THE HYPERBOLA

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane.

OBJECTIVES AND OUTCOMES:

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Rewrite the equation of a hyperbola in standard form.
- Solve applied problems involving hyperbolas.

CONTENTS

Figure 1 The hyperbolic shadows

In this section, we study the curve with two parts known as the hyperbola. In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called branches.

Figure 1 shows a cylindrical lampshade casting two shadows on a wall. These shadows indicate the distinguishing feature of hyperbolas. Although each branch might look like a parabola, its shape is actually quite different.

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Definition: A hyperbola is the set of points in a plane the difference of whose distances from two fixed points, called foci, is constant.

Figure 2 illustrates the two branches of a hyperbola. The line through the foci intersects the hyperbola at two points, called the vertices. The line segment that joins the vertices is the transverse axis. The midpoint of the transverse axis is the center of the hyperbola. Notice that the center lies midway between the vertices, as *well as midway between the foci.*

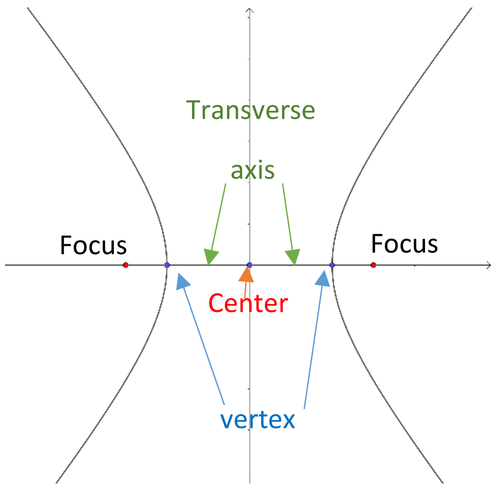

Figure 2 The two branches of a hyperbola

STANDARD FORM OF THE EQUATION OF A HYPERBOLA

The rectangular coordinate system enables us to translate a hyperbola's geometric definition into an algebraic equation. **Figure 3** is our starting point for obtaining an equation. We place the foci, F_1 and F_2 on the x −axis at the points $(-c, 0)$ and $(c, 0)$. Note that the center of this hyperbola is at the origin. We let (x, y) represent the coordinates of any point, P, on the hyperbola.

For any point (x, y) on the hyperbola, the absolute value of the difference of the distances from the two foci, $|d_2 - d_1|$, must be constant. We denote this constant by $2a$ just as we did for the ellipse. Thus, the point (x, y) is on the hyperbola if and only if

$$|d_2 - d_1| = 2a.$$

Using distance formula

$$\left|\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}\right| = 2a.$$

After eliminating radicals and simplifying, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

For convenience, let $b^2 = c^2 - a^2$. Substituting b^2 for $c^2 - a^2$ in the preceding equation, we obtain

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2 give us

$$\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

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$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This last equation is called the standard form of the equation of a hyperbola centered at the origin. There are two such equations.

The first is for a hyperbola in which the transverse x −axis lies on the second is for a hyperbola in which the transverse axis lies on the y −axis.

STANDARD FORMS OF THE EQUATIONS OF A HYPERBOLA

The standard form of the equation of a hyperbola with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad or \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Figure 3a illustrates that for the equation on the left, the transverse axis lies on the $x -$ axis. **Figure 3b** illustrates that for the equation on the right, the transverse axis lies on the $y -$ axis. The vertices are a units from the center and the foci are c units from the center. For both equations, $b^2 = c^2 - a^2$**.** Equivalently, $c^2 = a^2 + b^2$.

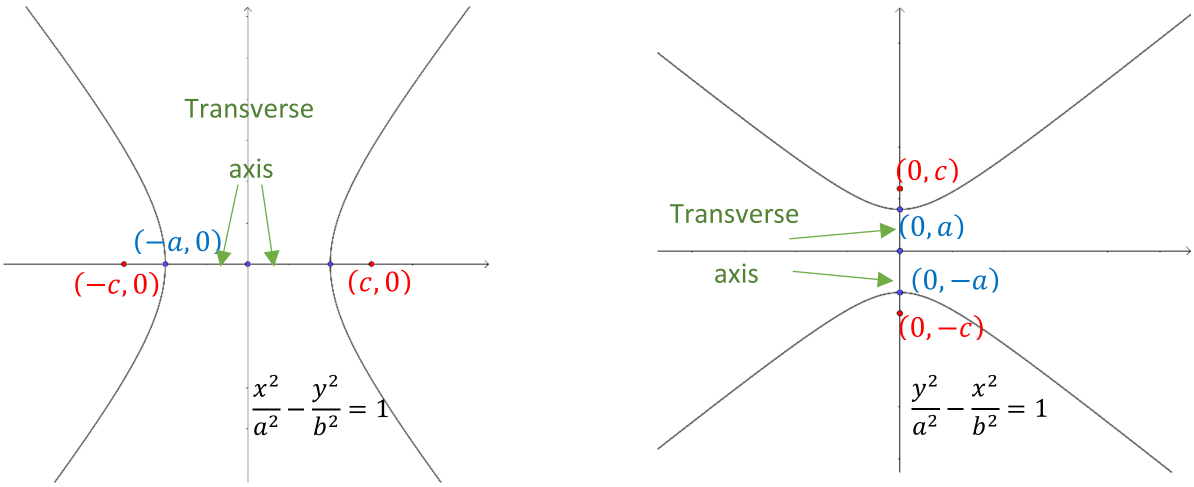


Figure 3 a) Transverse axis lies on the x-axis; b) Transverse axis lies on the y-axis

USING THE STANDARD FORM OF THE EQUATION OF A HYPERBOLA

We can use the standard form of the equation of a hyperbola to find its vertices and locate its foci. Because the vertices are a units from the center, begin by identifying a^2 in the equation. In the standard form of a hyperbola's equation, a^2 is the number under the variable whose term is preceded by a plus sign. If the x^2 −term is preceded by a plus sign, the transverse axis lies along the x −axis. Thus, the vertices are a units to the left and right of the origin. If the y^2 −term is preceded by a plus sign, the transverse axis lies along the y −axis. Thus, the vertices are a units above and below the origin.

