THE PARABOLA

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. In this section, we study parabolas and their applications, including parabolic shapes.

OBJECTIVES AND OUTCOMES:

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Rewrite equations of parabolas in standard form.
- Solve applied problems involving parabolas.

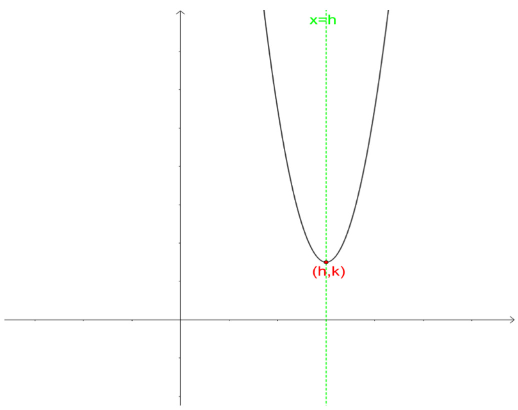
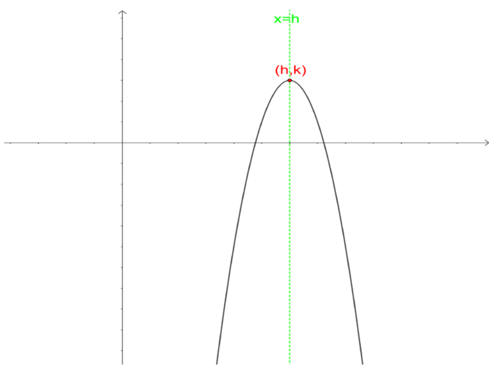
CONTENTS

DEFINITION OF A PARABOLA

Here is a summary of what you should already know about graphing parabolas.

Graphing $y = a(x-h)^2 + k$ and $y = ax^2 + bx + c$.

1. If $a > 0,$ the graph opens upward. If $a < 0,$ the graph opens downward.
2. The vertex of $y = a(x-h)^2 + k$ is (h, k).
3. The x −coordinate of the vertex of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

	
$y = a(x-h)^2 + k, a > 0.$	$y = a(x-h)^2 + k, a < 0.$

Parabolas can be given a geometric definition that enables us to include graphs that open to the left or to the right, as well as those that open obliquely. The definitions of ellipses and hyperbolas involved two fixed points, the foci. By contrast, the definition of a parabola is based on one point and a line.

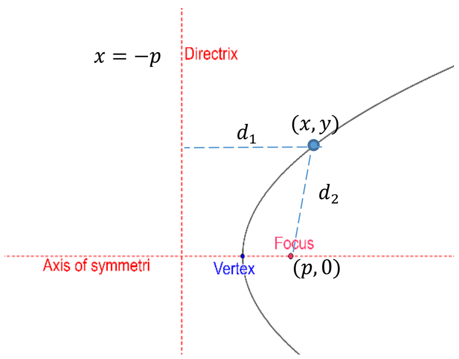


Figure 1

Definition: A parabola is the set of all points in a plane that are equidistant from a fixed line, called the directrix, and a fixed point not on the line, the focus. (see **Figure 1**).

In other words, if given a red line - the directrix, and a point - the focus, then (x, y) is a point on the parabola if the shortest distance from it to the focus and from it to the line is equal.

In **Figure 1**, find the line passing through the focus and perpendicular to the directrix. This is the axis of symmetry of the parabola. The vertex of the parabola is the point where the shortest distance to the directrix is at a minimum. Notice that the vertex is midway between the focus and the directrix. In addition, a parabola is formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone.

STANDARD FORM OF THE EQUATION OF A PARABOLA

The rectangular coordinate system enables us to translate a parabola's geometric definition into an algebraic equation. **Figure 1** is our starting point for obtaining an equation. We place the focus on the x −axis at the point $(p, 0)$. The directrix has an equation given by $x = -p$. The vertex, located midway between the focus and the directrix, is at the origin.

For any point (x, y) on the parabola, the distance d_1 to the directrix is equal to the distance d_2 to the focus. Thus, the point (x, y) is on the parabola if and only if

$$d_1 = d_2.$$

Using distance formula

$$\sqrt{(x+p)^2 + (y-y)^2} = \sqrt{(x-p)^2 + (y-0)^2}$$

and squaring both sides of the equation we find

$$(x+p)^2 = (x-p)^2 + y^2.$$
$$x^2 + 2xp + p^2 = x^2 - 2xp + p^2 + y^2$$
$$y^2 = 4px.$$

This last equation is called the standard form of the equation of a parabola with its vertex at the origin. There are two such equations, one for a focus on the x −axis and one for a focus on the y −axis.

