THE ELLIPSE

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. **Figure 1** illustrates the four conic sections: the circle, the ellipse, the parabola, and the hyperbola.

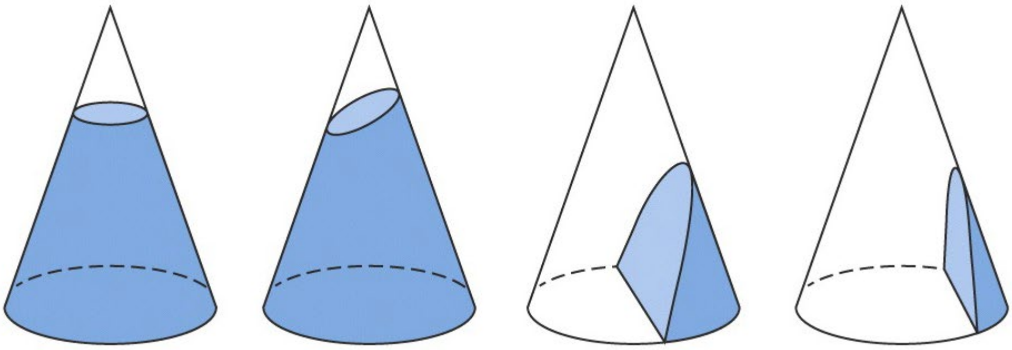

Figure 1 Obtaining the conic sections by intersecting a plane and a cone

OBJECTIVES AND OUTCOMES:

- Graph ellipses centred at the origin.
- Write equations of ellipses in standard form.
- Rewrite the equation of an ellipse in standard form.
- Graph an ellipse not centred at the origin.
- Solve application with ellipses.

CONTENTS

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In this section, we study the symmetric oval-shaped curve known as the ellipse. In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. We will use a geometric definition for an ellipse to derive its equation.

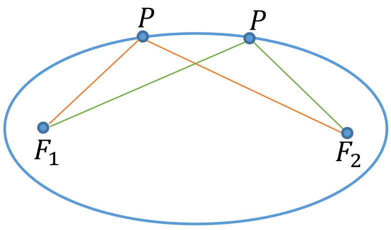

Figure 2 Ellipse

Definition: An ellipse is the set of all points P, in a plane the sum of whose distances from two fixed points, F_1 and F_1 is constant (see Figure 1). These two fixed points are called the foci (plural of focus). The midpoint of the segment connecting the foci is the center of the ellipse.

Figure 3 illustrates that an ellipse can be elongated in any direction. In this section, we will limit our discussion to ellipses that are elongated horizontally or vertically. The line through the foci intersects the ellipse at two points, called the **vertices** (singular: **vertex**). The line segment that joins the vertices is the **major axis**. Notice that the midpoint of the major axis is the center of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis at the center is called the **minor axis** of the ellipse.

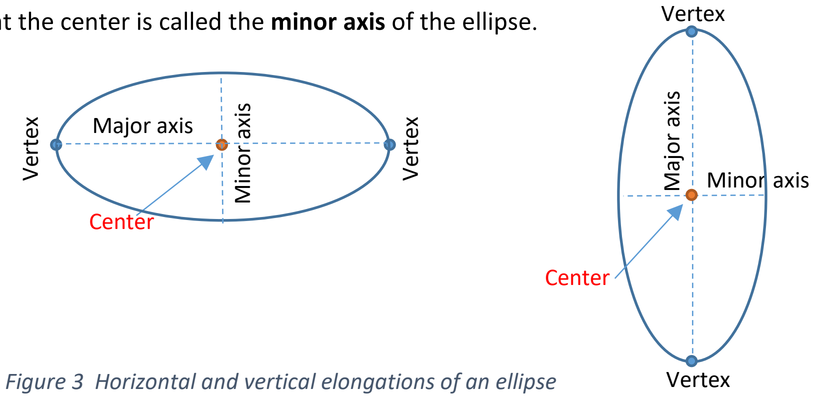

Figure 3 Horizontal and vertical elongations of an ellipse

STANDARD FORM OF THE EQUATION OF AN ELLIPSE

The rectangular coordinate system gives us a unique way of describing an ellipse. It enables us to translate an ellipse's geometric definition into an algebraic equation.