We know that the foci are c units from the center. The substitution that is used to derive the hyperbola's equation, $c^2 = a^2 + b^2$, is needed to locate the foci when a^2 and b^2 are known.

EXAMPLE 1 FINDING VERTICES AND FOCI FROM A HYPERBOLA'S EQUATION

Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
b. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Solution: Both equations are in standard form. We begin by identifying a^2 and b^2 in each equation.

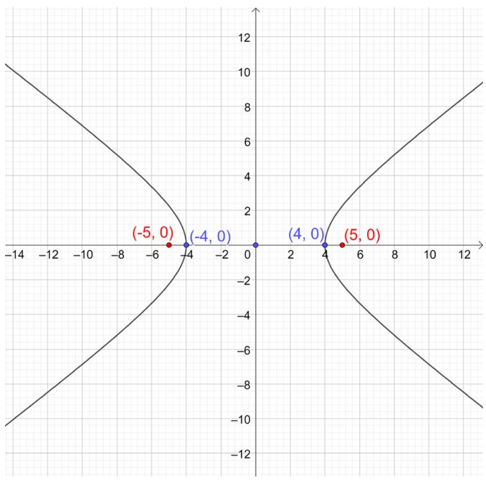
a. The first equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Because the x^2 −term is preceded by a plus sign, the transverse axis lies along the x −axis. Thus, the vertices are a units to the left and right of the origin. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 16, a = 4$. Thus, the vertices are $(-4,0)$ and $(4, 0)$, shown in **Figure 4a**.

We use $c^2 = a^2 + b^2$, to find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We know that $a^2 = 16$ and $b^2 = 9$; we need to find c^2 in order to find c.

Because $c^2 = 25, c = 5$. The foci are located at $(-5,0)$ and $(5, 0)$. They are shown in **Figure 4a**.



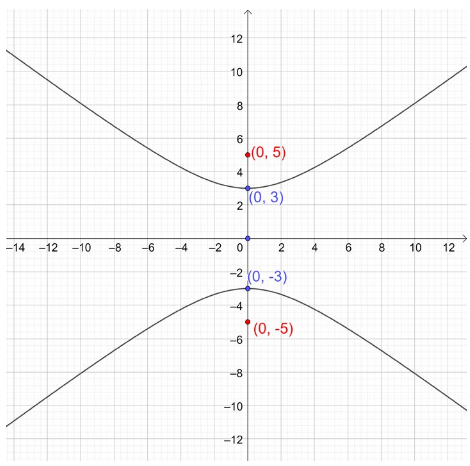


Figure 4 a) The graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ *b) The graph of* $\frac{y^2}{9} - \frac{x^2}{16} = 1$

b. The second given equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

Because the y^2 −term is preceded by a plus sign, the transverse axis lies along the y −axis. Thus, the vertices are a units above and below the origin. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 9, a = 3$. Thus, the vertices are $(0, -3)$ and $(0,3)$, shown in **Figure 4b**.

We use $c^2 = a^2 + b^2$, to find the foci, which are located at $(0, -c)$ and $(0, c)$. We know that $a^2 = 9$ and $b^2 = 16$; we need to find c^2 in order to find c.

Because $c^2 = 25, c = 5$. The foci are located at $(0, -5)$ and $(0,5)$. They are shown in **Figure 4b**.

In Example 1, we used equations of hyperbolas to find their foci and vertices. In the next example, we reverse this procedure.

EXAMPLE 2 FINDING THE EQUATION OF A HYPERBOLA FROM ITS FOCI AND VERTICES

Find the standard form of the equation of a hyperbola with foci at $(-3,0)$ and $(0, 3)$ and vertices $(0, -2)$ and $(0, 2)$, shown in **Figure 5**.

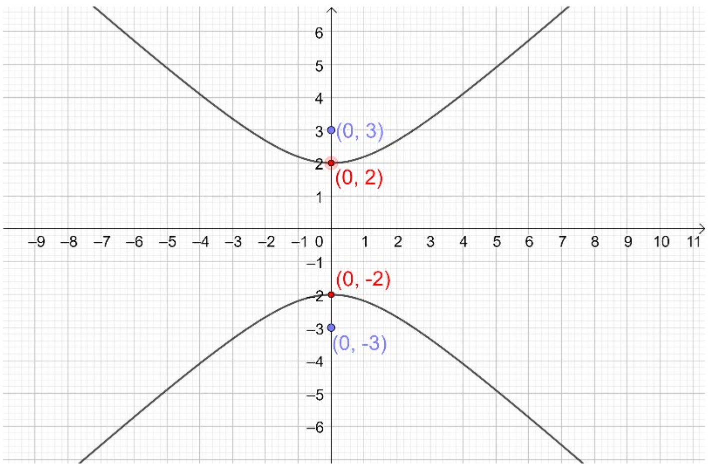


Figure 5

Solution: Because the foci are located at $(-3,0)$ and $(0, 3)$, on the y −axis the transverse axis lies on the y −axis. The center of the hyperbola is midway between the foci, located at $(0, 0)$.Thus, the form of the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, to either vertex, $(0, -2)$ or $(0, 2)$, is 2, so $a = 2$.

