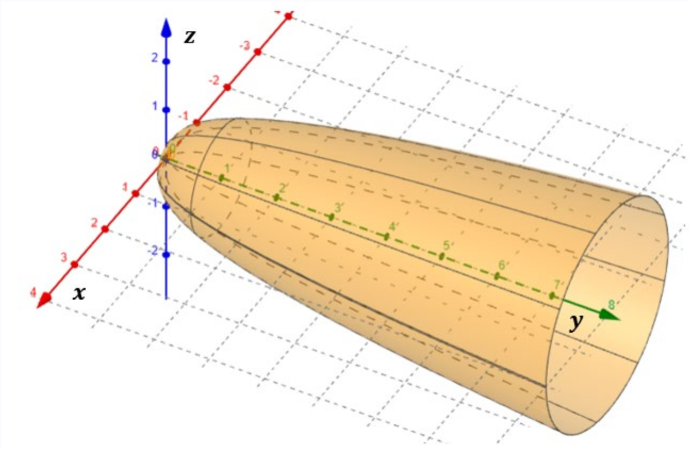

Figure 3.3

We use the formula

$$V_y = \pi \int_c^d \left(x(y)\right)^2 dy$$

From the equation $y = x^3$ we express $x = \sqrt[3]{y}$.

Then

$$V_y = \pi \int_0^8 \left(\sqrt[3]{y}\right)^2 dy = \pi \int_0^8 y^{\frac{2}{3}} dy = \pi \left.\frac{3y^{\frac{5}{3}}}{5}\right|_0^8 = \pi \frac{3\left(\sqrt[3]{8}\right)^5}{5} = \frac{96\pi}{5} \approx 60.32$$

Example 3.2

Let us consider the givens from Example 1.1 (see figure 1.2). Let the region bounded by the functions $y = x^3$ and $x = 2$ revolve around the y-axis (see figure 3.4). We will find the volume of such solid.

The region of the revolution is bounded by the line $x = 2$ on the right side and by the curve $y = x^3$ on the right side, therefore we use the formula

$$V = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2\right] dy$$

To find the interval of integration, we find the value of the function $y = x^3$ at $x = 2$:

$$y(2) = 8$$

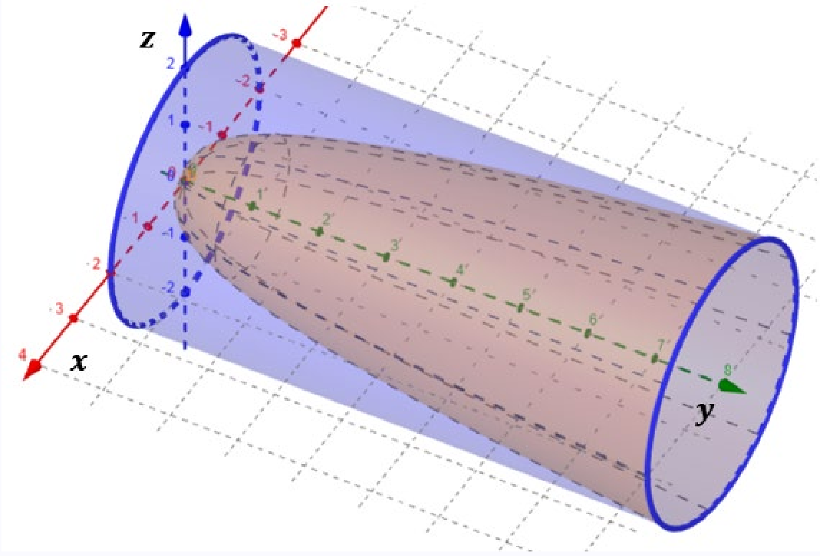

Figure 3.4

From the equation of the curve $y = x^3$ we find $x = \sqrt[3]{y}$.

Then

$$V_y = \pi \int_0^8 \left[(2)^2 - \left(\sqrt[3]{y}\right)^2\right] dy = \pi \int_0^8 \left[4 - y^{\frac{2}{3}}\right] dy = \pi \left(4y - \frac{3y^{\frac{5}{3}}}{5}\right)\Bigg|_0^8 =$$

$$= \pi \left(32 - \frac{3\left(\sqrt[3]{8}\right)^5}{5}\right) = \pi \left(32 - \frac{96}{5}\right) = \frac{64\pi}{5} \approx 40.2$$

Example 3.3

Let us consider the region from example 2.1 (see figure 2.3) now rotating about the y-axis. The area is bounded by the curve $y = x^3$ and the straight line $y = 4x$. To find the volume of the solid of revolution obtained by rotating this region about the y-axis (see figure 3.5), we will use the same formula as in the previous example. The region is bounded by $x = y/4$ and $x = \sqrt[3]{y}$. The variable y belongs to the interval $[0,8]$. The volume is

$$V_y = \pi \int_0^8 \left[\left(\sqrt[3]{y}\right)^2 - \left(\frac{y}{4}\right)^2\right] dy = \pi \int_0^8 \left[y^{\frac{2}{3}} - \frac{y^2}{16}\right] dy = \pi \left(\frac{3y^{\frac{5}{3}}}{5} - \frac{y^3}{16 \cdot 3}\right)\Bigg|_0^8 =$$

$$= \pi \left(\frac{3\left(\sqrt[3]{8}\right)^5}{5} - \frac{8^3}{48}\right) = \pi \left(\frac{96}{5} - \frac{32}{3}\right) = \frac{128\pi}{15} \approx 26.8$$