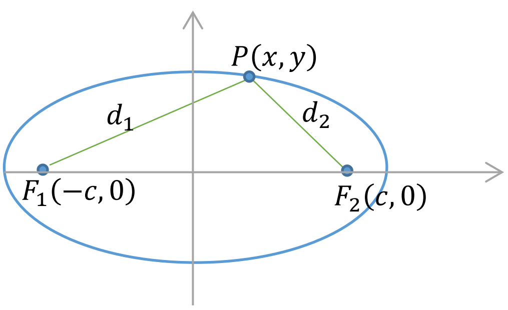

Figure 4 Ellipse

We start with **Figure 4** to obtain an ellipse's equation. We have placed an ellipse that is elongated horizontally into a rectangular coordinate system. The foci are on the $x - axis$ at $(-c, 0)$ and $(c, 0)$, as in **Figure 4.** In this way, the center of the ellipse is at the origin. We let represent the coordinates of any point on the ellipse. What does the definition of an ellipse tell us about the point in **Figure 4**?

For any point (x, y) on the ellipse, the sum of the distances to the two foci, $d_1 + d_2$, must be constant. As we shall see, it is convenient to denote this constant by $2a$. Thus, the point (x, y) is on the ellipse if and only if

$$d_1 + d_2, = 2a.$$

Use the distance formula

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

After eliminating radicals and simplifying, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Look at the triangle in **Figure 4.** Notice that the distance from F_1 to F_2 is $2c$. Because the length of any side of a triangle is less than the sum of the lengths of the other two sides, $2c < d_1 + d_{2.}$ Equivalently, $2c < 2a$ and $c < a$. Consequently, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Substituting b^2 for $a^2 - c^2$ in the preceding equation, we obtain

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}.$$

Then, simplifying, we get

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}.$$

This last equation is the **standard form of the equation of an ellipse centred at the origin**. There are two such equations, one for a horizontal major axis and one for a vertical major axis.

STANDARD FORMS OF THE EQUATIONS OF AN ELLIPSE

The **standard form of the equation of an ellipse** with center at the origin, and major and minor axes of lengths $2a$ and $2b$ (where and are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Figure 5 illustrates that the vertices are on the major axis, a units from the center. The foci are on the major axis, c units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$.

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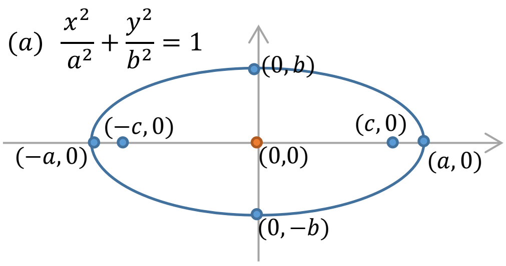

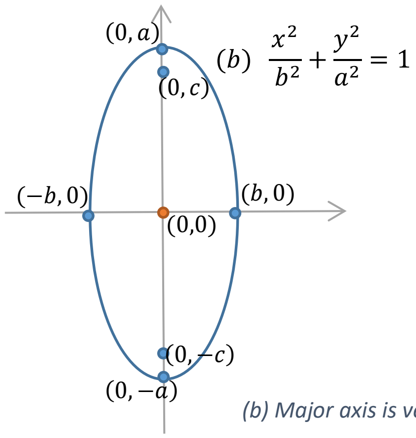

Figure 5 (a) Major axis is horizontal with length 2a; (b) Major axis is vertical with length 2a

The intercepts shown in **Figure 5 (a)** can be obtained algebraically. Let us do this for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\boldsymbol{x}$ **–intercepts: Set** $\boldsymbol{y=0}$	$\boldsymbol{y}$ **–intercepts: Set** $\boldsymbol{x=0}$
$$\frac{x^2}{a^2}=1$$ $$x^2=a^2$$ $$x=\pm a.$$ x –intercepts are $-a$ and a.The graph passes through $(-a,0)$ and $(a,0)$, which are the vertices.	$$\frac{y^2}{b^2}=1$$ $$y^2=b^2$$ $$y=\pm b.$$ y –intercepts are $-b$ and b.The graph passes through $(0,-b)$ and $(0,\mathrm{b})$.

USING THE STANDARD FORM OF THE EQUATION OF AN ELLIPSE

We can use the standard form of an ellipse's equation to graph the ellipse. Although the definition of the ellipse is given in terms of its foci, the foci are not part of the graph. A complete graph of an ellipse can be obtained without graphing the foci.

EXAMPLE 1 GRAPHING AN ELLIPSE CENTERED AT THE ORIGIN

Graph and locate the foci: $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Solution: The given equation is the standard form of an ellipse's equation with $a^2=9$ (the larger of the two denominators) and $b^2=4$ (the smaller of the two denominators). Because the denominator of the x^2 –term is greater than the denominator of the y^2 –term, the major axis is horizontal.

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- Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 9, \mathrm{a} = 3$. Thus, the vertices are $(-3,0)$ and $(3,0)$, shown in Figure 6.
- Now let us find the endpoints of the vertical minor axis. According to the standard form of the equation, these endpoints are $(0, -b)$ and $(0, b)$. Because $b^2 = 4$, $\mathrm{b} = 2$. Thus, the endpoints of the minor axis are $(0, -2)$ and $(0,2)$. They are shown in Figure 6.