$$\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1 \quad or \quad \frac{y^2}{4} - \frac{x^2}{b^2} = 1.$$

We must still find b^2. The distance from the center, $(0, 0)$, to either focus, $(-3,0)$ or $(0, 3)$, is 3. Thus, $c = 3$. Using $c^2 = a^2 + b^2$, we have

$$3^2 = 2^2 + b^2$$

and

$$b^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting 5 for b^2 in $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ gives us the standard form of the hyperbola's equation. The equation is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1.$$

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THE ASYMPTOTES OF A HYPERBOLA

As and get larger, the two branches of the graph of a hyperbola approach a pair of intersecting straight lines, called asymptotes. The asymptotes pass through the center of the hyperbola and are helpful in graphing hyperbolas.

Figure 6 shows the asymptotes for the graphs of hyperbolas centered at the origin. The asymptotes pass through the corners of a rectangle. Note that the dimensions of this rectangle are $2a$ by $2b$. The line segment of length $2b$ is the conjugate axis of the hyperbola and is perpendicular to the transverse axis through the center of the hyperbola.

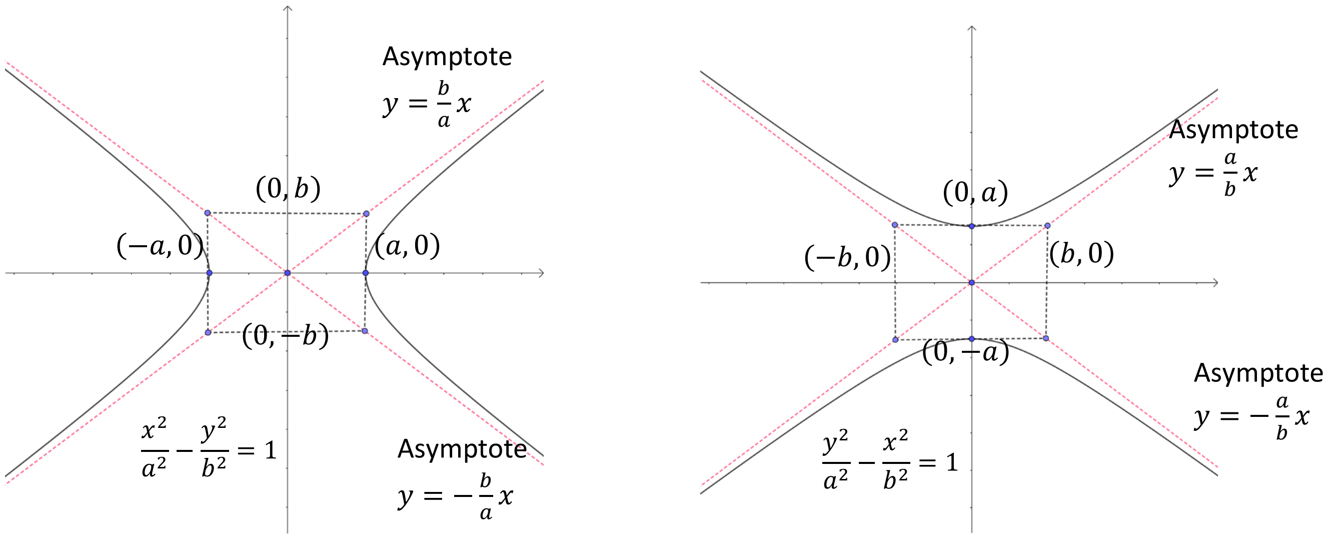

Figure 6 Asymptotes of a hyperbola

THE ASYMPTOTES OF A HYPERBOLA CENTERED AT THE ORIGIN

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a horizontal transverse axis and two asymptotes

$$y = -\frac{b}{a}x$$

and

$$y = \frac{b}{a}x.$$

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has a vertical transverse axis and two asymptotes

$$y = -\frac{a}{b}x$$

and

$$y = \frac{a}{b}x.$$

GRAPHING HYPERBOLAS CENTERED AT THE ORIGIN

Hyperbolas are graphed using vertices and asymptotes.

Graphing Hyperbolas:

1. Locate the vertices.
2. Use dashed lines to draw the rectangle centered at the origin with sides parallel to the axes, crossing one axis at $\pm a$ and the other at $\pm b$.
3. Use dashed lines to draw the diagonals of this rectangle and extend them to obtain the asymptotes.
4. Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

EXAMPLE 3 GRAPHING A HYPERBOLA

Graph and locate the foci: $\frac{x^2}{25} - \frac{y^2}{16} = 1.$ What are the equations of the asymptotes?