STANDARD FORMS OF THE EQUATIONS OF A PARABOLA

The standard form of the equation of a parabola with vertex at the origin is

$$y^2 = 4px$$

Or

$$x^2 = 4py.$$

Figure 2 illustrates that for the equation on the left, the focus is on the x–axis, which is the axis of symmetry. **Figure 3** illustrates that for the equation on the right, the focus is on the y −axis, which is the axis of symmetry.

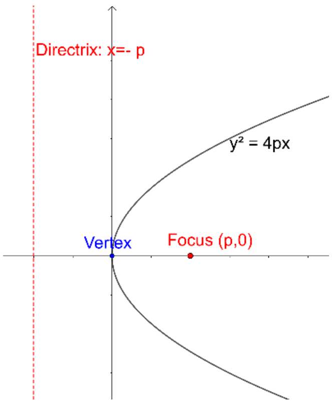


Figure 2 Parabola with the x −axis as the axis of symmetry. If $p > 0$, the graph opens to the right. If $p < 0$, the graph opens to the left

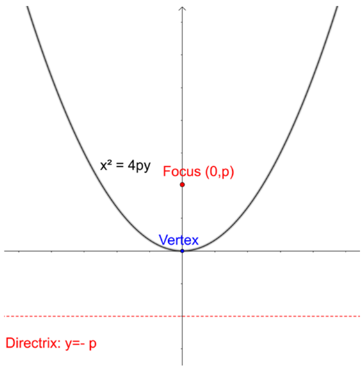


Figure 3 Parabola with the y −axis as the axis of symmetry. If $p > 0$, the graph opens to the upward. If $p < 0$, the graph opens to the downward

USING THE STANDARD FORM OF THE EQUATION OF A PARABOLA

We can use the standard form of the equation of a parabola to find its focus and directrix. Observing the graph's symmetry from its equation is helpful in locating the focus.

Although the definition of a parabola is given in terms of its focus and its directrix, the focus and directrix are not part of the graph. The vertex, located at the origin, is a point on the graph of $y^2 = 4px$ and $x^2 = 4py$. Example 1 illustrates how you can find two additional points on the parabola.

EXAMPLE 1 FINDING THE FOCUS AND DIRECTRIX OF A PARABOLA

Find the focus and directrix of the parabola given by $y^2 = 12x$. Then graph the parabola.

Solution: The given equation, $y^2 = 12x$ is in the standard form $y^2 = 4px$, so $4p = 12$.
We can find both the focus and the directrix by finding p.

$$4p = 12.$$

Diving both sides by 4 we get

$$p = 3.$$

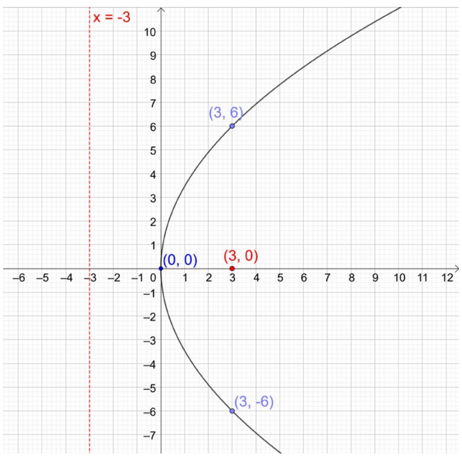


Figure 4 The graph of $y^2 = 12x$

Because p is positive, the parabola, with its x–axis symmetry, opens to the right. The focus is 3 units to the right of the vertex, $(0, 0)$. The focus, $(3, 0)$, and directrix, $x = -3$, are shown in **Figure 4**.

To graph the parabola, we will use two points on the graph that lie directly above and below the focus. Because the focus is at $(3, 0)$, substitute 3 for x in the parabola's equation, $y^2 = 12x$

$$y^2 = 12 \cdot 3$$
$$y^2 = 36.$$

Applying the square root property, we find

$$y = \pm\sqrt{36} = \pm 6.$$

The points on the parabola above and below the focus are $(3, 6)$ and $(3, -6)$. The graph is sketched in **Figure 4**.

In general, the points on a parabola $y^2 = 4px$ that lie above and below the focus, $(p, 0)$, are each at a distance $|2p|$ from the focus. This is because if $x = p$, then $y^2 = 4px = 4p^2$, so $y = \pm 2p$. The line segment joining these two points is called the latus rectum; its length is $|4p|$.

