ROTATION OF AXES

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. To recognize a conic section, you often need to pay close attention to its graph. Graphs powerfully enhance our understanding of algebra and trigonometry. However, for people who are blind—or sometimes, it is not possible visually impaired—to see a graph. Creating informative materials for the blind and visually impaired is a challenge for instructors and mathematicians. Many people who are visually impaired "see" a graph by touching a three-dimensional representation of that graph, perhaps while it is described verbally. Is it possible to identify conic sections in nonvisual ways? The answer is yes, and the methods for doing so are related to the coefficients in their equations. As we present these methods, think about how you learn them.

OBJECTIVES AND OUTCOMES:

- Identify conics without completing the square.
- Use rotation of axes formulas.
- Write equations of rotated conics in standard form.
- Identify conics without rotating axes.

CONTENTS

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IDENTIFYING CONIC SECTIONS WITHOUT COMPLETING THE SQUARE

Conic sections can be represented both geometrically (as intersecting planes and cones) and algebraically. In previous sections of this chapter, we have focused on the standard form equations for conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

We can use A and C the coefficients of x^2 and y^2, respectively, to identify a conic section without completing the square.
A nondegenerate conic section of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero, is

- A circle if $A = C$,
- A parabola if $AC = 0$.
- An ellipse if $A \neq C$ and $AC > 0$,
- A hyperbola if $AC < 0$.

EXAMPLE 1 IDENTIFYING A CONIC SECTION WITHOUT COMPLETING THE SQUARE

Identify the graph of each of the following nondegenerate conic sections:

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0$,
b. $x^2 + y^2 + 6x - 2y + 6 = 0$,
c. $y^2 + 12x + 2y - 23 = 0$,
d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0$.

Solution: We use A, the coefficient of x^2, and C, the coefficient of y^2, to identify each conic section.

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0,\ A = 4, C = -25$ and $AC = 4 \cdot (-25) = -100 < 0$. Because $AC < 0$, the graph of the equation is a hyperbola.
b. $x^2 + y^2 + 6x - 2y + 6 = 0,\ A = 1, C = 1$ and $A = C$. Because $A = C$, the graph of the equation is a circle.
c. $y^2 + 12x + 2y - 23 = 0,\ A = 0,\ C = 1$ and $AC = 0 \cdot 1 = 0$. Because $AC = 0$, the graph of the equation is a parabola.
d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0,\ A = 9,\ C = 25$ and $AC = 9 \cdot 25 = 225 > 0$. Because $AC > 0$, and $A \neq C$.the graph of the equation is an ellipse.

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Figure 1 shows the graph of

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

The graph looks like an ellipse, although its major axis neither lies along the x −axis or y −axis nor is parallel to the x −axis or y −axis. Do you notice anything unusual about the equation? It contains an x, y − term. However, look at what happens if we rotate the x − and y −axises through an angle of 30°. In the rotated x', y' −system, the major axis of the ellipse lies along the x' −axis. We can write the equation of the ellipse in this rotated x', y' −system as

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1.$$

Observe that there is no x', y' −term in the equation.

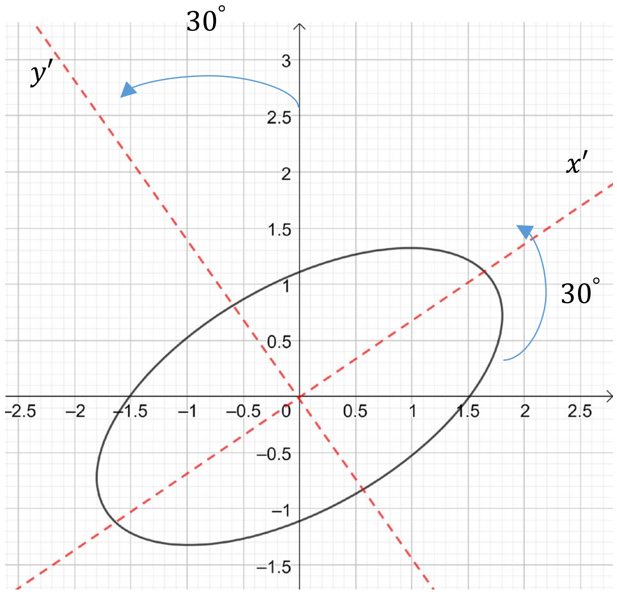

Figure 1 The graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$, *a rotated ellipse*

Except for degenerate cases, the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents one of the conic sections. However, due to the an x, y − term in the equation, these conic sections are rotated in such a way that their axes are no longer parallel to the x − and y −axises. To reduce these equations to forms of the conic sections with which you are already familiar, we use a procedure called rotation of axes.