Solution:

1. ***Locate the vertices.*** The given equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with $a^2 = 25$ and $b^2 = 16.$ Based on the standard form of the equation with the transverse axis on the x −axis, we know that the vertices are $(-a, 0)$ and $(a, 0).$ Because $a^2 = 25$ $a = 5.$ Thus, the vertices are $(-5,0)$ and $(5, 0)$, shown in **Figure 7a**.
2. ***Draw a rectangle.*** Because $a^2 = 25$ and $b^2 = 16,$ $a = 5$ and $b = 4.$ We construct a rectangle to find the asymptotes, using -5 and 5 on the x −axis (the vertices are located here) and -4 and 4 on the y −axis. The rectangle passes through these four points, shown using dashed lines in **Figure 7a.**
3. ***Draw extended diagonals for the rectangle to obtain the asymptotes.*** We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 7a**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are

$$y = \pm \frac{b}{a} x$$

or

$$y = \pm \frac{4}{5} x.$$

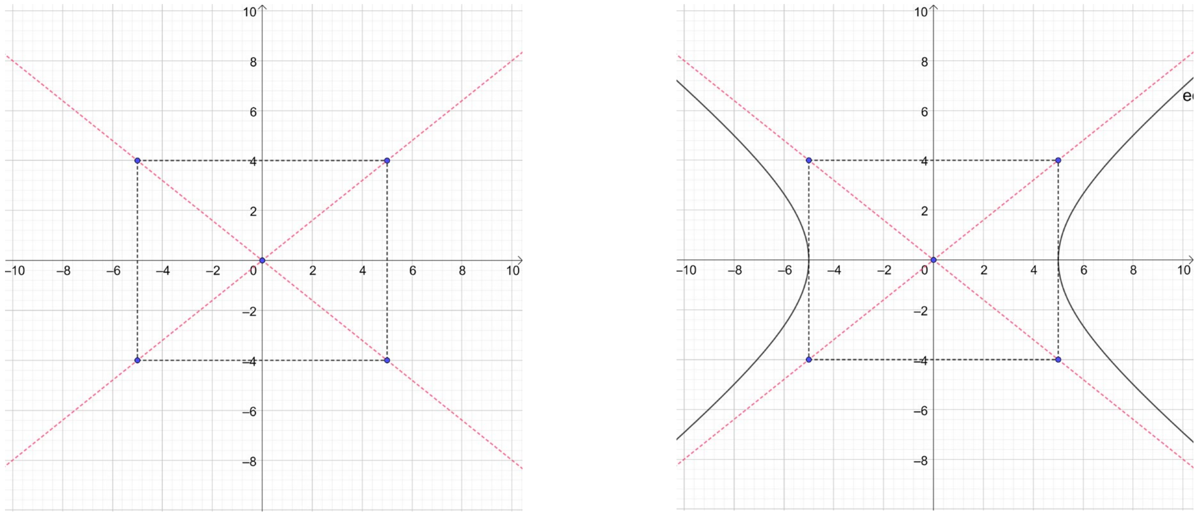

Figure 7 a) Preparing to graph $\frac{x^2}{25} - \frac{y^2}{16} = 1$; *b) The graph of* $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

4. ***Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.*** The hyperbola is shown in **Figure 7b**.

We now consider the foci, located at $(-c, 0)$ and $(c, 0)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 25 + 16 = 41.$$

Because $c^2 = 41, c = \sqrt{41}$. The foci are located at $(-\sqrt{41}, 0)$ and $(\sqrt{41}, 0)$, approximately $(-6.4, 0)$ and $(6.4, 0)$.

EXAMPLE 4 GRAPHING A HYPERBOLA

Graph and locate the foci: $9y^2 - 4x^2 = 36$. What are the equations of the asymptotes?

Solution: We begin by writing the equation in standard form. The right side should be 1, so we divide both sides by 36.

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36},$$

simplifying

$$\frac{y^2}{4} - \frac{x^2}{9} = 1.$$

Now we are ready to use our four-step procedure for graphing hyperbolas.

1. ***Locate the vertices.*** The equation that we obtained is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, with $a^2 = 4$ and $b^2 = 9$. Based on the standard form of the equation with the transverse axis on the y −axis, we know that the vertices are $(0, -a,)$ and $(0, a)$. Because $a^2 = 4$ $a = 2$. Thus, the vertices are $(0, -2)$ and $(0,2)$, shown in **Figure 8a**.

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2. *Draw a rectangle.* Because $a^2 = 4$ and $b^2 = 9$, $a = 2$ and $b = 3$. We construct a rectangle to find the asymptotes, using -2 and 2 on the y −axis (the vertices are located here) and -3 and 3 on the x −axis. The rectangle passes through these four points, shown using dashed lines in **Figure 8a.**
3. *Draw extended diagonals for the rectangle to obtain the asymptotes.* We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 8a**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are

$$y = \pm\frac{a}{b}x \text{ or } y = \pm\frac{2}{3}x.$$

4. *Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.* The hyperbola is shown in **Figure 8b**.