THE LATUS RECTUM AND GRAPHING PARABOLAS

The latus rectum of a parabola is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola. **Figure 5** shows that the length of the latus rectum for the graphs of $y^2 = 4px$ and $x^2 = 4py$ is $|4p|$.

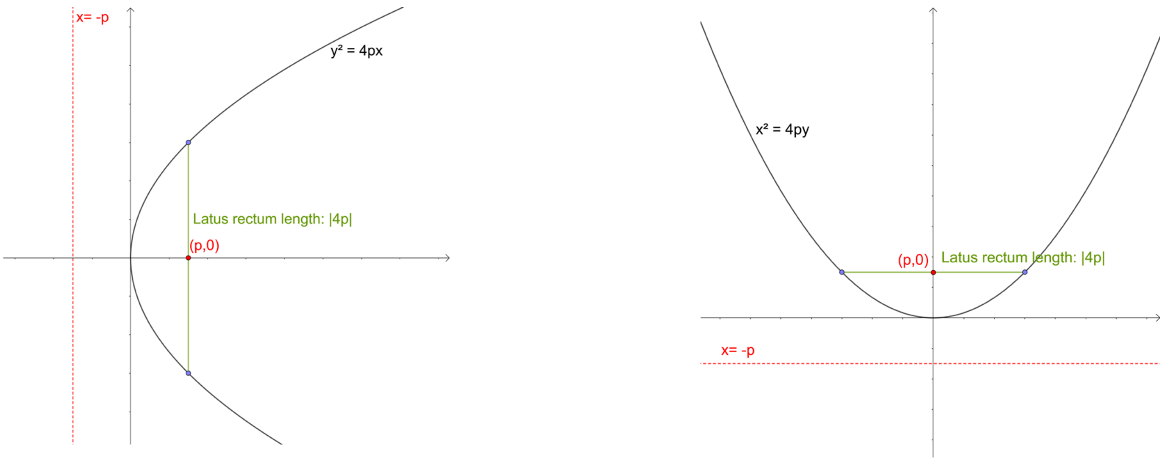


Figure 5 Endpoints of the latus rectum are helpful in determining a parabola's "width," o0r how it opens

EXAMPLE 2 FINDING THE FOCUS AND DIRECTRIX OF A PARABOLA

Find the focus and directrix of the parabola given by $x^2 = -8y$. Then graph the parabola.

Solution: The given equation, $x^2 = -8y$, is in the standard form $x^2 = 4py$, so $4p = -8$. We can find both the focus and directix by finding p

$$4p = -8,$$

Dividing both sides by 4 we get

$$p = -2.$$

Because is negative, the parabola, with its $y -$ axis symmetry, opens downward. The focus is 2 units below the vertex, $(0, 0)$.

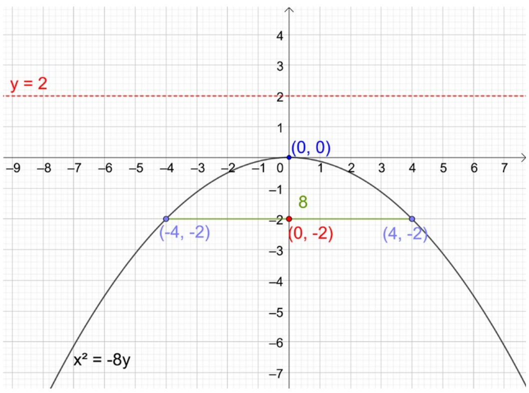


Figure 6 The graph of $x^2 = -8y$

Focus: $(0, p) = (0, -2)$

Directrix: $x = -p;\ y = 2$

The focus and directrix are shown in **Figure 6**.

To graph the parabola, we will use the vertex, $(0, 0)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-2)| = 8.$$

Because the graph has $y-$ axis symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(0,-2)$. The endpoints of the latus rectum are $(-4,-2)$ and $(4,-2)$. Passing a smooth curve through the vertex and these two

points, we sketch the parabola, shown in **Figure 6**.

In examples above, we used the equation of a parabola to find its focus and directrix. In the next example, we reverse this procedure.

EXAMPLE 3 FINDING THE EQUATION OF A PARABOLA FROM ITS FOCUS AND DIRECTRIX

Find the standard form of the equation of a parabola with focus $(5,0)$ and directrix $x=-5$, shown in **Figure 7**.

