# PARAMETRIC EQUATIONS

**DETAILED DESCRIPTION:**

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane. In this section, we look at ways of describing curves that reveal the where and the when of motion.

**OBJECTIVES AND OUTCOMES:**

- Use point plotting to graph plane curves described by parametric equations;
- Eliminate the parameter;
- Find parametric equations for functions;
- Understand the advantages of parametric representations.

**CONTENTS**



## PLANE CURVES AND PARAMETRIC EQUATIONS

You throw a ball from a height of 6 feet, with an initial velocity of 90 feet per second and at an angle of 40° with the horizontal. After seconds, the location of the ball, can be described by

$$x = (90\cos 40°)t \;\; and \; y = 6 + (90\sin 40°)t - 16t^{2}.$$

Because we can use these equations to calculate the location of the ball at any time $t$, we can describe the path of the ball. For example, to determine the location when $t = 1$ second, substitute 1 for $t$ in each equation

$$x = (90\cos 40°)t = (90\cos 40°)\cdot 1 \approx 68.9\,feet$$

$$y = 6 + (90\sin 40°)t - 16t^{2} = 6 + (90\sin 40°)\cdot 1 - 16\cdot(1)^{2} \approx 47.9\,feet.$$

This tells us that after one second, the ball has traveled a horizontal distance of approximately 68.9 feet, and the height of the ball is approximately 47.9 feet. **Figure 50** displays this information and the results for calculations corresponding to $t = 2$ seconds and $t = 3$ seconds.

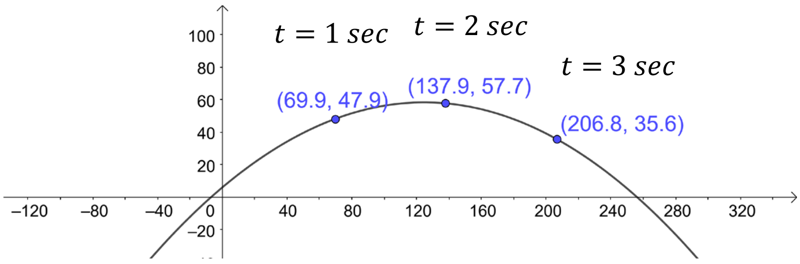


*Figure 1 The location of a thrown hall after 1,2, and 3 seconds*

The textboxes in **Figure 50** tell where the ball is located and when the ball is at a given point $(x, y)$ on its path. The variable $t$ called a parameter, gives the various times for the ball's location. The equations that describe where the ball is located express both and as functions of $t$ and are called parametric equations. The collection of points $(x, y)$ in **Figure 1** is called a plane curve.

## GRAPHING PLANE CURVES

Graphing a plane curve represented by parametric equations involves plotting points in the rectangular coordinate system and connecting them with a smooth curve.

**Graphing a Plane Curve Described by Parametric Equations**

1. Select some values of $t$ on the given interval;

2. For each value of $t$ use the given parametric equations to compute $x$ and $y$;

3. Plot the points $(x, y)$ in the order of increasing and connect them with a smooth curve.

Take a second look at **Figure 1**. Do you notice arrows along the curve? These arrows show the direction, or orientation, along the curve as $t$ increases. After graphing a plane curve described by parametric equations, use arrows between the points to show the orientation of the curve corresponding to increasing values of $t$.

### *EXAMPLE 1 GRAPHING A CURVE DEFINED BY PARAMETRIC EQUATIONS*

Graph the plane curve defined by the parametric equations:

$$x = t^2 - 1, \qquad y = 2t, \qquad -2 \leq t \leq 2.$$

Solution:

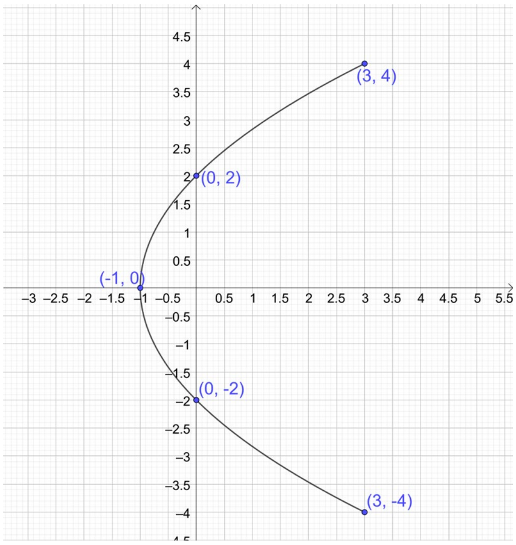


*Figure 2 The plane curve defined by by $x = t^2 - 1,\ y = 2t,$ $-2 \le t \le 2$*

1.**Select some values of on the given interval**. We will select integral values of on the interval $-2 \le t \le 2$. Let $t = -2, -1, 0, 1$ and $2$.