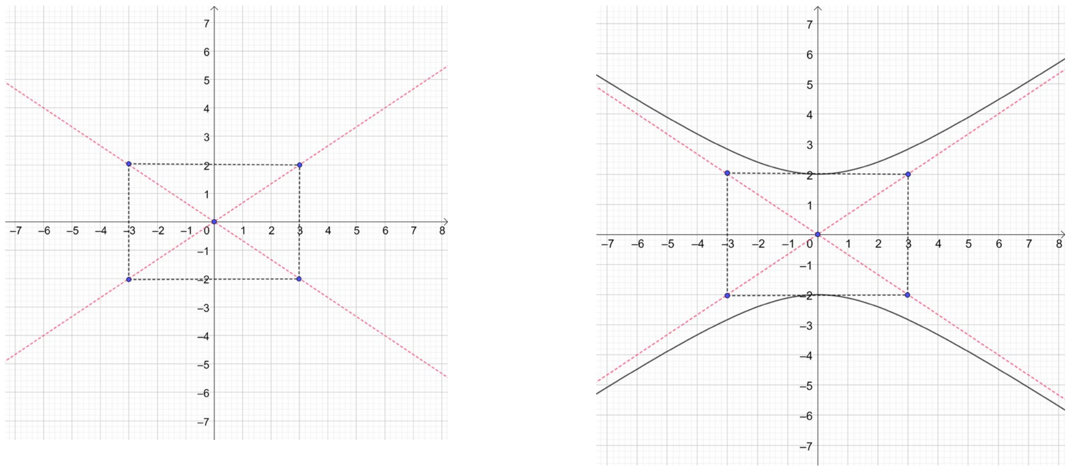


Figure 8 *a) Preparing to graph* $\frac{y^2}{4} - \frac{x^2}{9} = 1$; *b) The graph of* $\frac{y^2}{4} - \frac{x^2}{9} = 1$;

We now consider the foci, located at $(0, -c)$ and $(0, c)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 4 + 9 = 13.$$

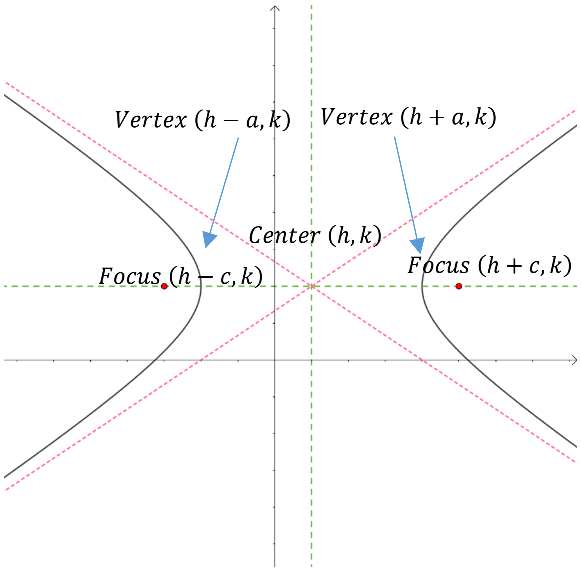
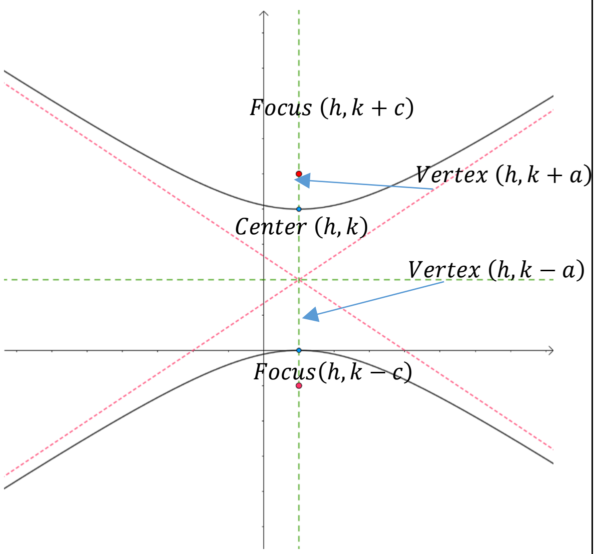
Because $c^2 = 13, c = \sqrt{13}$. The foci are located at $\left(0, -\sqrt{13}\right)$ and $\left(0, \sqrt{13}\right)$, approximately $(0, -3.6)$ and $(0,3.6)$.

TRANSLATIONS OF HYPERBOLAS

The graph of a hyperbola can be centered at (h, k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the hyperbola's equation.

Table 1 gives the standard forms of equations of hyperbolas centered at (h, k), and shows their graphs.

Table 1 Standard Forms of Equations of Hyperbolas Centered at (h,k)

Equation: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1;$ Vertices are a units right and a units left of center. Foci are c units right and c units left of center, where $c^2=a^2+b^2$.	Center: (h,k) Transverse Axis: Parallel to the x −axis; horizontal Vertices: $(h-a,k)$, $(h+a,k)$	
Equation: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1;$ Vertices are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2=a^2+b^2$.	Center:(h,k) Transverse Axis: Parallel to the y −axis; vertical Vertices: $(h,k-a)$, $(h,k+a)$	

EXAMPLE 5 GRAPHING A HYPERBOLA CENTERED AT (h, k)

Graph: $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$. Where are the foci located? What are the equations of the asymptotes?

Solution: In order to graph the hyperbola, we need to know its center (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and h is the number subtracted from y. Where is in

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

we see that $h = 2$ and $k = 3$. Thus, the center of the hyperbola, (h, k), is $(2, 3)$. We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.

1. *Locate the vertices.* To do this, we must identify a^2. The form of our equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

we see that $a^2 = 16$ and $b^2 = 9$. Based on the standard form of the equation with a horizontal transverse axis, the vertices are a units to the left and right of the center. Because $a^2 = 16$, $a = 4$. This means that the vertices are 4 units to the left and right of the center, $(2, 3)$. Four units to the left of $(2, 3)$ puts one vertex at $(2-4, 3)$ or $(-2, 3)$. Four units to the right of $(2, 3)$ puts the other vertex at $(2+4, 3)$, or $(6, 3)$. The vertices are shown in **Figure 9**.