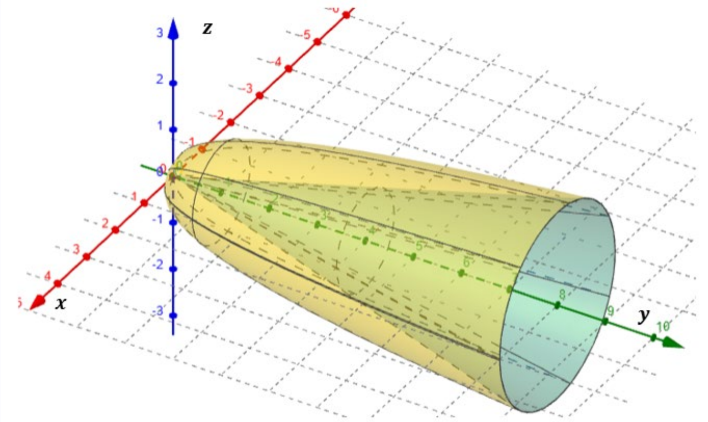

Figure 3.5

4. Revolution of parametrically given curves

1)Let the plane area be bounded by the line defined in parametric form $\boldsymbol{x = x(t)}$, $\boldsymbol{y = y(t)}$ and by the lines $\boldsymbol{x = a, x = b, y = 0}$.
If the corresponding values of the parameter t to the variable $\boldsymbol{x} \in [a, b]$ belong to the interval $[t_1, t_2]$, then the volume of the solid of revolution **around the x-axis** is calculated by the formula:

$$\boldsymbol{V_x = \pi \int_{t_1}^{t_2} \big(y(t)\big)^2 x'(t) dt}$$

2) In the case when the plane area, bounded by the line given in parametric form $\boldsymbol{x = x(t)}$, $\boldsymbol{y = y(t)}$ and by the lines $\boldsymbol{y = c}$, $\boldsymbol{y = d}$, $\boldsymbol{x = 0}$, revolves **about the y-axis**, the variable $\boldsymbol{y} \in [\boldsymbol{c}, \boldsymbol{d}]$, and corresponding parameter $\boldsymbol{t} \in [\boldsymbol{t_1}, \boldsymbol{t_2}]$, the volume of the solid of revolution is calculated by using the following formula:

$$\boldsymbol{V_y = \pi \int_{t_1}^{t_2} \big(x(t)\big)^2 y'(t) dt}$$

Example 4.1

The plane area is bounded by quarter of an ellipse presented in parametric form

$$\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

and by lines $x = 0$, $y = 0$ (see figure 4.1). Find the volume of the solid of revolution obtained by rotating this region a) about the x-axis; b) about the y-axis.

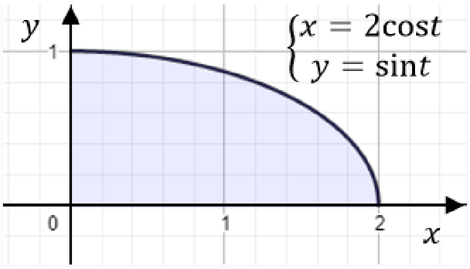

Figure 4.1

1) The region revolves about the x-axis (see figure 4.2):

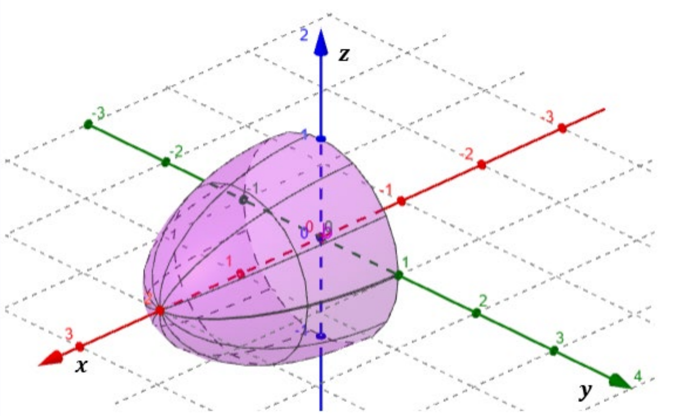

Figure 4.2

We will find the values of parameter t corresponding to the endpoints of a projection of quarter of ellipse on the x-axis, that is, $x = 0$ and $x = 2$:

If $x = 0$ then $\cos t = 0$ and $t = \pi/2$.

If $x = 2$ then $2\cos t = 2$ and $t = 0$.