Suppose that the x − and y −axises are rotated through a positive angle θ. resulting in a new x', y'coordinate system. This system is shown in **Figure 2**. The origin in the x', y' −system is the same as the origin in the x, y −system. Point P in **Figure 3** has coordinates (x, y) relative to the x, y −system and coordinates (x', y') relative to the x', y' −system. Our goal is to obtain formulas relating the old and new coordinates. Thus, we need to express x and y in terms of x', y', and θ.

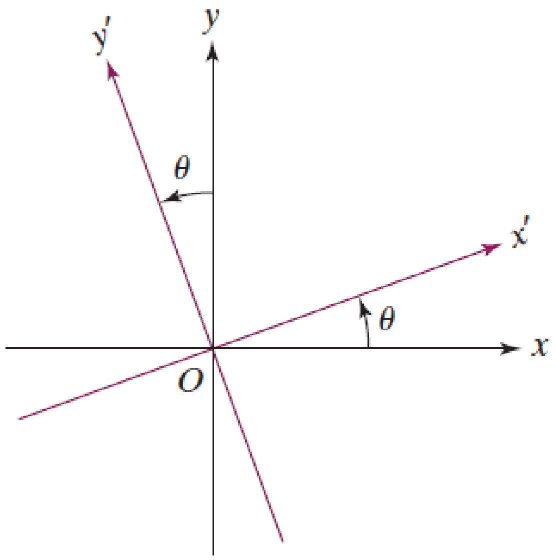

Figure 2 Rotating the x- and t-axes through a positive angle θ

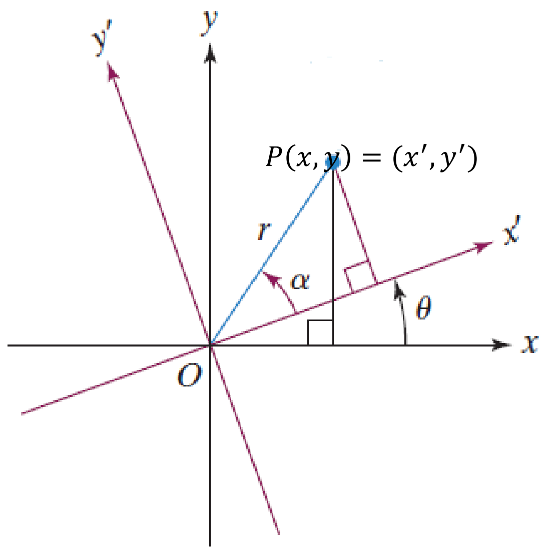

Figure 3 Describing point P relative to the xy-system and the rotated x'y'-system

Look at **Figure 3.** Notice that

r =the distance from the origin O to point P;

α =the angle from the positive x¿-axis to the ray from O through P.

Using the definitions of sine and cosine, we obtain

$$\cos\alpha = \frac{x'}{r} \rightarrow x' = r\cos\alpha\,;$$

$$\sin\alpha = \frac{y'}{r} \rightarrow y' = r\sin\alpha.$$

Those are from the right triangle with a leg along the x' −axis. And from the taller right triangle with a leg along x −axis we find

$$\cos(\theta+\alpha) = \frac{x}{r} \rightarrow x = r\cos(\theta+\alpha);$$

$$\sin(\theta+\alpha) = \frac{y}{r} \rightarrow y = r\sin(\theta+\alpha).$$

Thus,

$$x = r\cos(\theta+\alpha)$$

using the formula for the cosine of the sum of two angles and applying distributive property, and rearranging factors

$$x = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta = x'\cos\theta - y'\sin\theta.$$