2. *Draw a rectangle.* Because $a^2 = 16$ and $b^2 = 9$, $a = 4$ and $b = 3$. The rectangle passes through points that are 4 units to the right and left of the center (the vertices are located here) and 3 units above and below the center. The rectangle is shown using dashed lines in **Figure 9**.

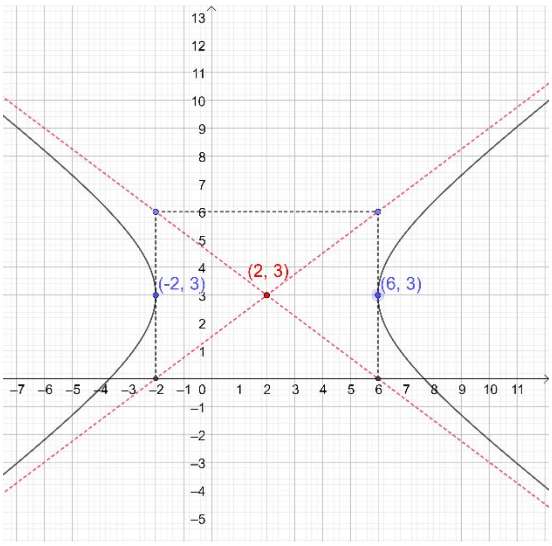


Figure 9 The graph of $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$

3. ***Draw the extended diagonals of the rectangle to obtain the asymptotes.*** We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 9**, to obtain the graph of the asymptotes. The equations of the asymptotes of the upshifted hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{3}{4}x$.

Thus, the asymptotes for the hyperbola that is shifted two units to the right and three units up, namely

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

have equations that can be expressed as

$$y - 3 = \pm\frac{3}{4}(x-2).$$

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4. *Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes*. The hyperbola is shown in **Figure** 9. We now consider the foci, located c units to the right and left of the center. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 16 + 9 = 25.$$

Because $c^2 = 25, c = 5.$ This means that the foci are 5 units to the left and right of the center, (2, 3). Five units to the left of $(2, 3)$ puts one focus at $(2 - 5, 3)$ or $(-3, 3).$ Five units to the right of $(2, 3)$ puts the other focus at $(2 + 5, 3)$, or $(7, 3).$

In our next example, it is necessary to convert the equation of a hyperbola to standard form by completing the square on x and $y.$

EXAMPLE 6 GRAPHING A HYPERBOLA CENTERED AT (h, k)

Graph: $4x^2 - 24x - 25y^2 + 250y - 489 = 0.$ Where are the foci located? What are the equations of the asymptotes?

Solution: We begin by completing the square on x and y.

This is given equation

$$4x^2 - 24x - 25y^2 + 250y - 489 = 0.$$

Group terms and add 489 to both sides

$$(4x^2 - 24x) + (-25y^2 + 250y) = 489;$$

$$4(x^2 - 6x + ?) - 25(y^2 + 10y + ?) = 489$$

Factor out 4 and -25, respectively, so coefficients of x^2 and y^2 are 1. Complete each square by adding the square of half the coefficient of x and y, respectively.

$$4(x^2 - 6x + 9) - 25(y^2 + 10y + 25) = 489 + 36 + (-625).$$

Factoring

$$4(x - 3)^2 - 25(y - 5)^2 = -100$$

and dividing both sides by -100

$$\frac{4(x-3)^2}{-100} - \frac{25(y-5)^2}{-100} = -\frac{100}{-100}.$$

Simplifying

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1.$$

We see that $h = 3$ and $k = 5.$ Thus, the center of the hyperbola, (h, k), is $(3, 5).$ Because the x^2 –term is being subtracted, the transverse axis is vertical and the hyperbola opens upward and downward.

We use our four-step procedure to obtain the graph of

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1.$$

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1. *Locate the vertices.* Based on the standard form of the equation with a vertical transverse axis, the vertices are a units above and below the center. Because $a^2 = 4,\ a = 2$**.** This means that the vertices are 2 units above and below the center, $(3, 5)$**.** This puts the vertices at $(3, 7)$ and $(3, 3)$, shown in **Figure 10**.

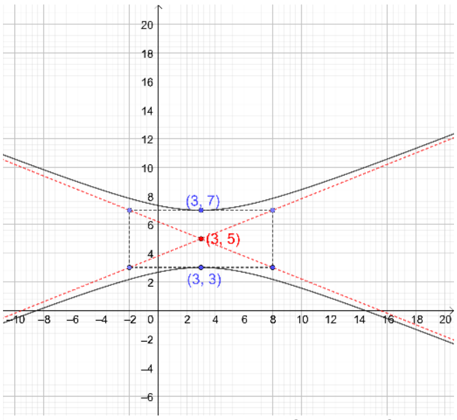

Figure 10 The graph of $\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$

2. *Draw a rectangle.* Because $b^2 = 25,\ a^2 = 4$ and $a = 2,\ b = 5$**.** The rectangle passes through points that are 2 units above and below the center (the vertices are located here) and 5 units to the right and left of the center. The rectangle is shown using dashed lines in **Figure 10**.