We find the volume of the solid of revolution obtained when the area revolves about the x-axis by using the formula:

$$V_x = \pi \int_{t_1}^{t_2} \left(y(t)\right)^2 x'(t)dt$$

We differentiate the function $x = 2cost$ with respect to the argument t

$$x' = (2\cos t)' = -2\sin t$$

The volume of the solid is

$$V_x = \pi \int_{\frac{\pi}{2}}^{0} (\sin t)^2(-2\sin t)dt = -2\pi \int_{\frac{\pi}{2}}^{0} \sin^3 t dt = 2\pi \int_{0}^{\frac{\pi}{2}} \sin^2 t \cdot \sin t dt =$$

$$= -2\pi \int_0^{\frac{\pi}{2}} (1 - \cos^2 t) d(\cos t) = -2\pi \left(\cos t - \frac{\cos^3 t}{3} \right) \Bigg|_0^{\frac{\pi}{2}} =$$

$$= -2\pi \left(\cos\frac{\pi}{2} - \frac{\cos^3 \frac{\pi}{2}}{3} \right) + 2\pi \left(\cos 0 - \frac{\cos^3 0}{3} \right) = 2\pi \left(1 - \frac{1}{3} \right) = \frac{4\pi}{3}$$

2) The region revolves about the y-axis (see figure 4.3)

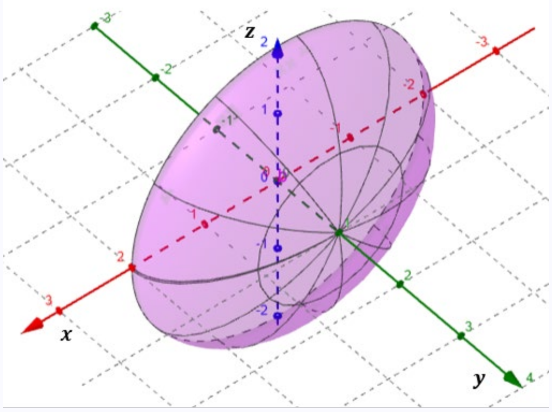

Figure 4.3

We find the values of parameter t that correspond to $y = 0$ and $y = 1$.

If $y = 0$ then $\sin t = 0$ and $t = 0$.

If $y = 3$ then $\sin t = 1$ and $t = \pi/2$.

We find the volume of the solid of revolution obtained when the area revolves about the y-axis by using the formula:

$$V_y = \pi \int_{t_1}^{t_2} \left(x(t)\right)^2 y'(t) dt$$

We differentiate $y' = (\sin t)' = \cos t$.

$$V_y = \pi \int_0^{\frac{\pi}{2}} (2\cos t)^2 (\cos t) dt = 4\pi \int_0^{\frac{\pi}{2}} \cos^2 t \cdot \cos t \, dt =$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (1 - \sin^2 t) d(\sin t) = 4\pi \left(\sin t - \frac{\sin^3 t}{3} \right) \Bigg|_0^{\frac{\pi}{2}} = 4\pi \left(1 - \frac{1}{3} \right) - 0 = \frac{8\pi}{3}$$

5. Exercises

1. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2 + 1$ and the straight lines $y = -1, x = 1, y = 0$ about the x-axis.

2. Find the volume of the solid obtained by rotating the region bounded by two sine functions $y = 3\sin x$ and $y = \sin x$ between $x = 0$ and $x = \pi$ about the x-axis.

3. Calculate the volume of the solid obtained by revolving about y-axis the area bounded by the hyperbola $xy = 4$ and the lines $y = 1, y = 4$, and $x = 0$.

4. Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = \ln x$ and the lines $y = 0,\ x = e$ about the y-axis.

5. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = \mathrm{x}^2$, the line $x + y = 2$, and $y = 0$ a) about the x-axis; b) about the y-axis.

6. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the part of asteroid given in parametric form $x = cos^3 t$ and $y = 2sin^3 t$ on the interval $t \in [-\frac{\pi}{2}; \frac{\pi}{2}]$.

7. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the part of line given in parametric form $x = 2tant$ and $y = 2cos^2 t,\ x = -2,\ x = 2,\ y = 0$.

6. Solutions

1. Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2 + 1$, straight lines $x = -1, x = 1, y = 0$ about the x-axis.

Solution

The region is bounded by a curve and straight lines (see figure 6.1)

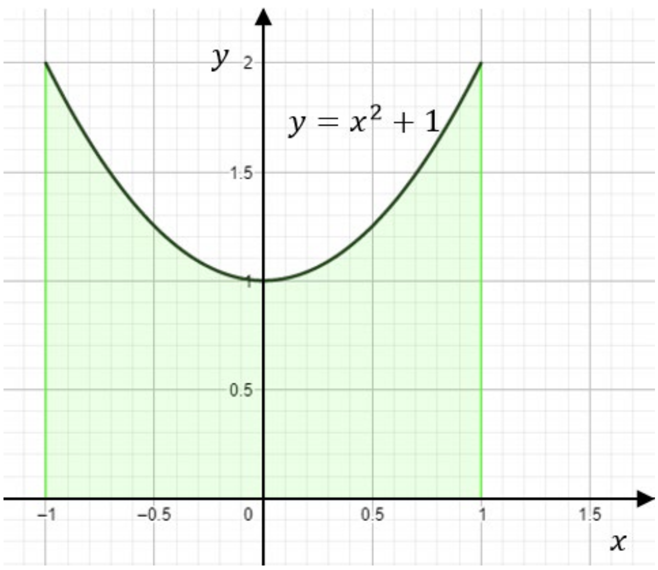

Figure 6.1

As the region is rotating about the x-axis we use the following formula to calculate the volume of the solid of revolution (see figure 6.2):