Solution: The focus is $(5,0)$. Thus, the focus is on $x-$axis. We use the standard from of the equation in which there is $x-$axis, namely $y^2=4px$.

We need to determine the value of p. **Figure 7** shows that the focus is 5 units to the right of vertex, $(0,0)$. Thus, p is positive and $p=5$. We substitute 5 for p in $y^2=4px$ to obtain the standard form of the equation of the parabola. The equation is

$$y^2=4\cdot 5x \quad or \quad y^2=20x.$$

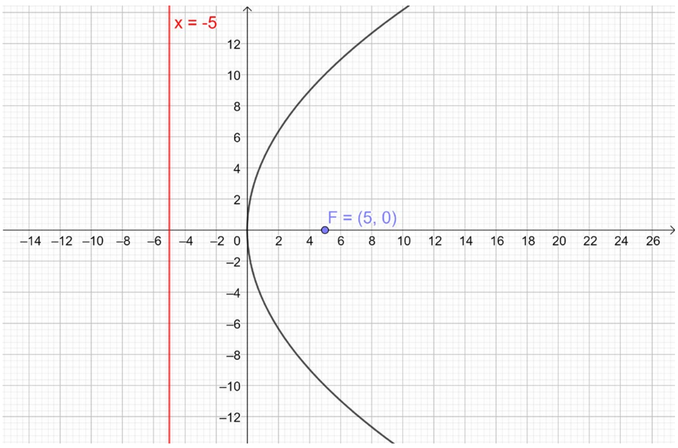


Figure 7

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TRANSLATIONS OF PARABOLAS

The graph of a parabola can have its vertex at (h,k) rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x-h$ and y with $y-k$ in the standard form of the parabola's equation. **Table 1** gives the standard forms of equations of parabolas with vertex at (h,k). **Figure 8** shows their graphs.

Table 1 Standard Forms of Equations of Parabolas with Vertex at (h,k)

Equation	Vertex	Axis of Symmetry	Focus	Directrix	Description
$(y-k)^2=4p(x-h)$	(h,k)	Horizontal	$(h+p,k)$	$x=h-p$	If $p>0$, opens to the right. If $p<0$, opens to the left.
$(x-h)^2=4p(y-k)$	(h,k)	Vertical	$(h,k+p)$	$y=k-p$	If $p>0$, opens to the upward. If $p<0$, opens to the downward.

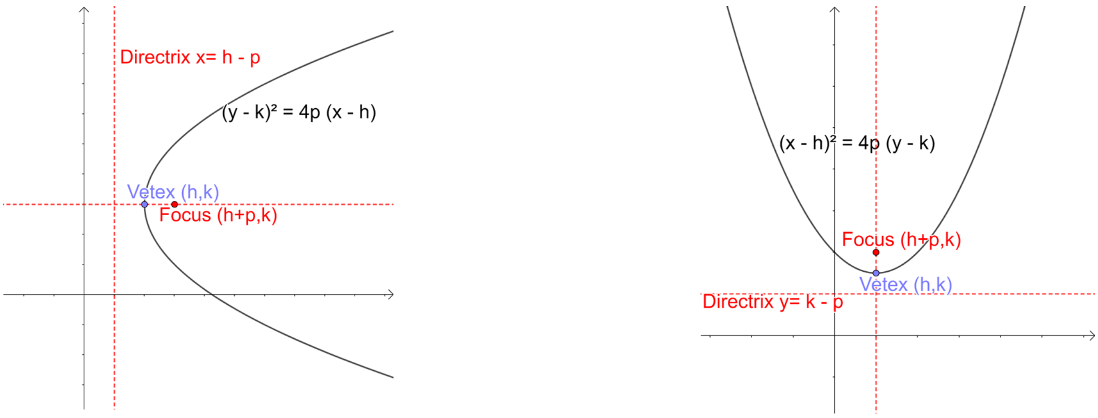

Figure 8 Graphs of parabolas with vertex at (h,k) and $p>0$

EXAMPLE 4 GRAPHING A PARABOLA WITH VERTEX AT (h,k)

Find the vertex, focus, and directrix of the parabola given by

$$(x-3)^2=8(y+1).$$

Then graph the parabola.