2.**For each value of $t$ use the given parametric equations to compute $x$ and $y$.** We organize our work in a table. The first column lists the choices for the parameter $t$. The next two columns show the corresponding values for $x$ and $y$. The last column lists the ordered pair $(x, y)$.

| $t$ | $x = t^2 - 1$ | $y = 2t$ | $(x, y)$ |
|---|---|---|---|
| $-2$ | $(-2)^2 - 1 = 3$ | $2(-2) = -4$ | $(3, -4)$ |
| $-1$ | $(-1)^2 - 1 = 2$ | $2(-1) = -2$ | $(2, -2)$ |
| $0$ | $(0)^2 - 1 = -1$ | $2 \cdot 0 = 0$ | $(-1,0)$ |
| $1$ | $(1)^2 - 1 = 0$ | $2 \cdot 1 = 2$ | $(0,2)$ |
| $2$ | $(2)^2 - 1 = 3$ | $2 \cdot 2 = 4$ | $(3,4)$ |

3. **Plot the points $(x, y)$ in the order of increasing $t$ and connect them with a smooth curve.** The plane curve defined by the parametric equations on the given interval is shown in **Figure 2**.The arrows show the direction, or orientation, along the curve as $t$ varies from $-2$ to $2$.

**Eliminating the parameter**

The graph in **Figure 51** shows the plane curve for $x = t^2 - 1,\ y = 2t,\ -2 \le t \le 2$. Even if we examine the parametric equations carefully, we may not be able to tell that the corresponding plane curve is a portion of a parabola. By eliminating the parameter, we can write one equation in $x$ and $y$ that is equivalent to the two parametric equations. Begin with the parametric equations

$$x = t^2 - 1$$
$$y = 2t.$$

Solve for $t$ in the second of the equations

$$t = \frac{y}{2}.$$

Substitute the expression for $t$ in the other parametric equation

$$x = t^2 - 1 = \left(\frac{y}{2}\right)^2 - 1 = \frac{y^2}{4} - 1.$$

Co-funded by the
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The rectangular equation (the equation in $x$ and ), $x = \frac{y^2}{4} - 1$, can be written as $y^2 = 4(x+1)$. This is the standard form of the equation of a parabola with vertex at $(-1,0)$ and axis of symmetry along the $x$ −axis. Because the parameter $t$ is restricted to the interval $[-2,2]$, the plane curve in **Figure 2** shows only a part of the parabola. Our discussion illustrates a second method for graphing a plane curve described by parametric equations. Eliminate the parameter $t$ and graph the resulting rectangular equation in $x$ and $y$. However, you may need to change the domain of the rectangular equation to be consistent with the domain for the parametric equation in $x$. This situation is illustrated in the next example.

*EXAMPLE 2 FINDING AND GRAPHING THE RECTANGULAR EQUATION OF A CURVE DEFINED PARAMETRICALLY*

Sketch the plane curve represented by the parametric equations

$$x = \sqrt{t}$$

$$y = \frac{1}{2}t + 1$$

by eliminating the parameter.

**Solution**: We eliminate the parameter $t$ and then graph the resulting rectangular equation.

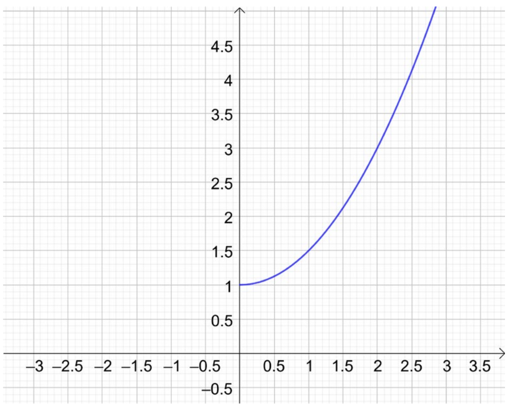


*Figure 3 The plane curve for $x = \sqrt{t}$ and $y = \frac{1}{2}t + 1$*

Using

$$x = \sqrt{t}$$

and squaring both sides,

$$t = x^2.$$

Using previous and $y = \frac{1}{2}t + 1$, we find

$$y = \frac{1}{2}x^2 + 1.$$

Because $t$ is not limited to a closed interval, you might be tempted to graph the entire bowl-shaped parabola whose equation is $y = \frac{1}{2}x^2 + 1$. However, take a second look at the parametric equation for $x$:

$x = \sqrt{t}$. This equation is defined only when $t \geq 0$. Thus, $x$ is nonnegative. The plane curve is the parabola given by $y = \frac{1}{2}x^2 + 1$ with the domain restricted to $x \geq 0$. The plane curve is shown in **Figure 3.** Eliminating the parameter is not always a simple matter. In some cases, it may be not possible. When this occurs, you can use point plotting to obtain a plane curve. Trigonometric identities can be helpful in eliminating the parameter. For example, consider the plane curve defined by the parametric equations

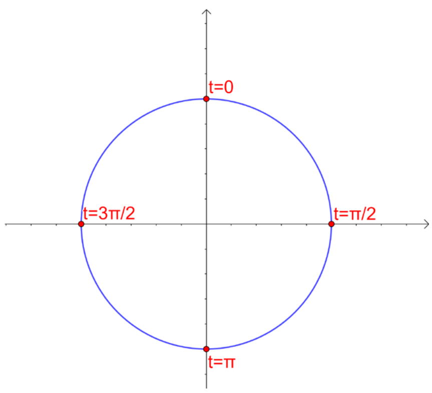


*Figure 4 The plane curve defined by $x = \sin t, \quad y = \cos t, \quad 0 \leq t < 2\pi$*

$$x = \sin t, \qquad y = \cos t, \qquad 0 \leq t < 2\pi.$$

We use the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to eliminate the parameter. Square each side of each parametric equation and then add

$$x^2 = \sin^2 t, \qquad y^2 = \cos^2 t$$
$$x^2 + y^2 = \sin^2 t + \cos^2 t .$$

Using a Pythagorean identity, we write this equation as

$$x^2 + y^2 = 1.$$

The plane curve is a circle with center (0, 0) and radius 1. It is shown in **Figure 4**.

*EXAMPLE 3 FINDING AND GRAPHING THE RECTANGULAR EQUATION OF A CURVE DEFINED PARAMETRICALLY*

Sketch the plane curve represented by the parametric equations

$$x = 5\cos t\,, \quad y = 2\sin t\,, \quad 0 \le t \le \pi$$

by eliminating the parameter.

**Solution**: We eliminate the parameter using the identity $\sin^2 x + \cos^2 x = 1.$ To apply the identity, divide the parametric equation for $x$ by 5 and the parametric equation for $y$ by 2

$$\frac{x}{5} = \cos t\,, \qquad \frac{y}{2} = \sin t.$$

Square and add these two equations

$$\frac{x^2}{25} = \cos^2 t$$

$$\frac{y^2}{4} = \sin^2 t$$

$$\cos^2 t + \sin^2 t = \frac{x^2}{25} + \frac{y^2}{4}.$$

Using a Pythagorean identity, we write this equation as

$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

This rectangular equation is the standard form of the equation for an ellipse centered at $(0,0)$. The ellipse is shown in **Figure 5a**. However, this is not the plane curve. Because $t$ is restricted to the interval $[0,\pi]$, the plane curve is only a portion of the ellipse. Use the starting and ending values for $t$, $0$ and $\pi$ respectively, and a value of $t$ in the interval $(0,\pi)$ to find which portion to include.

| $t = 0$ | $t = \frac{\pi}{2}$ | $t = \pi$ |
|---|---|---|
| $x = 5\cos t = 5\cos 0 = 5$ <br> $y = 2\sin t = 2\sin 0 = 0$ | $x = 5\cos t = 5\cos\frac{\pi}{2} = 0$ <br> $y = 2\sin t = 2\sin\frac{\pi}{2} = 2$ | $x = 5\cos t = 5\cos\pi = -5$ <br> $y = 2\sin t = 3\sin\pi = 0$ |

Points on the plane curve include $(5,0)$, which is the starting point, $(0,2)$, and $(-5,0)$, which is the ending point. The plane curve is the top half of the ellipse, shown in **Figure 5b**.