Similarly,

$$y = r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = x'\sin\theta + y'\cos\theta.$$

Rotation of Axes Formulas

Suppose an $xy -$ coordinate system and an $x'y' -$ coordinate system have the same origin and θ is the angle from the positive $x -$ axis to the positive $x' -$ axis. If the coordinates of point P are (x, y) in the $xy -$system and (x', y') in the rotated $x'y' -$ system, then

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\ \theta + y' \cos\theta.$$

EXAMPLE 2 ROTATING AXES

Write the equation $xy = 1$ in terms of rotated $x'y' -$ system if the angle of rotatio from the $x -$ axis to the $x' -$ axis is 45°. Express the wquatioon in standard form. Use the rortated system to graph $xy = 1$.

Solution: With $\theta = 45^\circ$, the rotation formulas for x and y are

$$x = x' \cos\theta - y' \sin\theta = x' \cos 45^\circ - y' \sin 45^\circ = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y').$$

$$y = x' \sin\theta - y' \cos\theta = x' \sin 45^\circ - y' \cos 45^\circ = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y').$$

Now substitute these expressions for x and y in the given equation, $xy = 1$.

This is the given equation

$$xy = 1.$$

Substituting the expressions for x and y from the rotation formula we get

$$\frac{\sqrt{2}}{2}(x' - y') \cdot \frac{\sqrt{2}}{2}(x' + y') = 1,$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}(x' - y') \cdot (x' + y') = 1,$$

$$\frac{1}{2}(x'^2 - y'^2) = 1.$$

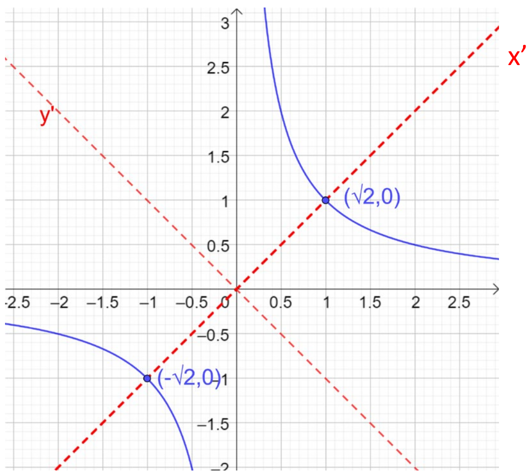

Figure 4 The graph of $xy = 1$

Writing the equation in standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we find

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1.$$

This equation expresses $xy = 1$ in terms of the rotated $x'y' -$ system. You can see that this is the standard form of the equation of a hyperbola. The hyperbola's center is at $(0, 0)$, with the transverse axis on the $x' -$ axis. The vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 2$, the vertices are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. located on the $x' -$ axis. Based on the standard form of the hyperbola's equation, the equations for the asymptotes are

$$y' = \pm\frac{b}{a}x'$$

or

$$y' = \pm\frac{\sqrt{2}}{\sqrt{2}}x' = x'.$$

The equations of the asymptotes can be simplified to $y' = x'$ and $y' = -x'$, which correspond to the original x and y −axes. The graph of the hyperbola is shown in **Figure 4**.

USING ROTATIONS TO TRANSFORM EQUATIONS WITH xy − TERMS TO STANDARD EQUATIONS OF CONIC SECTIONS

We have noted that the appearance of the term $Bxy\ (B \neq 0)$ in the general second-degree equation indicates that the graph of the conic section has been rotated. A rotation of axes through an appropriate angle can transform the equation to one of the standard forms of the conic sections in x' and y' in which no $x'y'$ −term appears.