$$V_x = \pi \int_a^b (f(x))^2 dx$$

Then

$$V = \pi \int_{-1}^{1} (x^2+1)^2 dx = \pi \int_{-1}^{1} (x^4+2x^2+1) dx =$$

$$= \pi \int_{-1}^{1} (x^4+2x^2+1) dx = \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_{-1}^{1} =$$

$$= \pi \left(\frac{1}{5} + \frac{2}{3} + 1 \right) - \pi \left(-\frac{1}{5} - \frac{2}{3} - 1 \right) = \frac{56\pi}{15} \approx 11.73$$

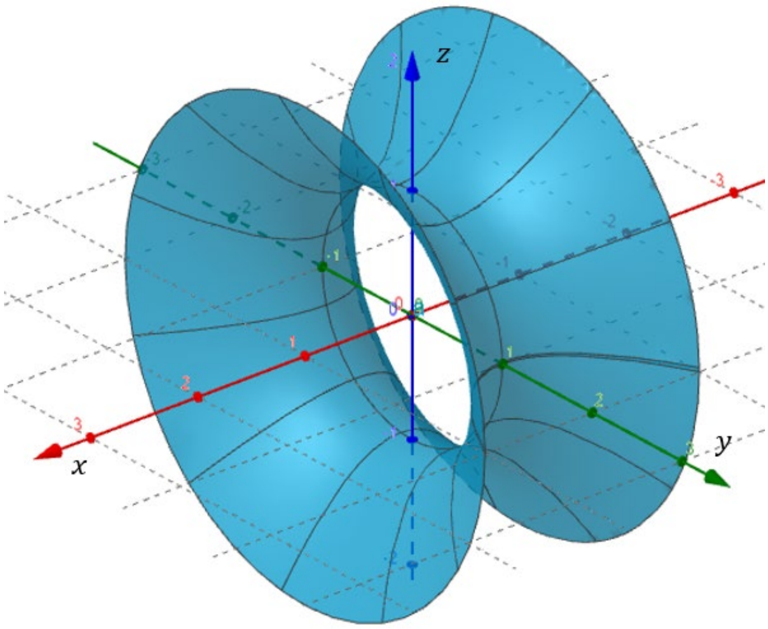

Figure 6.2

2. Find the volume of the solid obtained by rotating the region bounded by two sine functions $y = 3\sin x$ and $y = \sin x$ between $x = 0$ and $x = \pi$ about the x-axis.

Solution

Let us construct the given region (see figure 6.3):

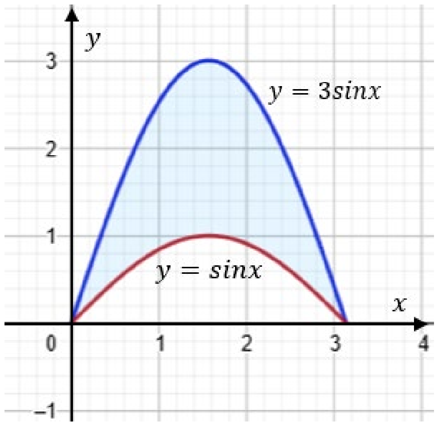

Figure 6.3

The region, revolving about x-axis, is bounded on the top by the curve $y = 3\sin x$ and on the bottom by the curve $y = \sin x$ between $x = 0$ and $x = \pi$. Therefore, to calculate the volume of the solid of revolution (see figure 6.4) we use:

$$V_x = \pi\int_a^b [(f_2(x))^2 - (f_1(x))^2]dx$$

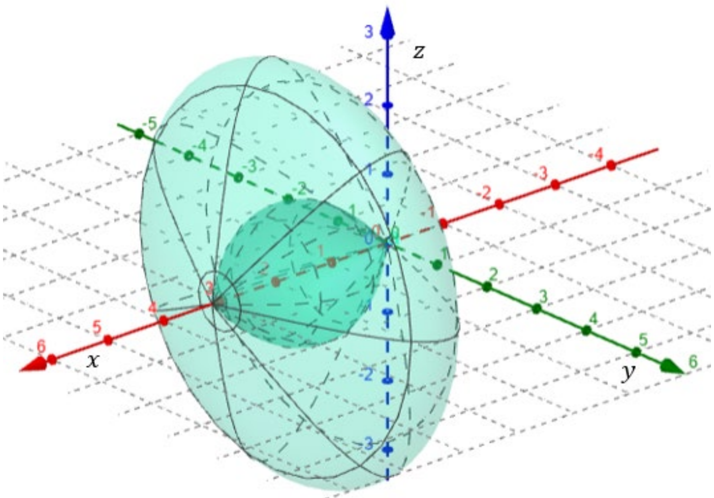

Figure 6.4

$$V_x = \pi\int_0^\pi [(3\sin x)^2 - (\sin x)^2]dx = \pi\int_0^\pi [9\sin^2 x - \sin^2 x]dx = \pi\int_0^\pi 8\sin^2 x dx =$$