Solution: In order to find the focus and directrix, we need to know the vertex. In the standard forms of equations with vertex at (h, k), h is the number subtracted from x and k is the number subtracted from y.

$$(x-3)^2 = 8(y-(-1)).$$

We see that $h = 3$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (3, -1)$. Now that we have the vertex, $(3, -1)$, we can find both the focus and directrix by finding p. The equation

$$(x-3)^2 = 8(y+1)$$

Is in the standard form $(x-h)^2 = 4p(y-k)$. Because x is the square term, there is vertical symmetry and the parabola's equation is a function. Because $4p = 8, p = 2$. Based on the standard form of the equation, the axis of symmetry is vertical. With a positive value for and a vertical axis of symmetry, the parabola opens upward. Because $p = 2$, the focus is located 2 units above the vertex, $(3, -1)$. Likewise, the directrix is located 2 units below the vertex.

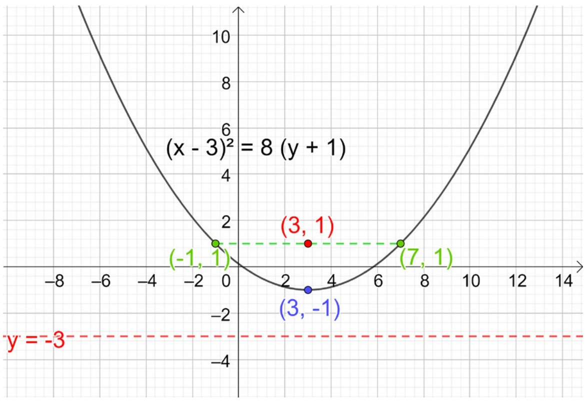

Figure 9 The graph of $(x-3)^2 = 8(y-(-1))$.

Focus: $(h, k+p) = (3, -1+2) = (3,1)$;

Directrix: $y = k - p, \quad y = -1 - 2 = -3.$

Thus, the focus is $(3, 1)$ and the directrix is $y = -3$. They are shown in **Figure 9**. To graph the parabola, we will use the vertex, $(3, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4 \cdot 2| = |8| = 8.$$

Because the graph has vertical symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(3, 1)$. The endpoints of the latus rectum are $(3-4, 1)$, or $(-1, 1)$, and $(3+4, 1)$, or $(7, 1)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in **Figure 9**.

In some cases, we need to convert the equation of a parabola to standard form by completing the square on x or y, whichever variable is squared. Let's see how this is done.

EXAMPLE 5 GRAPHING A PARABOLA WITH VERTEX AT (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$y^2 + 2y + 12x - 23 = 0.$$

Then graph the parabola.

Solution: We convert the given equation to standard form by completing the square on the variable y. We isolate the terms involving y on the left side. This is the given equation

$$y^2 + 2y + 12x - 23 = 0.$$

Isolate the terms involving y

$$y^2 + 2y = -12x + 23.$$

Complete square by adding the square og half the coefficient of y

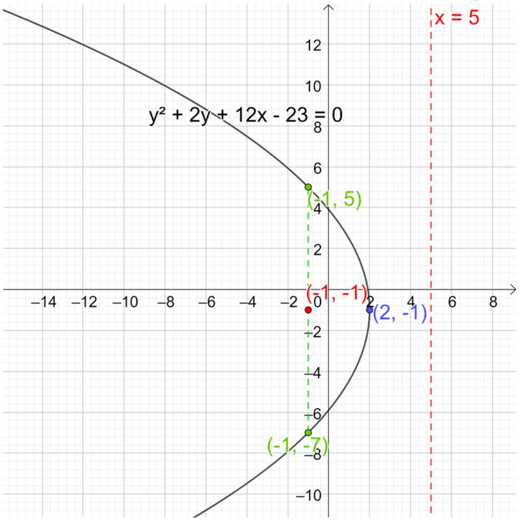

Figure 10 The graph of $y^2 + 2y + 12x - 23 = 0$

$$y^2 + 2y + 1 = -12x + 23 + 1.$$

Factoring

$$(y + 1)^2 = -12x + 24.$$

To express the equation $(y - 1)^2 = -12x + 24$ to identify the vertex, (h, k), and the value p needed to locate the focus and directrix

$$\left(y - (-1)\right)^2 = -12(x - 2).$$

We see that $h = 2$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (2, -1)$. Because $4p = -12$, $p = -3$. Based on the standard form of the equation, the axis of symmetry is horizontal. With a negative value for p and a horizontal axis of symmetry, the parabola opens to the left. Because, $p = -3$, the focus is located 3 units to the left of the vertex, $(2, -1)$. Likewise, the directrix is located 3 units to the right of the vertex.