<u>Amount of Rotation Formula</u>

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0$$

can be rewritten as an equation in x' and y' without an $x'y'$ −term by rotating the axes through angle θ, where

$$\cos 2\theta = \frac{A-C}{B}.$$

Before we learn to apply this formula, let's see how it can be derived. We begin with the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0.$$

Then we rotate the axes through an angle θ. In terms of the rotated $x'y'$ −system the general second-degree equation can be written as

$$(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta)\,x'^2 + [B(\cos^2\theta - \sin^2\theta) + 2(C-A)\sin\theta\cos\theta]x'y'$$
$$+(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta)y'^2 + (D\cos\theta + E\sin\theta)x' + (-D\sin\theta + E\cos\theta)y' + F = 0.$$

After a lot of simplifying that involves expanding and collecting like terms, you will obtain the following equation:

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta)$$
$$+E(x'\sin\theta + y'\cos\theta) + F = 0.$$

If this looks somewhat ghastly, take a deep breath and focus only on the $x'y'$ −term. We want to choose θ so that the coefficient of this term is zero. This will give the required rotation that results in no $x'y'$ −term.

Setting coefficient of the $x'y'$ −term equal to 0 and using the double-angle formulas, we get

$$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0,$$

$$B\cos 2\theta + (C - A)\sin 2\theta = 0.$$

Subtracting $(C - A)$ and 2θ from both sides and simlifying, we find

$$B\cos 2\theta = -(C - A)\sin 2\theta$$

$$B\cos 2\theta = (A - C)\sin 2\theta.$$

Dividing both sides by $B\sin 2\theta$

$$\frac{B\cos 2\theta}{B\sin 2\theta} = \frac{(A - C)\sin 2\theta}{B\sin 2\theta}$$

Simplifying and applying a quotient identity, we get

$$\cot 2\theta = \frac{(A - C)}{B}.$$

If $\cot 2\theta$ is positive, we will select θ so that $0° < \theta < 45°.$ If $\cot 2\theta$ is negative, we will select so that $45° < \theta < 90°.$ Thus $\theta,$ the angle of rotation, is always an acute angle.

Here is a step-by-step procedure for writing the equation of a rotated conic section in standard form:

WRITING THE EQUATION OF A ROTATED CONIC IN STANDARD FORM

1. Use a given equation

   $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0$$

   to find $\cot 2\theta$

   $$\cot 2\theta = \frac{(A - C)}{B}.$$

2. Use the expression for $\cot 2\theta$ to determine θ the angle of rotation.
3. Substitute θ in the rotation formulas

   $$x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta.$$

   and simplify.
4. Substitute the expressions for x and y from the rotation formulas in the given equation and simplify. The resulting equation should have no $x'y'$ −term.
5. Write the equation involving x' and y' in standard form.

   Using the equation in step 5, you can graph the conic section in the rotated $x'y'$ −system.

EXAMPLE 3 WRITING THE EQUATION OF A ROTATED CONIC SECTION IN STANDARD FORM

Rewrite the equation

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

in a rotated $x'y'$ −system without an $x'y'$ −term. Express the equation in the standard form of a conic section. Graph the conic section in the rotated system.

Solution:

1. *Use the given equation to find* $\cot 2\theta$. We need to identify the constants A, B and C in the given equation.
   Because $A = 7, B = -6\sqrt{3}$ and $C = 13$, the appropriate angle θ through which to rotate the axes satisfiies the equation

$$\cot 2\theta = \frac{(A-C)}{B} = \frac{7-13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \ or \frac{\sqrt{3}}{3}.$$

2. *Use the expression for* $\cot 2\theta$ *to determine the angle of rotation.* We have $\cot 2\theta = \frac{\sqrt{3}}{3}$. Based on our knowledge of exact values for trigonometric functions, we conclude that $2\theta = 60°$. Thus, $\theta = 30°$.
3. *Substitute* θ *in the rotation formulas and simplify.* Substituting $30°$ for θ,

$$x = x' \cos 30° - y' \sin 30° = x' \cdot \frac{\sqrt{3}}{3} - y' \cdot \frac{1}{2} = \frac{\sqrt{3}x' - y'}{2},$$

$$y = x' \sin 30° + y' \cos 30° = x' \cdot \frac{1}{2} - y' \cdot \frac{\sqrt{3}}{3} = \frac{x' + \sqrt{3}y'}{2}.$$

4. *Substitute the expressions for* x *and* y *from the rotation formulas in the given equation and simplify.*
   The given equation