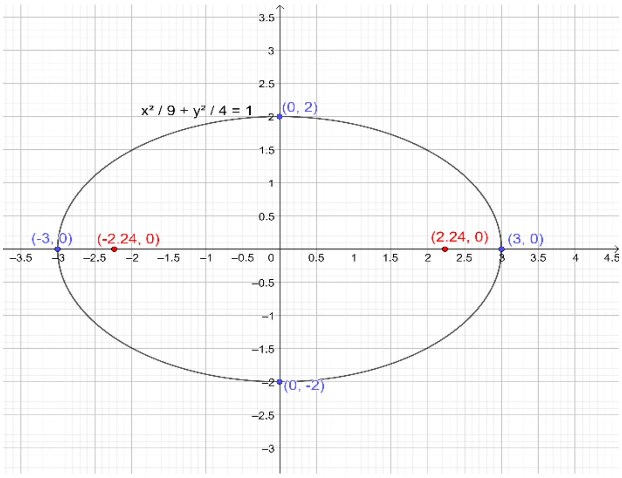

Figure 6 The graph of $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$

- Finally, we find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 9$ and $b^2 = 4$. Thus,

$$c^2 = a^2 - b^2 = 9 - 4 = 5.$$

- Because $c^2 = 5, c = \sqrt{5}$. The foci, $(-c, 0)$ and $(c, 0)$ are located at $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$. They are shown in Figure 6.
- You can sketch the ellipse in **Figure 6** by locating endpoints on the major and minor axes.

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1.$$

Endpoints of the major axis are 3 units to the right and left of the center. Endpoints of the minor axis are 2 units up and down from the center.

EXAPLE 2 GRAPHING AN ELLIPSE CENTERED AT THE ORIGIN

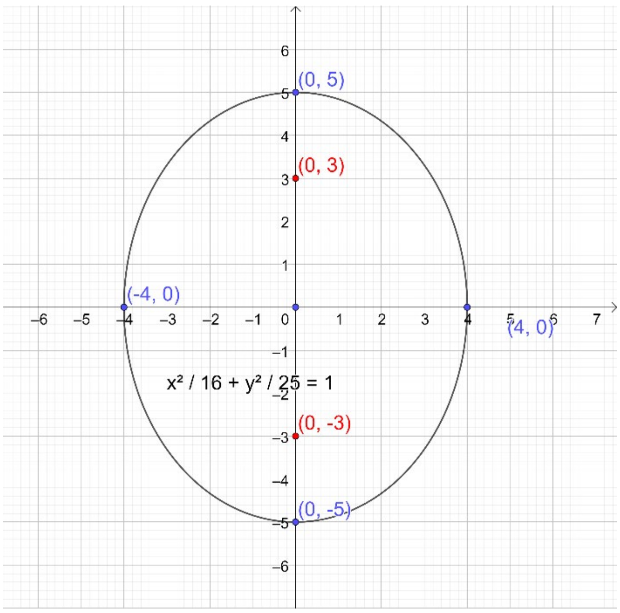

Figure 7 The graph of $25x^2 + 16y^2 = 400.$

Graph and locate the foci: $25x^2 + 16y^2 = 400.$

Solution: We begin by expressing the equation in standard form. Because we want 1 on the right side, we divide both sides by 400.

$$\frac{25x^2}{400} + \frac{16y^2}{400} = \frac{400}{400};$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1.$$

$b^2 = 16$ – this is the smaller of the two denominators; $a^2 = 25$ – this is the larger of the two denominators.

- The equation is the standard form of an ellipse's equation with $a^2 = 25$ and $b^2 = 16$. Because the denominator of the $y^2 -$ term is greater than the denominator of the $x^2 -$ term, the major axis is vertical. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 25$, $a = 5$. Thus, the vertices are $(0, -5)$ and $(0, 5)$, shown in Figure 8.

Now let us find the endpoints of the horizontal minor axis. According to the standard form of the equation, these endpoints are $(-b, 0)$ and $(b, 0)$. Because $b^2 = 16$, $b = 4$. Thus, the endpoints of the minor axis are $(-4, 0)$ and $(4, 0)$. They are shown in Figure 8.

- Finally, we find the foci, which are located at $(0, -c)$ and $(0, c)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 25$ and $b^2 = 16$. Thus,
- $c^2 = a^2 - b^2 = 25 - 16 = 9.$
- Because $c^2 = 9, c = 3$. The foci, $(0, -c)$ and $(0, c)$ are located at $(0, -3)$ and $(0,3)$. They are shown in **Figure 7**.
- You can sketch the ellipse in Figure 7 by locating endpoints on the major and minor axes.