Focus: $(h + p, k) = (2 + (-3), -1) = (-1, -1)$;

Directrix: $x = h - p, \quad x = 2 - (-3) = 5$.

Thus, the focus is $(-1, -1)$ and the directrix is $x = 5$. They are shown in **Figure 10**. To graph the parabola, we will use the vertex, $(2, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-3)| = |-12| = 12.$$

Because the graph has horizontal symmetry, the latus rectum extends 6 units above and 6 units below the focus, $(-1, -1)$. The endpoints of the latus rectum are $(-1, -1 + 6)$, or $(-1,5)$ and $(-1, -1 - 6)$ or $(-1, -7)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola shown in **Figure 10**.

APPLICATIONS

Parabolas have many applications. Cables hung between structures to form suspension bridges form parabolas. Arches constructed of steel and concrete, whose main purpose is strength, are usually parabolic in shape.

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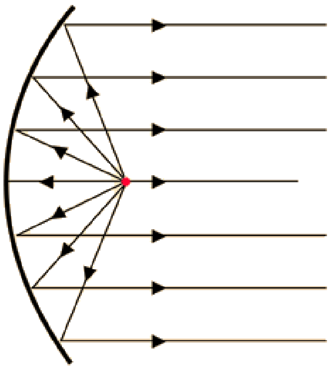
Figure 11 Suspension and arch bridges

We have seen that comets in our solar system travel in orbits that are ellipses and hyperbolas. Some comets follow parabolic paths. Only comets with elliptical orbits, such as Halley's Comet, return to our part of the galaxy.



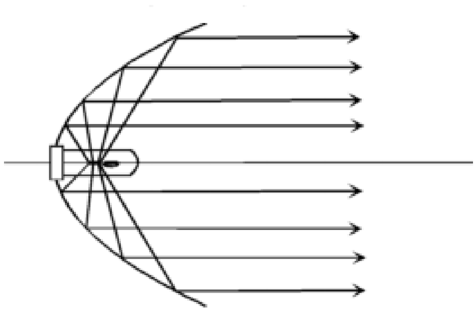


Figure 12 a) Parabolic surface reflecting light *b)Light from the focus is reflected parallel to the axis of symmetry*

If a parabola is rotated about its axis of symmetry, a parabolic surface is formed. **Figure 12a** shows how a parabolic surface can be used to reflect light. Light originates at the focus. Note how the light is reflected by the parabolic surface, so that the outgoing light is parallel to the axis of symmetry. The reflective properties of parabolic surfaces are used in the design of searchlights [see **Figure 12b**], automobile headlights, and parabolic microphones.

EXAMPLE 6 USING THE REFLECTION PROPERTY OF PARABOLAS

An engineer is designing a using a parabolic reflecting mirror and a light source, shown in **Figure 13**. The casting has a diameter of 4 inches and a depth of 2 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

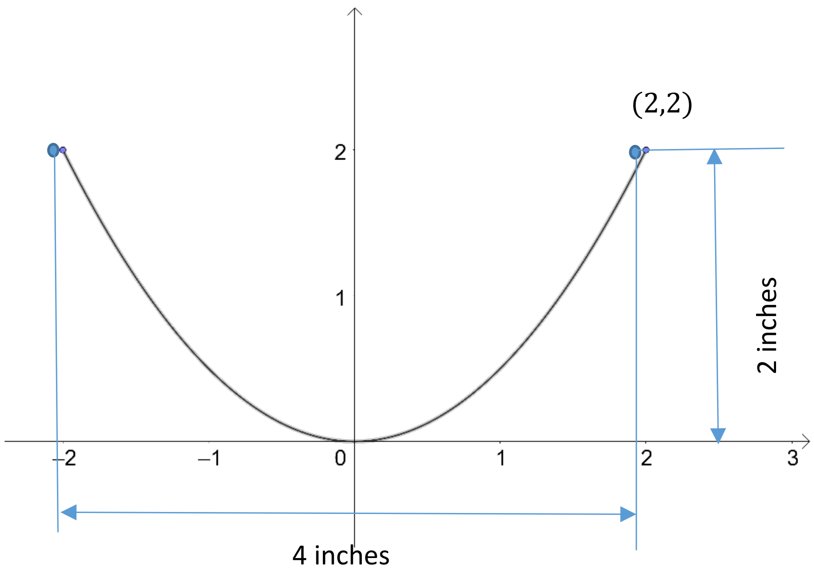


Figure 1 Designing a flashlight

Solution: We position the parabola with its vertex at the origin and opening upward (see **Figure 13**). Thus, the focus is on the y −axis, located at $(0, p)$. We use the standard form of the equation in which there is y −axis symmetry, namely $x^2 = 4py$. We need to find p. Because $(2,2)$ lies on the parabola, we let $x = 2$ and $y = 2$ in $x^2 = 4py$.