3. *Draw extended diagonals of the rectangle to obtain the asymptotes.* We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 10**, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{y^2}{4} - \frac{x^2}{25} = 1$ are $y = \pm\frac{a}{b}x$ or $y = \pm\frac{2}{5}x$**.** Thus, the asymptotes for the hyperbola that is shifted three units to the right and five units up, namely

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$$

have equations that can be expressed as $y - 5 = \pm\frac{2}{5}(x - 3)$**.**

4. *Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.* The hyperbola is shown in **Figure 10**. We now consider the foci, located c units above and below the center, $(3, 5)$**.** We find c using $c^2 = a^2 + b^2$**.**

$$c^2 = 4 + 25 = 29.$$

Because $c^2 = 29, c = \sqrt{29}$**.** This means that the foci are located at $(3, 5 + \sqrt{29})$ and $(3, 5 - \sqrt{29})$.

APPLICATIONS

Hyperbolas have many applications. When a jet flies at a speed greater than the speed of sound, the shock wave that is created is heard as a sonic boom. The wave has the shape of a cone. The shape formed as the cone hits the ground is one branch of a hyperbola. Halley's Comet, a permanent part of our solar system, travels around the sun in an elliptical orbit. Other comets pass through the solar system only once, following a hyperbolic path with the sun as a focus. Hyperbolas are of practical importance in fields ranging from architecture to navigation. Cooling towers used in the design for nuclear power plants have cross sections that are both ellipses and hyperbolas. Three-dimensional solids whose cross sections are hyperbolas are used in some rather unique architectural creations. A hyperbolic mirror is used in some telescopes. Such a mirror has the property that a light ray directed at one focus will be reflected to the other focus.

Long-range navigation (LORAN) is a radio navigation system developed during World War II. The system enables a pilot to guide aircraft by maintaining a constant difference between the aircraft's distances from two fixed points: the master station and the slave station.

EXAMPLE 7 AN APPLICATION INVOLVING HYPERBOLAS

An explosion is recorded by two microphones that are 2 miles apart. Microphone M_1 received the sound 4 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

Solution: We begin by putting the microphones in a coordinate system. Because $1\ mile = 5280\ feet$, we place M_1 $5280\ feet$ on a horizontal axis to the right of the origin and M_2 $5280\ feet$ on a horizontal axis to the left of the origin. Figure 11 illustrates that the two microphones are 2 miles apart.

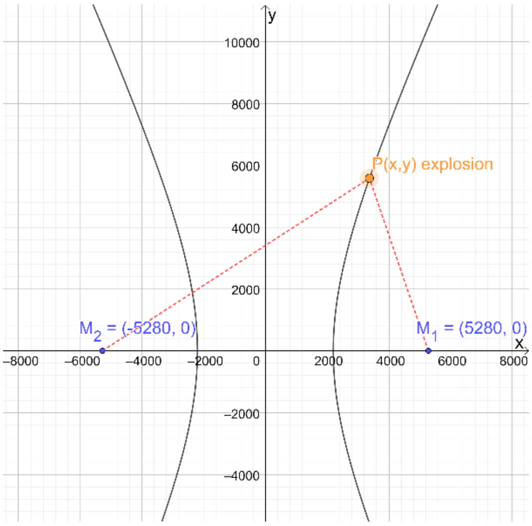

Figure 11 Locating an explosion on the branch of a hyperbola

We know that M_2 received the sound 4 seconds after M_1. Because sound travels at $1100\ feet$ per second, the difference between the distance from P to M_1 and the distance from P to M_2 is $4400\ feet$. The set of all points P (or locations of the explosion) satisfying these conditions fits the definition of a hyperbola, with microphones M_1 and M_2 at the foci. Use the standard form of the hyperbola's equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$P(x, y)$, the explosion point, lies on the hyperbola. We must find a^2 and b^2. The difference between the distances, represented by $2a$ in the derivation of the hyperbola's equation, is $4400\ feet$. Thus, $2a = 4400$ and $a = 2200$. Substitute 2200 for a into hyperbola's equation

$$\frac{x^2}{2200^2} - \frac{y^2}{b^2} = 1,$$

then square 2200

$$\frac{x^2}{4\,840\,000} - \frac{y^2}{b^2} = 1.$$

We must still find b^2. We know that $a = 2200$. The distance from the center $(0,0)$, to either focus, $(-5280,0)$ or $(5280,0)$, is 5280. Thus, $c = 5280$. Using $c^2 = a^2 + b^2$, we have

$$5280^2 = 2200^2 + b^2$$

and

$$b^2 = 5280^2 - 2200^2 = 23\,038\,400.$$

The equation of the hyperbola with a microphone at each focus is

$$\frac{x^2}{4\,840\,000} - \frac{y^2}{23\,038\,400} = 1.$$

We can conclude that the explosion occurred somewhere on the right branch (the branch closer to M_1) of the hyperbola given by this equation.

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In Example 7, we determined that the explosion occurred somewhere along one branch of a hyperbola, but not exactly where on the hyperbola. If, however, we had received the sound from another pair of microphones, we could locate the sound along a branch of another hyperbola. The exact location of the explosion would be the point where the two hyperbolas intersect.

PRACTICE EXERCISES

Find the vertices and locate the foci of each hyperbola with the given equation.

1. $\frac{x^2}{4}-\frac{y^2}{1}=1;$
2. $\frac{y^2}{4}-\frac{x^2}{1}=1;$
3. $\frac{x^2}{1}-\frac{y^2}{4}=1;$
4. $\frac{y^2}{1}-\frac{x^2}{4}=1;$

Find the standard form of the equation of each hyperbola satisfying the given conditions.