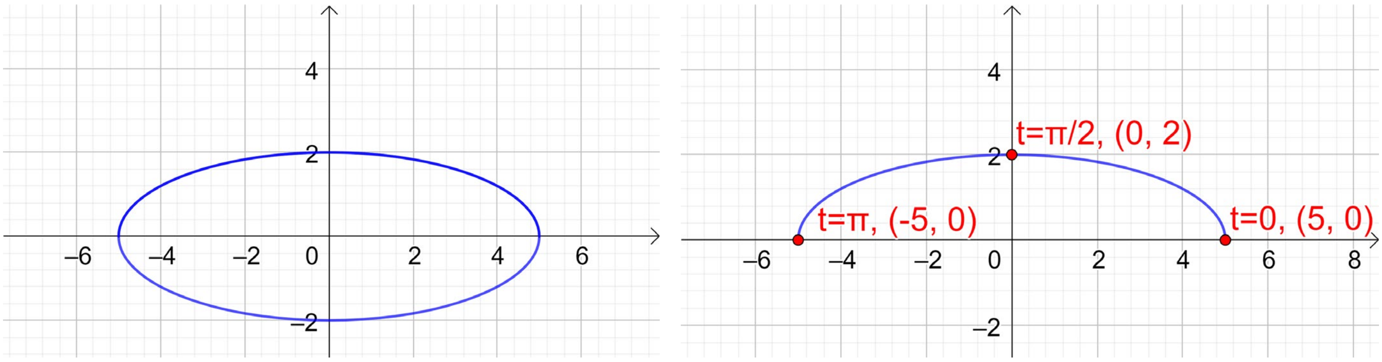


*Figure 5 a) The graph of* $\frac{x^2}{25}+\frac{y^2}{4}=1$ *b) The plane curve for* $x=5\cos t,\quad y=2\sin t,\quad 0\le t\le \pi$

## FINDING PARAMETRIC EQUATIONS

Infinitely many pairs of parametric equations can represent the same plane curve. If the plane curve is defined by the function $y=f(x)$, here is a procedure for finding a set of parametric equations:

One set of parametric equations for the plane curve defined by $y=f(x)$, is

$$x=t,\qquad y=f(t),$$

in which $t$ is in the domain of $f$.

### *EXAMPLE 4 FINDING PARAMETRIC EQUATIONS*

Find a set of parametric equations for the parabola whose equation is $y=9-x^2$.

Solution: Let $x=t$. Parametric equations for $y=f(x)$ are $x=t$ and , $y=f(t)$. Thus, parametric equations for $y=9-x^2$ are

$$x=t,\qquad y=9-t^2.$$

You can write other sets of parametric equations for by starting with a different parametric equation for $y=9-x^2$. Here are three more sets of parametric equations for $y=9-x^2$:

- If $x=t^3$, $y=9-(t^3)^2=9-t^6$. Parametric equations are $x=t^3$ and $t=9-t^6$;
- If $x=t+1$, $y=9-(t+1)^2=8-t^2-2t$. Parametric equations are $x=t+1$ and $t=8-t^2-2t$;
- If $x=\frac{t}{2}$, $y=9-\left(\frac{t}{2}\right)^2=9-\frac{t^2}{4}$. Parametric equations are $x=\frac{t}{2}$ and $t=9-\frac{t^2}{4}$.

Can you start with any choice for the parametric equation for $x$? The answer is no. The substitution for $x$ must be a function that allows $x$ to take on all the values in the domain of the given rectangular equation. For example, the domain of the function $y = 9 - x^2$ is the set of all real numbers. If you incorrectly let $x = t^2$, these values of $x$ exclude negative numbers that are included in $y = 9 - x^2$. The parametric equations

$$x = t^2, \qquad y = 9 - (t^2)^2 = 9 - t^4$$

do not represent $y = 9 - x^2$ because only points for which $x \geq 0$ are obtained.

## ADVANTAGES OF PARAMETRIC EQUATIONS OVER RECTANGULAR EQUATIONS

Parametric equations are frequently used to represent the path of a moving object. If $t$ represents time, parametric equations give the location of a moving object and tell when the object is located at each of its positions. Rectangular equations tell where the moving object is located but do not reveal when the object is in a particular position.

When using technology to obtain graphs, parametric equations that represent relations that are not functions are often easier to use than their corresponding rectangular equations. It is far easier to enter the equation of an ellipse given by the parametric equations

$$x = 2 + 3\cos t\,, \quad y = 3 + 2\sin t$$

than to use the rectangular equivalent

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1.$$

The rectangular equation must first be solved for $y$, and then entered as two separate equations before a graphing utility reveals the ellipse.