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

   Substitute the expressions for x and y

$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\frac{\sqrt{3}x' - y'}{2} \cdot \frac{x' + \sqrt{3}y'}{2} + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0.$$

   Square and multiply

$$7\left(\frac{3x'^2 - 2\sqrt{3}x'y' + y'^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2}{4}\right) + 13\left(\frac{x'^2 + 2\sqrt{3}x'y' + 3y'^2}{4}\right) - 16 = 0.$$

   Multiply both sides by 4

$$7(3x'^2 - 2\sqrt{3}x'y' + y'^2) - 6\sqrt{3}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2) + 13(x'^2 + 2\sqrt{3}x'y' + 3y'^2) - 64 = 0.$$

   Distribute throughout parentheses

$$21x'^2 - 14\sqrt{3}x'y' + 7y'^2 - 18x'^2 - 12\sqrt{3}x'y' + 18y'^2 + 13x'^2 + 26\sqrt{3}x'y' + 39y'^2 - 64 = 0.$$

   Combine like terms

$$16x'^2 + 64y'^2 - 64 = 0.$$

5. *Write the equation involving x' and y' in standard form.* We can express $16x'^2 + 64y'^2 - 64 = 0$, an equation of an ellipse, in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   This equation describes the ellipse relative to a system rotated through $30°$

$$16x'^2 + 64y'^2 - 64 = 0.$$

   Add 64 to both sides

$$16x'^2 + 64y'^2 = 64.$$

   Divide both sides by 64

$$\frac{16x'^2}{64} + \frac{64y'^2}{64} = \frac{64}{64}.$$

   Simplify

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1.$$

The equation $\frac{x'^2}{4} + \frac{y'^2}{1} = 1$ is the standard form of the equation of an ellipse. The major axis is on the x' −axis and the vertices are $(-2,0)$ and $(2,0)$. The minor axis in on the y' −axis with endpoints $(0,-1)$ and $(0,1)$. The graph of the ellipse is shown in Figure 1.

EXAMPLE 4 GRAPHING THE EQUATION OF A ROTATED CONIC

Graph relative to rotated $x'y'$ −system in which the equation has no $x'y'$ −term:

$$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0.$$

Solution:

1. *Use the given equation to find* $\cot 2\theta$. With $A = 16, B = -24, C = 9$ we have

$$\cot 2\theta = \frac{(A - C)}{B} = \frac{16 - 9}{-24} = -\frac{7}{24}.$$

2. *Use the expression for* $\cot 2\theta$ *to determine* $\sin\theta$ *and* $\cos\theta$. A rough sketch showing $\cot 2\theta$ is given in **Figure 5.**

   Because θ is always acute and $\cot 2\theta$ is negative, 2θ is in II quadrant. The third side of the triangle is found using $r = \sqrt{x^2 + y^2}$. Thus, $r = \sqrt{(-7)^2 + 24^2} = 25$. By the definition of the cosine function,

$$\cos\theta = \frac{x}{r} = \frac{-7}{25} = -\frac{7}{25}.$$

   Now we use identities find values for $\sin\theta$ *and* $\cos\theta$.

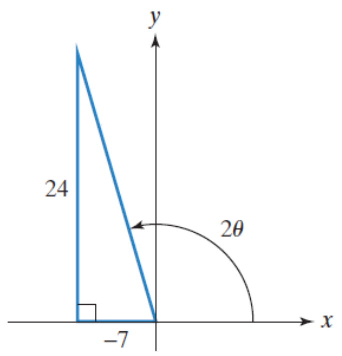

Figure 5 Using $\cot 2\theta$ *to find* $\cos 2\theta$

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}} = \sqrt{\frac{32}{50}} = \frac{4}{5};$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\left(-\frac{7}{25}\right)}{2}} = \sqrt{\frac{18}{50}} = \frac{3}{5}.$$

3. Substitute $\sin\theta$ *and* $\cos\theta$ *in the rotation formulas and simplify*

$$x = x'\cdot\frac{3}{5} - y'\cdot\frac{4}{5} = \frac{3x'-4y'}{5},$$

$$y = x'\cdot\frac{4}{5} + y'\cdot\frac{3}{5} = \frac{4x'+3y'}{5}.$$

4. Substitute the expressions for x and y form the rotation formulas in the given equation and simplify

$$16\left(\frac{3x'-4y'}{5}\right)^2 - 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right) + 9\left(\frac{4x'+3y'}{5}\right)^2 + 110\left(\frac{3x'-4y'}{5}\right) - 20\left(\frac{4x'+3y'}{5}\right) + 100 = 0.$$

Expanding, multiplying both sides of equation by 25, and combining like terms we obtain

$$y'^2 + 2x' - 4y' + 4 = 0,$$

an equation that has no $x'y'$ −term.