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1.$$

Endpoints of the minor axis are 4 units to the right and left of the center. Endpoints of the major axis are 5 units up and down from the center.

EXAPLE 3 FINDING THE EQUATION OF AN ELLIPSE FROM ITS FOCI AND VERTICES

Find the standard form of the equation of an ellipse with foci at $(-1,0)$ and $(1,0)$ and vertices $(-2,0)$ and $(2,0)$.

Solution: Because the foci are located at $(-1,0)$ and $(1, 0)$, on the x-axis, the major axis is horizontal. The center of the ellipse is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, ti either vertex, $(-2,0)$ or $(2,0)$, is 2. Thus, $a = 2$.

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1 \quad or \quad \frac{x^2}{4} + \frac{y^2}{b^2} = 1.$$

We must still find b^2. The distance from the center, $(0, 0)$, to either foci $(-1,0)$ or $(1,0)$, is 1, so $c = 1$. Using $c^2 = a^2 - b^2$, we have

$$1^2 = 2^2 - b^2$$

and

$$b^2 = 2^2 - 1^2 = 3.$$

Substituting 3 for b^2 in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives us the standard form of the ellipse's equation

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

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TRANSLATIONS OF ELLIPSES

Horizontal and vertical translations can be used to graph ellipses that are not centered at the origin. Figure 8 illustrates that the graphs of

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Have the same size and shape. However, the graph of the first equation is centered at (h,k) rather than at the origin.

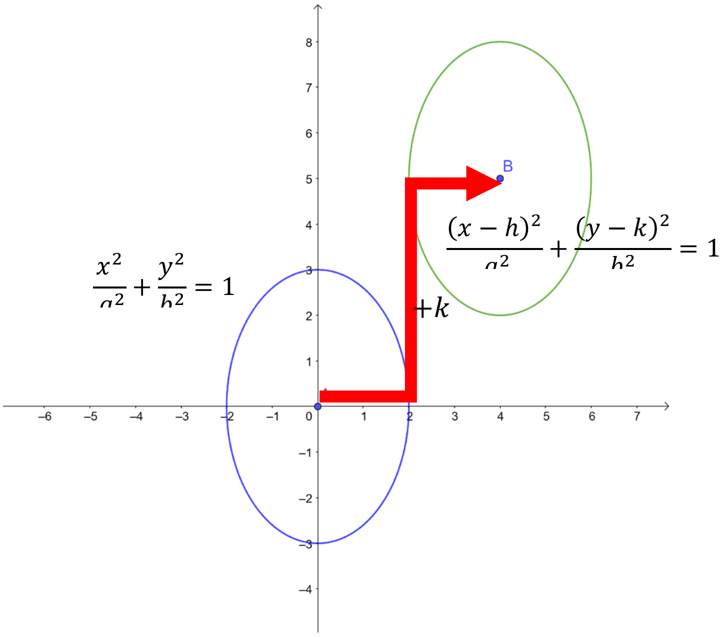


Figure 8 Translations an ellipse's graph

Table 1 gives the standard forms of equations of ellipses centered at $(h,k)$and shows their graphs.

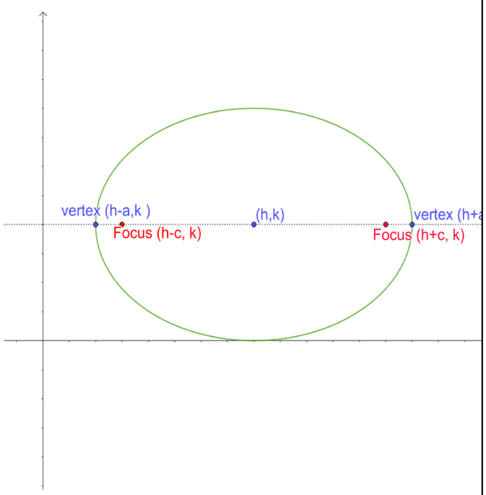
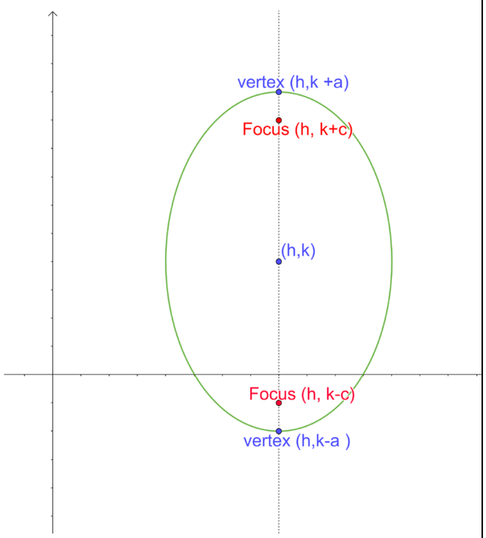
EQUATION	CENTER	GRAPH
Equation: $$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ $$a^2>b^2$$ Foci are c units right and c units left of center, where $c^2=a^2-b^2$.	Center:(h,k) **Major axis**: Parallel to the x-axis, horizontal **Vertices**: $(h-a,k)$ $(h+a,k)$	
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$ $$a^2>b^2$$ Foci are c units above and c units below of center, where $c^2=a^2-b^2$.	Center: (h,k) **Major axis**: Parallel to the y-axis, vertical **Vertices**: $(h,k-a)$ $(h,k+a)$	