$$= 8\pi\int_0^\pi \frac{1}{2}(1 - \cos 2x)dx = 4\pi\int_0^\pi (1 - \cos 2x)dx = 4\pi\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^\pi =$$

$$= 4\pi\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] = 4\pi^2 \approx 39.48$$

3. Calculate the volume of the solid obtained by revolving about the y-axis the area bounded by the hyperbola $xy = 4$ and the lines $y = 1$, $y = 4$, and $x = 0$.

Solution

The region is shown in figure 6.5

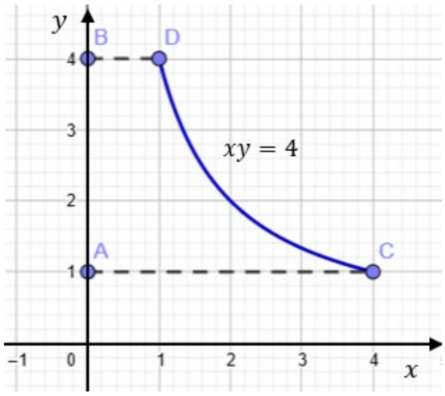

Figure 6.5

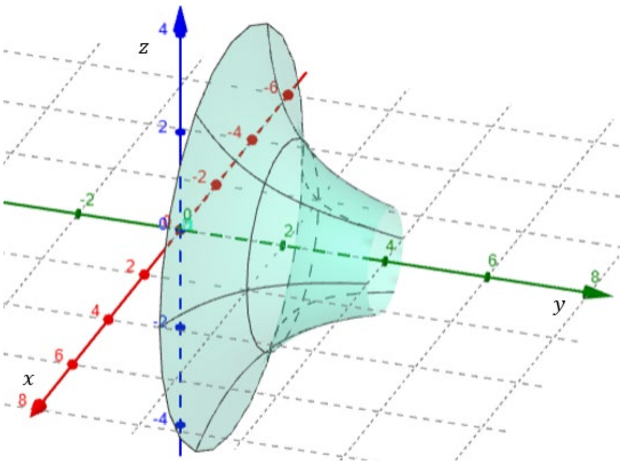

Figure 6.6

The region revolving about the y-axis is bounded on the right by the curve $y = 4/x$ and on the left by the y-axis between $y = 1$ and $= 4$. Therefore, to calculate the volume of the solid of the revolution (see figure 6.6) we use:

$$V_y = \pi \int_c^d (x(y))^2 dy$$

The solution is

$$V_y = \pi \int_1^4 \left(\frac{4}{y}\right)^2 dx = \pi \int_1^4 \frac{16}{y^2} dx = 16\pi \int_1^4 y^{-2} dx = 16\pi \frac{y^{-1}}{-1}\bigg|_1^4 = -16\pi \frac{1}{y}\bigg|_1^4 =$$

$$= -16\pi\left(\frac{1}{4} - 1\right) = 12\pi$$

4. Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = \ln x$ and the lines $y = 0$ and $x = e$ about the y-axis.

Solution

We construct the region:

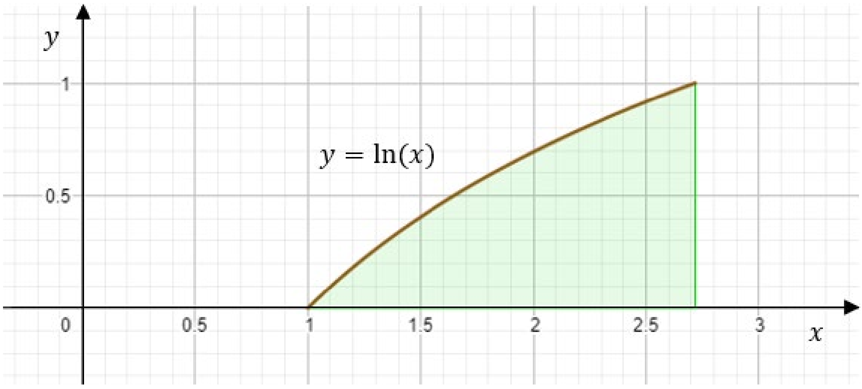

Figure 6.7

The region is bounded by the line $\mathrm{x} = \mathrm{e}$ on the right and by $\mathrm{y} = \ln x$ on the right side of the area. In this case, to calculate the volume of the solid obtained by rotating the region about the y-axis (see figure 6.8), we use the following formula:

$$V_y = \pi\int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2\right] dy$$