5. **Write the equation involving x' and y' in standard form.** With only one variable that is squared, we have the equation of parabola. We need to write the equation in standard form $(y-k)^2 = 4p(x-h)$.

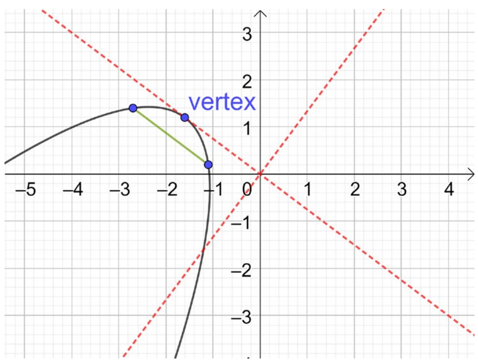

Figure 6 The graph of $(y'-2)^2 = -2x'$ *in a rotated* $x'y'$ *−system*

This is the equation without the $x'y'$ −term

$$y'^2 + 2x' - 4y' + 4 = 0.$$

Isolate the terms involving y'

$$y'^2 - 4y' = -2x' - 4.$$

Complete the square by adding the square of half the coefficient of y' and factor

$$y'^2 - 4y' + 4 = -2x' - 4 + 4.$$

$$(y'-2)^2 = -2x'.$$

We see that $h = 0$ and $k = 2$. Thus, the vertex of the parabola in the $x'y'$ −system is $(h,k) = (0,2)$.

We can use the $x'y'$ −system to graph the parabola. Using a calculator to solve $\sin\theta = \frac{4}{5}$, we find that $\theta = \arcsin\frac{4}{5} \approx 53°$. Rotate the axes through approximately is $53°$. With $4p = -2$ and $p = -\frac{1}{2}$, the parabola's focus is $\frac{1}{2}$ unit to the left of the vertex, $(0,2)$. Thus, the focus in the $x'y'$ −system is $\left(-\frac{1}{2}, 2\right)$.

To graph the parabola, we use the vertex, $(0,2)$, and the two endpoints of the latus rectum

$$length\ of\ latus\ rectum = |4p| = |-2| = 2.$$

The latus rectum extends 1 unit above and 1 unit below the focus, $\left(-\frac{1}{2}, 2\right)$. Thus, the endpoints of the latus rectum in the $x'y'$ −system are $\left(-\frac{1}{2}, 3\right)$ and $\left(-\frac{1}{2}, 1\right)$. Using the rotated system, pass a smooth curve through the vertex and the two endpoints of the latus rectum. The graph of the parabola is shown in **Figure 6**.

IDENTIFYING CONIC SECTIONS WITHOUT ROTATING AXES

We now know that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0$$

can be rewritten as

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0,$$

in a rotated $x'y'$ −system. A relationship between the coefficients of the two equations are given by

$$B^2 - 4AC = -4A'C'.$$

We also know that A' and C' can be used to identify the graph of the rotated equation. Thus, $B^2 - 4AC$ can also be used to identify the graph of the general second-degree equation.

A nondegenerate conic section of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is

- a parabola if $B^2 - 4AC = 0$
- an ellipse or circle if $B^2 - 4AC < 0$, and
- a hyperbola if $B^2 - 4AC > 0$.

EXAMPLE 5 IDENTIFYING A CONIC SECTION WITHOUT ROTATING AXES

Identify the graph of

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0.$$

Solution: We use A, B and C to identify the conic section. We see, that $A = 11, B = 10\sqrt{3}$ and $C = 1$.