5. Foci: $(0,-3)$, $(0,3)$; vertices: $(0,-1)$, $(0,1)$;
6. Foci: $(0,-6)$, $(0,6)$; vertices: $(0,-2)$, $(0,2)$;
7. Foci: $(-4,0)$, $(4,0)$; vertices: $(-3,0)$, $(3,0)$;
8. Foci: $(-7,0)$, $(7,0)$; vertices: $(-5,0)$, $(5,0)$;
9. Endpoints of transverse axis: $(0,-6)$, $(0,6)$; asymptote: $y=2x$;
10. Endpoints of transverse axis: $(-4,0)$, $(4,0)$; asymptote: $y=2x$;
11. Center: $(4,-2)$; Focus: $(7,-2)$; vertex: $(6,-2)$;
12. Center: $(-2,1)$; Focus: $(-2,6)$; vertex: $(-2,4)$.

Use vertices and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

13. $\frac{x^2}{9}-\frac{y^2}{25}=1;$
14. $\frac{x^2}{100}-\frac{y^2}{64}=1;$
15. $\frac{y^2}{16}-\frac{x^2}{36}=1;$
16. $\frac{x^2}{16}-\frac{y^2}{25}=1;$
17. $\frac{x^2}{144}-\frac{y^2}{81}=1;$
18. $\frac{y^2}{25}-\frac{x^2}{64}=1;$
19. $4y^2-x^2=1;$
20. $9x^2-25y^2=36;$
21. $9y^2-25x^2=225;$
22. $9y^2-x^2=1;$
23. $4x^2-25y^2=100;$
24. $16y^2-9x^2=144;$
25. $y=\pm\sqrt{x^2-3};$
26. $y=\pm\sqrt{x^2-2}.$

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Find the standard form of the equation of each hyperbola.

27. $9x^2 - 4y^2 - 18x + 8y - 31 = 0$
28. $16x^2 - 4y^2 + 64x - 24y - 36 = 0$
29. $y^2 - x^2 - 4y + 2x - 6 = 0$
30. $4y^2 - 16x^2 - 24y + 96x - 172 = 0$
31. $9y^2 - x^2 + 18y - 4x - 4 = 0$

Use the center, vertices, and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

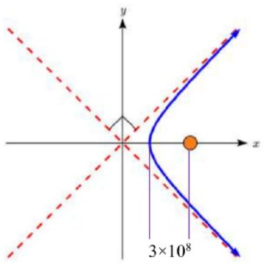
32. $\frac{(x+4)^2}{9} - \frac{(y+3)^2}{16} = 1;$
33. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{25} = 1;$
34. $\frac{(x+3)^2}{25} - \frac{y^2}{16} = 1;$
35. $\frac{(x+2)^2}{9} - \frac{y^2}{25} = 1;$
36. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1;$
37. $\frac{(y-2)^2}{36} - \frac{(x+1)^2}{49} = 1;$
38. $(x-3)^2 - 4(y+3)^2 = 4;$
39. $(x-1)^2 - (y-2)^2 = 3;$
40. $(x+3)^2 - 9(y-4)^2 = 9;$
41. $(y-2)^2 - (x+3)^2 = 5.$

Convert each equation to standard form by completing the square on x and y. Then graph the hyperbola. Locate the foci and find the equations of the asymptotes.

42. $x^2 - y^2 - 2x - 4y - 4 = 0;$
43. $4x^2 - y^2 + 32x + 6y + 39 = 0;$
44. $16x^2 - y^2 + 64x - 2y + 67 = 0;$
45. $-4x^2 + 9y^2 + 24x - 18y - 63 = 0;$
46. $4x^2 - 9y^2 - 16x + 54y - 101 = 0;$
47. $4x^2 - 9y^2 + 8x - 18y - 6 = 0;$
48. $4x^2 - 25y^2 - 32x + 164 = 0;$
49. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

APPLICATION EXERCISES

50. Two microphones that are 1 mile apart record an explosion. Microphone M_1 received the sound 2 seconds before Microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.
51. Radio towers A and B, 200 kilometres apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 microseconds before the signal from A.
    a. Assuming that the radio signals travel 300 meters per microsecond, determine the equation of the hyperbola on which the ship is located.
    b. If the ship lies due north of tower B how far out at sea is it?
52. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250\ 000$, where and are in yards. How far apart are the houses at their closest point?
53. Stations A and B are 100 kilometres apart and send a simultaneous radio signal to a ship. The signal from A arrives 0.0002 seconds before the signal from B. If the signal travels 300,000 kilometres per second, find an equation of the hyperbola on which the ship is positioned if the foci are located at A and B.
54. Anna and Julia are standing 3050 feet apart when they see a bolt of light strike the ground. Anna hears the thunder 0.5 seconds before Julia does. Sound travels at 1100 feet per second. Find an equation of the hyperbola on which the lighting strike is positioned if Anna and Julia are located at the foci.
55. A comet passes through the solar system following a hyperbolic trajectory with the sun as a focus. The closest it gets to the sun is 3×108 miles. The figure shows the trajectory of the comet, whose path of entry is at a right angle to its path of departure. Find an equation for the comet's trajectory. Round to two decimal places



56. Write the standard form equation for the ship's location $P(x)$ in the diagram below. Assume that two stations, 300 miles apart, are positioned as pictured