Table 1 Standard Forms of Equations of Ellipses Centered at (h,k)

EXAMPLE 4 GRAPHING AN ELLIPSE CENTERED AT (h, k)

Graph:

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

Where are the foci located?

Solution: To graph the ellipse, we need to know its center, (h, k). In the standard forms of equations centered at (h, k) is the number subtracted from x and is the number subtracted from y.

$$\frac{(x-1)^2}{4}+\frac{\left(y-(-2)\right)^2}{9}=1.$$

We see that $h=1$ and $k=-2$. Thus, the center of the ellipse, (h, k) is $(1,-2)$. We can graph the ellipse by locating endpoints on the major and minor axes. To do this, we must identify a^2 and b^2.

$$b^2=4-\ \ \textit{the smaller of the two deniminators};$$

$$a^2=9-\ \ \textit{the larger of the two deniminators}.$$

The larger number is under the expression involving y. This means that the major axis is vertical and parallel to the y –axis.

We can sketch the ellipse by locating endpoints on the major and minor axes.

$$\frac{(x-1)^2}{2^2}+\frac{(y+2)^2}{3^2}=1.$$

We categorize the observations in the textboxes above as follows:

For a Vertical Major Axis with Center $(1,-2)$	
Vertices	Endpoints of Minor Axis
$(1,-2+3)=(1,1)$	$(1+2,-2)=(3,1)$
$(1,-2-3)=(1,-5)$	$(1-2,-2)=(-1,1)$

Using the center and these four points, we can sketch the ellipse shown in **Figure 9**. With $c^2=a^2-b^2$ we have $c^2=9-4=5$. So the foci are located $\sqrt{5}$ units above and below the center, at $\left(1,-2+\sqrt{5}\right)$ and $\left(1,-2-\sqrt{5}\right)$.

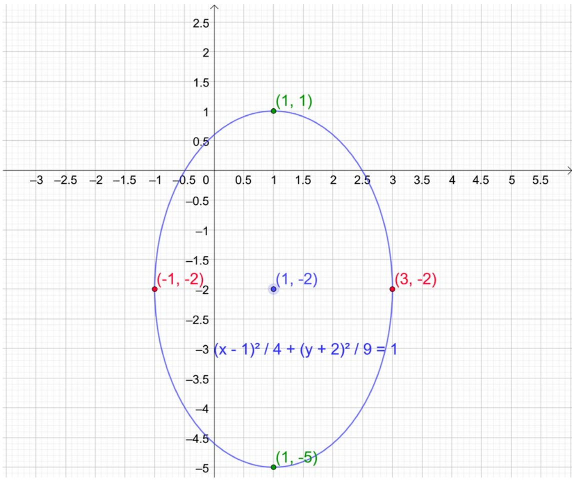


Figure 9 The graph of an ellipse centered at (1, -2)

REMARK

In some cases, it is necessary to convert the equation of an ellipse to standard form by completing the square on x and y. For example, suppose that we wish to graph the ellipse whose equation is

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

Because we plan to complete the square on both on x and y, we need to rearrange terms so that

- x −terms are arranged in descending order;
- y −terms are arranged in descending order;
- The constant term appears on the right.

This is the given equation

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0$$

Group terms and add 11 to both sides

$$9x^2 - 18x + 4y^2 + 16y = 11$$

To complete the square, coefficients of $x^2$and y^2 must be 1. Factor out 9 and 4, respectively

$$9(x^2 - 2x + ?) + 4(y^2 + 4y + ?) = 11.$$

Complete each square by adding the square of half the coefficient of x and y, respectively.

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16.$$

Factor

$$9(x-1)^2 + 4(y+2)^2 \;=\; 36.$$

Divide both sides by 36

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} \;=\; \frac{36}{36}.$$

Simplifying

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} \;=\; 1.$$

The equation is now in standard form. This is precisely the form of the equation that we graphed in Example 4.