Substitute 2 for x and 2 for y

$$2^2 = 4p \cdot 2.$$

Simplifying and dividing both sides of equation by 8 and reducing the resulting

fraction we get

$$4 = 8p,$$

$$p = \frac{1}{2}.$$

We substitute $\frac{1}{2}$ for p in $x^2 = 4py$ to obtain the standard form of the equation of the parabola. The equation of the parabola used to shape the mirror is

$$x^2 = 4 \cdot \frac{1}{2} y$$

or

$$x^2 = 2y.$$

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The light source shold be placed at the focus, $(0,p)$. Because $p=\frac{1}{2}$, the light should be placed at $\left(0,\frac{1}{2}\right)$, or $\frac{1}{2}$ inch above the vertex.

PRACTICE EXERCISES

Find the focus and directrix of each parabola with the given equation. $y^2=4x$;

1. $x^2=4y$;
2. $x^2=-4y$;
3. $y^2=-4x$.

Find the focus and directrix of the parabola with the given equation. Then graph the parabola.

4. $y^2=16x$;
5. $y^2=-8x$;
6. $y^2=16x$;
7. $x^2=-16y$;
8. $y^2-6x=0$;
9. $8x^2+4y=0$;
10. $y^2=4x$;
11. $y^2=-16x$;
12. $x^2=8y$;
13. $x^2=-20y$;
14. $x^2-6y=0$;
15. $8y^2+4x=0$.

Find the standard form of the equation of each parabola satisfying the given conditions.

16. Focus: $(7,0)$; Directrix: $x=-7$;
17. Focus: $(9,0)$; Directrix: $x=-9$;
18. Focus: $(-5,0)$; Directrix: $x=5$;
19. Focus: $(-10,0)$; Directrix: $x=10$;
20. Focus: $(0,15)$; Directrix: $y=-15$;
21. Focus: $(0,20)$; Directrix: $y=-20$;
22. Focus: $(0,-25)$; Directrix: $y=25$;
23. Focus: $(0,-15)$; Directrix: $y=15$;
24. Vertex: $(2,-3)$; Focus: $(2,-5)$;
25. Vertex: $(5,-2)$; Focus: $(7,-2)$;

26. Focus: $(3, 2)$; Directrix: $x = -1$;
27. Focus: $(2, 4)$; Directrix: $x = -4$;
28. Focus: $(-3, 4)$; Directrix: $y = 2$;
29. Focus: $(7, -1)$; Directrix: $y = -9$.

Find the vertex, focus, and directrix of each parabola with the given equation.

30. $(y-1)^2 = 4(x-1)$;
31. $(y-1)^2 = -4(x-1)$;
32. $(x+1)^2 = 4(y+1)$;
33. $(x+1)^2 = -4(y+1)$;

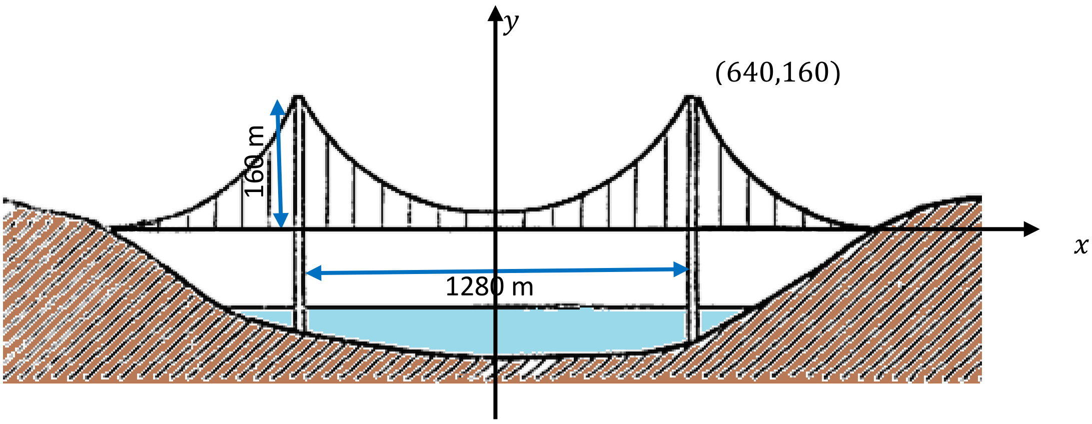
Find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