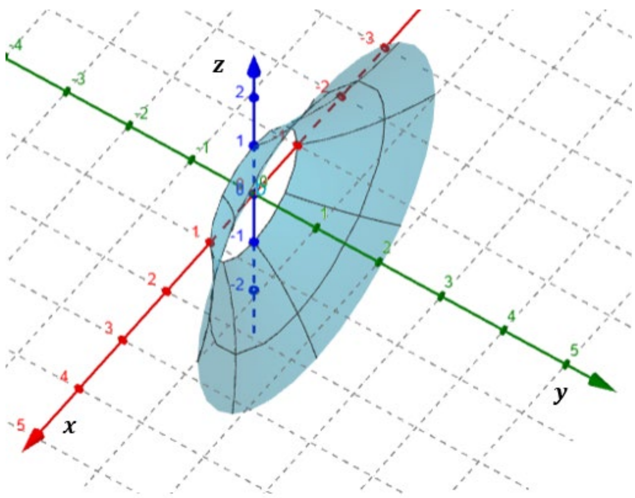

Figure 6.8

To compose an integral, we express the function x with respect to the argument y, that is, $x = e^y$

$$V_y = \pi \int_0^1 [e^2 - (e^y)^2]dy = \pi \int_0^1 [e^2 - e^{2y}]dy = \pi \left(e^2 y - \frac{1}{2}e^{2y}\right)\Big|_0^1$$
$$= \pi\left(e^2 - \frac{1}{2}e^2 - \left(0 - \frac{1}{2}\right)\right) = \frac{\pi}{2}(e^2+1) \approx 13.18$$

5. Calculate the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x + y = 2$, and $y = 0$ a) about the x-axis; b) about the y-axis.

Solution

The region is shown in figure 6.9.

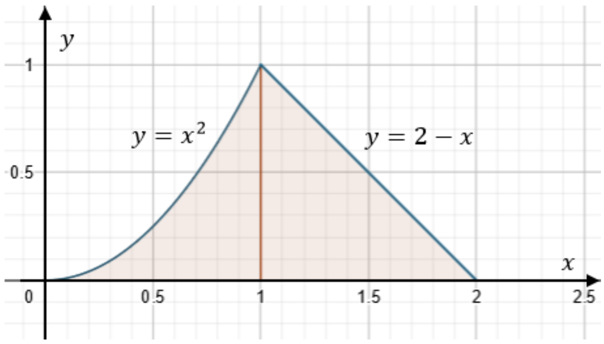

Figure 6.9

a) The region revolves about the x-axis (see figure 6.10).

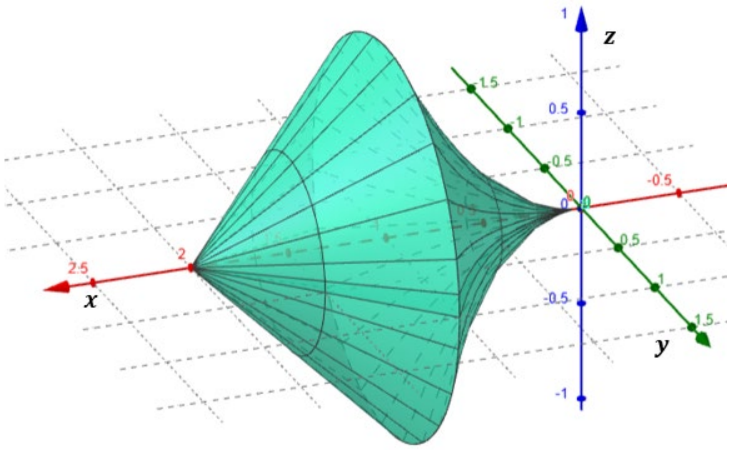

Figure 6.10

In this case, the volume of the obtained solid of revolution is equal to the sum of volumes of two solids V_1 and V_2

$$V_x = V_{x_1} + V_{x_2},$$

where V_{x_1} is the volume of the solid generated by the parabola $y = x^2$ rotated around the x-axis on the interval $0 \le x \le 1$ and V_{x_2} is the volume of the solid generated by the line $x + y = 2$ rotated around the x-axis on the interval $1 \le x \le 2$.

Therefore,

$$V_x = V_{x_1} + V_{x_2} = \pi \int_0^1 (x^2)^2\, dx + \pi \int_1^2 (2-x)^2\, dx = \pi \int_0^1 x^4\, dx + \pi \int_1^2 (4-4x+x^2)\, dx$$

$$= \pi \frac{x^5}{5}\Big|_0^1 + \pi \left(4x - 2x^2 + \frac{x^3}{3}\right)\Big|_1^2 = \frac{\pi}{5} + \pi\left(8 - 8 + \frac{8}{3}\right) - \pi\left(4 - 2 + \frac{1}{3}\right) = \frac{8\pi}{15} \approx 1.68$$

b) The region revolves about the y-axis (see figure 6.11).