APPLICATIONS

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. German scientist Johannes Kepler (1571–1630) showed that the planets in our solar system move in elliptical orbits, with the sun at a focus. Earth satellites also travel in elliptical orbits, with Earth at a focus. One intriguing aspect of the ellipse is that a ray of light or a sound wave emanating from one focus will be reflected from the ellipse to exactly the other focus. A whispering gallery is an elliptical room with an elliptical, dome-shaped ceiling. People standing at the foci can whisper and hear each other quite clearly, while persons in other locations in the room cannot hear them. Statuary Hall in the U.S. Capitol Building is elliptical. President John Quincy Adams, while a member of the House of Representatives, was aware of this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, easily eavesdropping on the *private conversations of other House members located near the other focus.*

The elliptical reflection principle is used in a procedure for disintegrating kidney stones. The patient is placed within a device that is elliptical in shape. The patient is placed so the kidney is centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone. The convergence of the ultrasound waves at the kidney stone causes vibrations that shatter it into fragments. The small pieces can then be passed painlessly through the patient’s system. The patient recovers in days, as opposed to up to six weeks if surgery is used instead.

Ellipses are often used for supporting arches of bridges and in tunnel construction.

EXAMPLE 5 AN APPLICATION INVOLVING AN ELLIPTICAL ORBIT

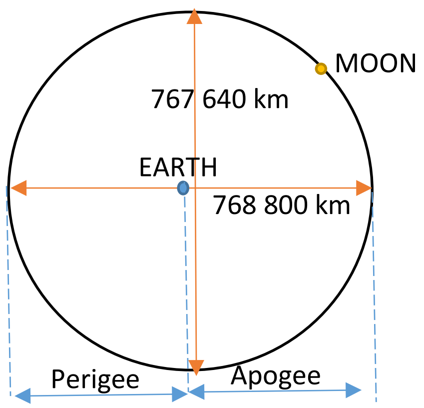


Figure 10

The Moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in **Figure 10**. The major and minor axes of the orbit have lengths of 768,800 kilometres and 767,640 kilometres, respectively. Find the greatest and smallest distances (the apogee and perigee), respectively from Earth's center to the Moon's center.

Solution: Note that Earth is not the center of the Moon's orbit.

Because $2a = 768\,800$ and $2b = 767\,640$, we have $a = 384\,400$ and $b = 383\,820$ which implies that

$$c^2 = a^2 - b^2 = (384\,400\,)^2 - (383\,820)^2 = 445\,567\,600.$$

Because $c^2 = 445\,567\,600\,;\ c = \sqrt{445\,567\,600} \approx 21\,108.$

So, the greatest distance between the center of Earth and the center on the Moon is

$$a + c = 384\,400 + 21\,108 = 405\,508\ kilometers.$$

And the smallest distance is

$$a - c = 384\,400 - 21\,108 = 363\,292\ kilometers.$$

ECCENTRICITY

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of eccentricity.

Definition: The eccentricity of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

Note that $0 < e < 1$ for every ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio $\frac{c}{a}$ is small, as shown in **Figure 11a**. On the other hand, for an elongated ellipse, the foci are close to the vertices, and the ratio $\frac{c}{a}$ is close to 1, as shown in **Figure 11b**.

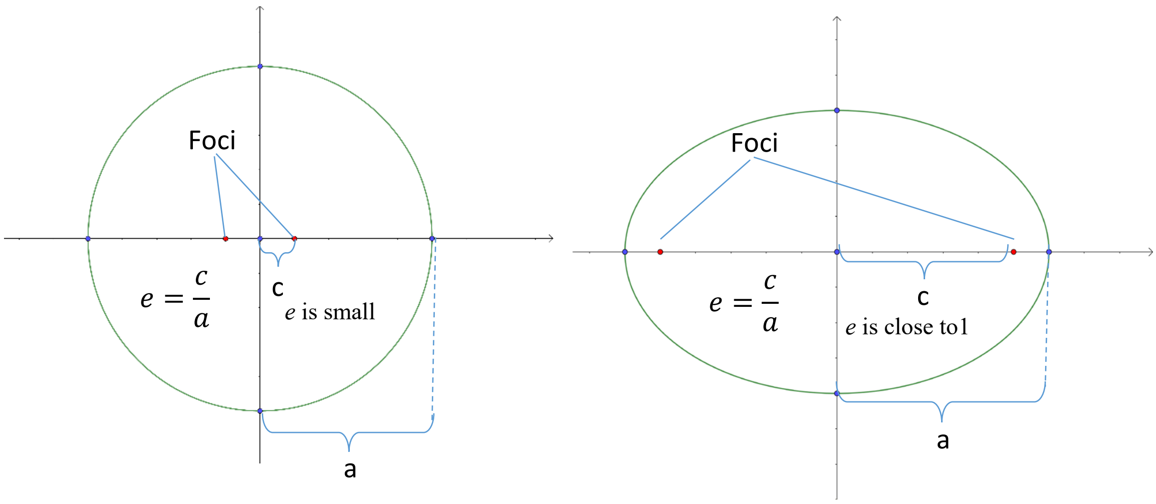


Figure 10 *a)* *b)*

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the nine planetary orbits are as follows.