34. $(x-2)^2 = 8(y-1)$;
35. $(x+1)^2 = -8(y+1)$;
36. $(y+3)^2 = 12(x+1)$;
37. $(y+1)^2 = 8x$;
38. $(x+2)^2 = 4(y+1)$;
39. $(x+2)^2 = -8(y+2)$;
40. $(y+4)^2 = 12(x+2)$;
41. $(y-1)^2 = -8x$.

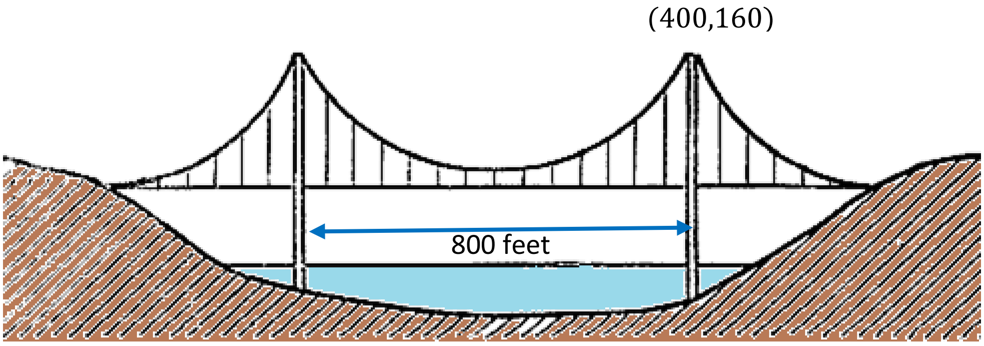
Convert each equation to standard form by completing the square on x or y. Then find the vertex, focus, and directrix of the parabola. Finally, graph the parabola.

42. $x^2 - 2x - 4y + 9 = 0$;
43. $y^2 - 2y + 12x - 35 = 0$;
44. $x^2 + 6x - 4y + 1 = 0$;
45. $x^2 + 6x + 8y + 1 = 0$;
46. $y^2 - 2y - 8x + 1 = 0$;
47. $x^2 + 8x - 4y + 8 = 0$.

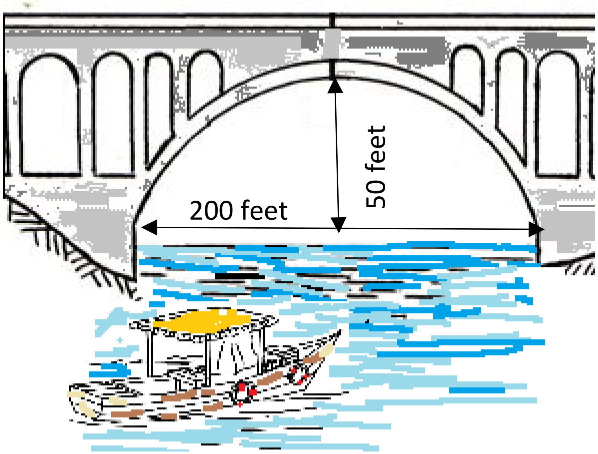
1. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 4 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
2. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 8 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?
3. A satellite dish, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish has a diameter of 12 feet and a depth of 2 feet. How far from the base of the dish should the receiver be placed?
4. In Exercise 3, if the diameter of the dish is halved and the depth stays the same, how far from the base of the smaller dish should the receiver be placed?
5. The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 meters apart and rise 160 meters above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower? Round to the nearest meter.



6. The towers of a suspension bridge are 800 feet apart and rise 160 feet above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 feet from a tower?



7. The parabolic arch shown in the figure is 50 feet above the water at the center and 200 feet wide at the base. Will a boat that is 30 feet tall clear the arch 30 feet from the center?



8. A satellite dish in the shape of a parabolic surface has a diameter of 20 feet. If the receiver is to be placed 6 feet from the base, how deep should the dish be?
9. A domed ceiling is a parabolic surface. Ten meters down from the top of the dome, the ceiling is 15 meters wide. For the best lighting on the floor, a light source should be placed at the focus of the parabolic surface. How far from the top of the dome, to the nearest tenth of a meter, should the light source be placed?