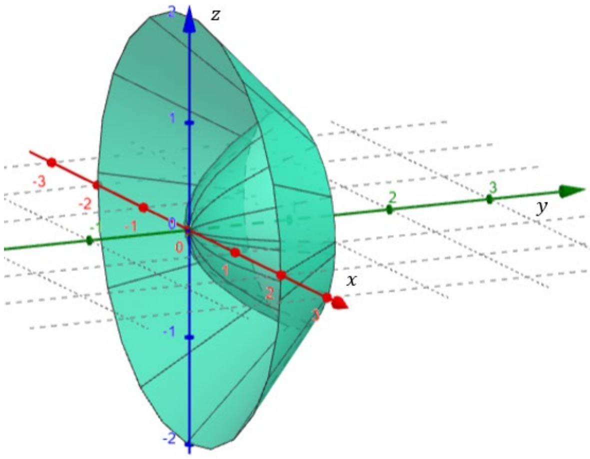

Figure 6.11

The region is located between $y = 0$ and $y = 1$ and is bounded by the line $x + y = 2$ on the right. On the left the parabola $y = x^2$ bounds the region. We express the functions in terms of the argument y:

$$\text{from } x + y = 2 \quad \text{follows} \quad x = 2 - y,$$

$$\text{from } y = x^2 \quad \text{follows} \quad x = \sqrt{y}.$$

Then

$$V_y = \pi \int_c^d \left[\left(x_2(y)\right)^2 - \left(x_1(y)\right)^2\right] dy =$$

$$= \pi \int_0^1 \left[(2-y)^2 - \left(\sqrt{y}\right)^2\right] dy = \pi \int_0^1 [4 - 4y + y^2 - y]dy =$$

$$= \pi \int_0^1 [4 - 5y + y^2]dy = \pi\left(4y - \frac{5y^2}{2} + \frac{y^3}{3}\right)\Big|_0^1 = \pi\left(4 - \frac{5}{2} + \frac{1}{3}\right) = \frac{11\pi}{6} \approx 5.76$$

6. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the part of asteroid given in parametric form $x = \cos^3 t$ and $y = 2\sin^3 t$, $t \in [-\frac{\pi}{2}; \frac{\pi}{2}]$.

Solution

The region is shown in figure 6.12. We apply the following formula to calculate the volume of the solid (see figure 6.13)

$$V_y = \pi \int_{t_1}^{t_2} \left(x(t)\right)^2 y'(t) dt$$

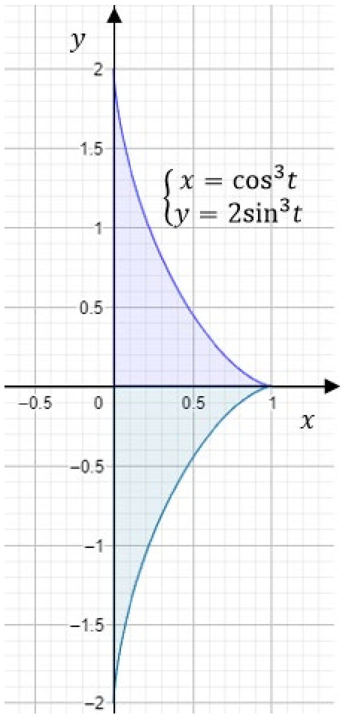

Figure 6.12

To create an integral for calculation of the volume of the solid (see figure 6.13) we need to find the derivative of the function y with respect to the argument t

$$y' = (2\sin^3 t)' = 6\sin^2 t \cos t$$

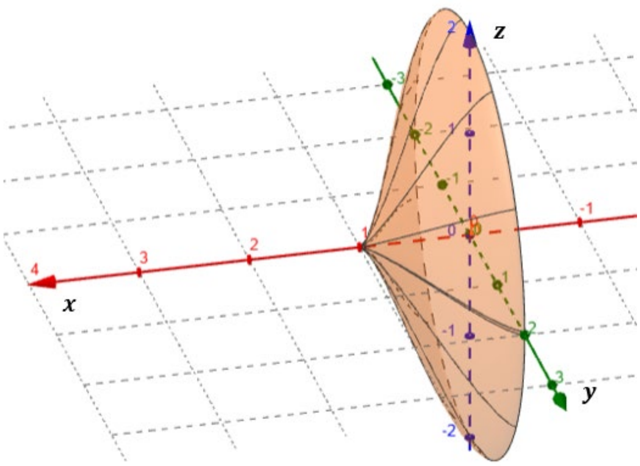

Figure 6.13

$$V_y = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^3 t)^2 (6\sin^2 t \cos t) dt = 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 t \sin^2 t \cdot \cos t dt =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 t)^3 \sin^2 t \cdot d(\sin t) =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - 3\sin^2 t + 3\sin^4 t - \sin^6 t) \cdot \sin^2 t \, d(\sin t) =$$

$$= 6\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 t - 3\sin^4 t + 3\sin^6 t - \sin^8 t) d(\sin t) =$$

$$= 6\pi \left(\frac{\sin^3 t}{3} - \frac{3\sin^5 t}{5} + \frac{3\sin^7 t}{7} - \frac{\sin^9 t}{9} \right) \Bigg|_0^{\frac{\pi}{2}} = 6\pi \left(\frac{1}{3} - \frac{3}{5} + \frac{3}{7} - \frac{1}{9} \right) - 0 = \frac{32\pi}{105} \approx 0.96$$

7. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the part of line given in parametric form

$$\begin{cases} x = 2\tan t \\ y = 2\cos^2 t \end{cases}; \; x = -2; \;\; x = 2; \;\; y = 0.$$

Solution

The area is shown in figure 6.14. The solid of revolution is presented in figure 6.15.