Mercury: $e \approx 0.2056$,

Venus: $e \approx 0.0068$,

Earth: $e \approx 0.0167$,

Mars: $e \approx 0.0934$,

Jupiter: $e \approx 0.0484$.

Saturn: $e \approx 0.0542$,

Uranus: $e \approx 0.0472$,

Neptune: $e \approx 0.0086$,

Pluto: $e \approx 0.2488$,

EXAMPLE 5 AN APPLICATION INVOLVING AN ELLIPSE

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet (see **Figure 12**). Your truck has a width of 10 feet and a height of 9 feet. Will your truck clear the opening of the archway?

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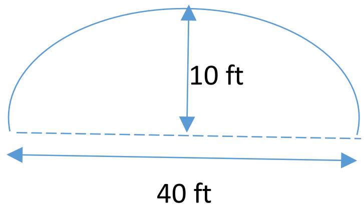

Figure 12 A semielliptical archway

Solution: Because your truck's width is 10 feet, to determine the clearance, we must find the height of the archway 5 feet from the center. If that height is 9 feet or less, the truck will not clear the opening.

In **Figure 15**, we've constructed a coordinate system with the x −axis on the ground and the origin at the center of the archway. Also shown is the truck, whose height is 9 feet.

Using the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ we can experess the equation of the blue archway in **Figure 12** as

$$\frac{x^2}{20^2} + \frac{y^2}{10^2} = 1$$

or

$$\frac{x^2}{400} + \frac{y^2}{100} = 1.$$

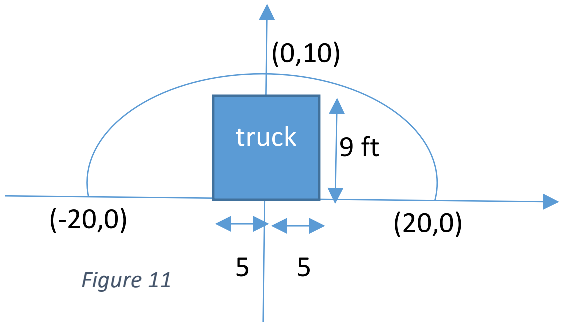

Figure 11

As shown in **Figure 16**, the edge of the 10-foot-wide truck corresponds to $x = 5$. We find the height of the archway 5 feet from the center by substituting 5 for x and solving for y.

$$\frac{5^2}{400} + \frac{y^2}{100} = 1.$$

Square 5

$$\frac{25}{400} + \frac{y^2}{100} = 1.$$

Clear fractions by multiplying both sides by 400

$$400\left(\frac{25}{400} + \frac{y^2}{100}\right) = 400 \cdot 1.$$

Using the distributive property and simplifying we get

$$25 + 4y^2 = 400,$$

then subtracting 25 from both sides

$$4y^2 = 375.$$

Deviding both sides by 4 and taking onli the positive square root (the archway is above the x −axis, so y is nonnegative.

$$y^2 = \frac{375}{4}; \qquad y = \sqrt{\frac{375}{4}} \approx 9.68.$$

Thus, the height of the archway 5 feet from the center is approximately 9.68 feet. Because truck's height is 9 feet, there is enough room for the truck to clear the archway.

PRACTICE EXERCISES

Graph each ellipse and locate the foci.

1. $\frac{x^2}{16}+\frac{y^2}{4}=1;$
2. $\frac{x^2}{9}+\frac{y^2}{36}=1;$
3. $\frac{x^2}{25}+\frac{y^2}{64}=1;$
4. $\frac{x^2}{49}+\frac{y^2}{81}=1;$
5. $\frac{x^2}{\frac{9}{4}}+\frac{y^2}{\frac{25}{4}}=1;$
6. $x^2=1-4y^2;$
7. $25x^2+4y^2=100;$
8. $5x^2+16y^2=64;$
9. $7x^2=35-5y^2;$
10. $\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{25}{16}}=1.$

Find the standard form of the equation of each ellipse and give the location of its foci.

11. $.2x^2+9y^2+16x-90y+239=0;$
12. $5x^2+y^2-3x+40=0;$
13. $x^2+4y^2+10x-16y+25=0;$
14. $36x^2+4y^2-4y-44=0;$
15. $16x^2+100y^2+64x-300y-111=0;$
16. $2x^2+3y^2-4x-5y+1=0.$

Find the standard form of the equation of each ellipse satisfying the given conditions.