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$$B^2 - 4AC = \left(10\sqrt{3}\right)^2 - 4 \cdot 11 \cdot 1 = 256 > 0.$$

Because $B^2 - 4AC > 0$, the graph of the equation is a hyperbola.

Identify each equation without completing the square

1. $y^2 - 4x + 2y + 21 = 0;$
2. $y^2 - 4x - 4y = 0;$
3. $4x^2 - 9y^2 - 8x - 36y - 68 = 0;$
4. $9x^2 + 25y^2 - 54x - 200y + 256 = 0;$
5. $4x^2 + 4y^2 + 12x + 4y + 1 = 0;$
6. $100x^2 - 7y^2 + 90y - 368 = 0;$
7. $9x^2 + 4y^2 - 36x + 8y + 31 = 0;$
8. $y^2 + 8x - 6y + 25 = 0.$

Write each equation in terms of a rotated $x'y'$ −system using θ the angle of rotation. Write the equation involving x' and y' in standard form

9. $xy = -1,\ \theta = 45°;$
10. $xy = -4,\ \theta = 45°;$
11. $x^2 - 4xy + y^2 - 3 = 0,\ \theta = 45°;$
12. $13x^2 - 10xy + 13y^2 - 72 = 0,\ \theta = 45°;$
13. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0,\ \theta = 30°;$
14. $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0,\ \theta = 60°.$

Write the appropriate rotation formulas so that in a rotated system the equation has no $x'y'$ −term

15. $x^2 + xy + y^2 - 10 = 0;$
16. $x^2 + 4xy + y^2 - 3 = 0;$
17. $3x^2 - 10xy + 3y^2 - 32 = 0;$
18. $5x^2 - 8xy + 5y^2 - 9 = 0;$
19. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0;$
20. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0;$
21. $10x^2 + 24xy + 17y^2 - 9 = 0;$
22. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0;$
23. $x^2 + 4xy - 2y^2 - 1 = 0;$
24. $3xy - 4y^2 + 18 = 0;$
25. $34x^2 - 24xy + 41y^2 - 25 = 0;$
26. $6x^2 - 6xy + 14y^2 - 45 = 0.$

Rewrite the equation in a rotated $x'y'-$ system without an $x'y'-$ term.

- Use the appropriate rotation formulas from Exercises 15–26.
- Express the equation involving x' and y' in the standard form of a conic section.
- Use the rotated system to graph the equation

27. $x^2 + xy + y^2 - 10 = 0$;
28. $x^2 + 4xy + y^2 - 3 = 0$;
29. $3x^2 - 10xy + 3y^2 - 32 = 0$;
30. $5x^2 - 8xy + 5y^2 - 9 = 0$;
31. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$;
32. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$;
33. $10x^2 + 24xy + 17y^2 - 9 = 0$;
34. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$;
35. $x^2 + 4xy - 2y^2 - 1 = 0$;
36. $3xy - 4y^2 + 18 = 0$;
37. $34x^2 - 24xy + 41y^2 - 25 = 0$;
38. $6x^2 - 6xy + 14y^2 - 45 = 0$.

Identify each equation without applying a rotation of axes.

39. $5x^2 - 2xy + 5y^2 - 12 = 0$;
40. $10x^2 + 24xy + 17y^2 - 9 = 0$;
41. $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0$;
42. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$;
43. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0$;
44. $4xy + 3y^2 + 4x + 6y - 1 = 0$.

In the next exercises

- If the graph of the equation is an ellipse, find the coordinates of the vertices on the minor axis.
- If the graph of the equation is a hyperbola, find the equations of the asymptotes.
- If the graph of the equation is a parabola, find the coordinates of the vertex.

Express answers relative to an $x'y'-$ system in which the given equation has no $x'y'-$ term. Assume that the $x'y'-$ system has the same origin as the $xy-$ system.

45. $5x^2 - 6xy + 5y^2 - 8 = 0$;
46. $2x^2 - 4xy + 5y^2 - 36 = 0$;
47. $x^2 - 4xy + 4y^2 + 5\sqrt{5}y - 10 = 0$;
48. $x^2 + 4xy - 2y^2 - 6 = 0$.