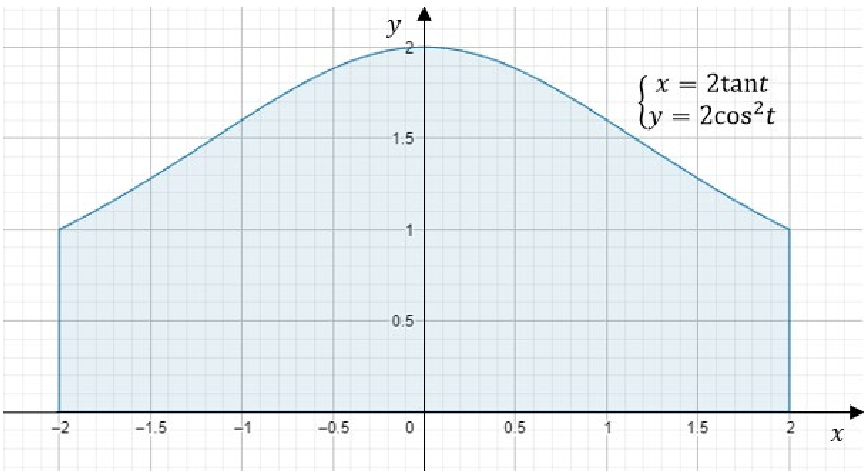

Figure 6.14

The given area is symmetric according to the y-axis, therefore we will find the volume for the interval $0 \le x \le 2$ and multiply the result by 2:

$$V_x = 2 \cdot V_{x_1}$$

where V_{x_1} is volume for $0 \le x \le 2$. We use the formula

$$V_x = \pi \int_{t_1}^{t_2} \left(y(t)\right)^2 x'(t) dt$$

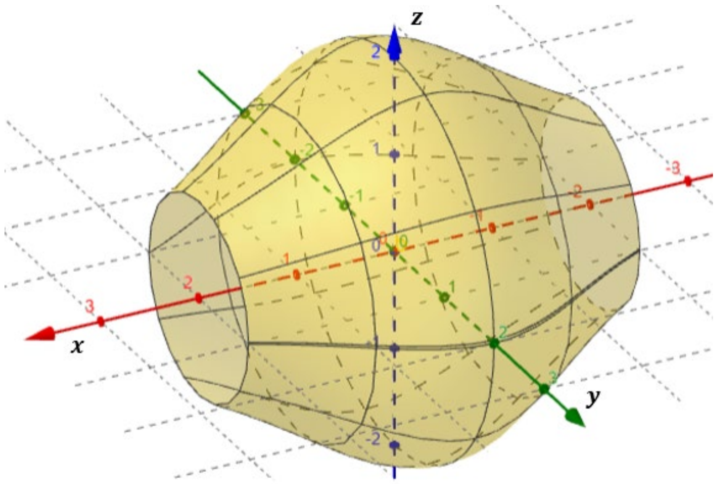

Figure 6.15

Let us recall the domain of tangent function $x \in \mathbb{R}$, $x \neq \frac{\pi}{2} + \pi n$, where n is an integer. For the given case it is necessary to have the tangent limits within interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where the tangent function is continuous. The values of a parameter t corresponding to $x = 0$ and $x = 2$ are the following:

If $x = 0$ then $2\tan t = 0$ and $t = 0$.

If $x = 2$ then $\tan t = 1$ and $t = \pi/4$.

So we get that the parameter t belongs to the interval that is the subinterval of the domain of a tangent function $t \in [0, \frac{\pi}{4}] \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We differentiate

$$x'(t) = (2\tan t)' = \frac{2}{\cos^2 t}$$

Then

$$V_{x_1} = \pi \int_0^{\frac{\pi}{4}} (2\cos^2 t)^2 \cdot \frac{2}{\cos^2 t} dt = 8\pi \int_0^{\frac{\pi}{4}} \cos^4 t \cdot \frac{1}{\cos^2 t} dt =$$

$$= 8\pi \int_0^{\frac{\pi}{4}} \cos^2 t \, dt = 8\pi \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2t}{2} dt =$$

$$= 4\pi \int_0^{\frac{\pi}{4}} (1 + \cos 2t) dt = 4\pi \left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{4}} = 4\pi\left(\frac{\pi}{4} - \frac{1}{2}\right) = \pi^2 - 2\pi \approx 3.59$$

$$V_x = 2V_{x_1} = 2\pi(\pi - 2) \approx 7.17$$