17. Foci: $(-5,0),(5,0)$; vertices: $(-8,0),(8,0)$;
18. Foci: $(-2,0),(2,0)$; vertices: $(-6,0),(6,0)$;
19. Foci: $(0,-4),(0,4)$; vertices: $(0,-7),(0,7)$;
20. Foci: $(0,-3),(0,3)$; y –intercepts: -3 and 3;
21. Foci: $(0,-2),(0,2)$; y –intercepts: -2 and 2;
22. Major axis horizontal with length 8; length of minor axis – 4; center: $(0,0)$;
23. Major axis horizontal with length 12; length of minor axis – 6; center: $(0,0)$;
24. Major axis vertical with length 10; length of minor axis – 4; center: $(-2,3)$;
25. Major axis vertical with length 20; length of minor axis – 10; center: $(2,-3)$;
26. Endpoints of major axis: $(7,9)$ and $(7,3)$; Endpoints of minor axis: $(5,6)$ and $(9,6)$;
27. Endpoints of major axis: $(2,2)$ and $(8,2)$; Endpoints of minor axis: $(5,3)$ and $(5,1)$.

Graph each ellipse and give the location of its foci

28. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1;$
29. $\frac{(x-1)^2}{16}+\frac{(y+2)^2}{9}=1;$
30. $(x+3)^2+4(y-2)^2=16;$
31. $(x-3)^2+9(y+2)^2=18;$
32. $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{25}=1;$

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33. $\frac{(x-3)^2}{9}+\frac{(y+1)^2}{16}=1;$
34. $\frac{x^2}{25}+\frac{(y-2)^2}{36}=1;$
35. $\frac{(x-4)^2}{4}+\frac{y^2}{25}=1;$
36. $9(x-1)^2+4(y+3)^2=36;$
37. $36(x+4)^2+(y+3)^2=36.$

Convert each equation to standard form by completing the square on x and y. Then graph the ellipse and give the location of its foci.

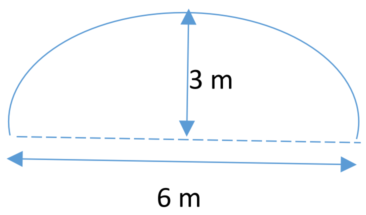
38. $9x^2+25y^2-36x+50y-164=0;$
39. $4x^2+9y^2-32x+36y+64=0;$
40. $9x^2+16y^2-18x+64y-71=0;$
41. $x^2+4y^2+10x-8y+13=0;$
42. $4x^2+y^2+16x-6y-39=0;$
43. $4x^2+25y^2-24x+100y+36=0.$

Find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

44. $\begin{cases} x^2+y^2=1 \\ x^2+9y^2=9 \end{cases};$
45. $\begin{cases} x^2+y^2=25 \\ 25x^2+y^2=25 \end{cases};$
46. $\begin{cases} x^2+y^2=25 \\ 25x^2+y^2=25 \end{cases};$
47. $\begin{cases} \frac{x^2}{25}+\frac{y^2}{9}=1 \\ y=3 \end{cases};$
48. $\begin{cases} 4x^2+y^2=4 \\ 2x-y=2 \end{cases};$
49. $\begin{cases} \frac{x^2}{4}+\frac{y^2}{36}=1 \\ x=-2 \end{cases};$
50. $\begin{cases} 4x^2+y^2=4 \\ x+y=3 \end{cases}.$

APPLICATION EXERCISES

51. Will a truck that is 2.4 m wide carrying a load that reaches 2.1 m feet above the ground clear the semielliptical arch on the one-way road that passes under the bridge shown in the figure?



52. A semielliptic archway has a height of 6 m and a width of 15 m. Can a truck 4 m high and 3 m wide drive under the archway without going into the other lane?
53. If an elliptical whispering room has a height of 6 m and a width of 30 m, where should two people stand if they would like to whisper back and forth and be heard?
54. A semielliptical arch over a tunnel for a one-way road through a mountain has a major axis of 15 m and a height at the center of 3 m.
    - Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.
    - Find an equation of the semielliptical arch over the tunnel.
    - You are driving a moving truck that has a width of 2.4 m and a height of 2,7 m. Will the moving truck clear the opening of the arch?
55. Halley's comet has an elliptical orbit, with the sun at one focus. The eccentricity of the orbit is approximately 0.967. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.)
    - Find an equation of the orbit. Place the center of the orbit at the origin, and place the major axis on the x −axis.
    - Use a graphing utility to graph the equation of the orbit.
    - Find the greatest (aphelion) and smallest (perihelion) distances from the sun's center to the comet's center.
56. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 50 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

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