Parametric equations and a value for the parameter $t$ are given. Find the coordinates of the point on the plane curve described by the parametric equations corresponding to the given value of $t$

1. $x = 3 - 5t,\ y = 4 + 2t,\ t = 1;$
2. $x = 7 - 4t,\ y = 5 + 6t,\ t = 1;$
3. $x = t^2 + 1,\ y = 5 - t^3,\ t = 2;$
4. $x = t^2 + 3,\ y = 6 - t^3,\ t = 2;$
5. $x = 4 + 2\cos t\,,\ y = 3 + 5\sin t\,,\ t = \frac{\pi}{2};$
6. $x = 2 + 3\cos t\,,\ y = 4 + 2\sin t\,,\ t = \pi;$
7. $x = 60t\cos 30°\,,\ y = 5 + 60t\cos 30° - 16t^2,\ t = 2;$
8. $x = 80t\cos 45°\,,\ y = 6 + 80t\cos 45° - 16t^2,\ t = 2.$

Use point plotting to graph the plane curve described by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of $t$

9. $x = t + 2,\ y = t^2, -2 \le t \le 2;$
10. $x = t - 1,\ y = t^2, -2 \le t \le 2;$
11. $x = t - 2,\ y = 2t + 1, -2 \le t \le 3;$
12. $x = t - 3,\ y = 2t + 2, -2 \le t \le 2;$
13. $x = t + 1,\ y = \sqrt{t}, t \ge 0;$
14. $x = \sqrt{t},\ y = t - 1, 0 \le t;$
15. $x = \cos t\,,\ y = \sin t\,, 0 \le t < 2\pi;$
16. $x = -\sin t\,,\ y = -\cos t\,, 0 \le t < 2\pi;$
17. $x = t^2,\ y = t^3, -\infty < t < \infty;$
18. $x = t^2 + 2,\ y = t^3 - 1, -\infty < t < \infty;$
19. $x = 2t,\ y = |t - 1|, -\infty < t < \infty;$
20. $x = |t + 1|,\ y = t - 2, -\infty < t < \infty.$

Eliminate the parameter $t$. Then use the rectangular equation to sketch the plane curve represented by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of $t$ (If an interval for $t$ is not specified, assume that $-\infty < t < \infty$ )

21. $x = t,\ y = 2t;$
22. $x = t,\ y = -2t;$
23. $x = 2t - 4,\ y = 4t^2;$

24. $x = t - 2,\ y = t^2$;
25. $x = \sqrt{t},\ y = t - 1$;
26. $x = \sqrt{t},\ y = t + 1$;
27. $x = 2\sin t\,,\ y = 2\cos t\,,\ 0 \le t < 2\pi$;
28. $x = 2\sin t\,,\ y = 2\cos t\,,\ 0 \le t < 2\pi$;
29. $x = 1 + 3\cos t\,,\ y = 2 + 3\sin t\,,\ 0 \le t < 2\pi$;
30. $x = -1 + 2\cos t\,,\ y = 1 + 2\sin t\,,\ 0 \le t < 2\pi$;
31. $x = 2\cos t\,,\ y = 3\sin t\,,\ 0 \le t < 2\pi$;
32. $x = 3\cos t\,,\ y = 5\sin t\,,\ 0 \le t < 2\pi$;
33. $x = 1 + 3\cos t\,,\ y = -1 + 2\sin t\,,\ 0 \le t \le \pi$;
34. $x = 2 + 4\cos t\,,\ y = -1 + 3\sin t\,,\ 0 \le t \le \pi$;
35. $x = t^2 + 2,\ y = t^2 - 2$;
36. $x = 2^t,\ y = 2^{-t}, t \ge 0$;
37. $x = e^t,\ y = e^{-t}, t \ge 0$;
38. $x = \sqrt{t} + 2,\ y = \sqrt{t} - 2$.

Eliminate the parameter. Write the resulting equation in standard form.

39. A circle: $x = h + r\cos t\,,\ y = k + r\sin t\,$;
40. An ellipse: $x = h + a\cos t\,,\ y = k + b\sin t\,$;
41. A hyperbola: $x = h + a\sec t\,,\ y = k + b\tan t\,$;

Use your answers from Exercises 39–41 and the parametric equations given in Exercises 39–41 to find a set of parametric equations for the conic section or the line

42. Circle: Center: $(3, 5)$, Radius: 6;
43. Circle: Center: $(4, 6)$, Radius: 9
44. Ellipse: Center: $(-2, 3)$, Vertices: 5 units to the left and right of the center; Endpoints of Minor Axis: 2 units above and below the center;
45. Ellipse: Center: $(4, -1)$, Vertices: 5 units above and below the center; Endpoints of Minor Axis: 3 units to the left and right of the center;
46. Hyperbola: Vertices: $(4, 0)$ and $(-4, 0)$, Foci: $(6, 0)$ and $(-6, 0)$;
47. Hyperbola: Vertices: $(0, 4)$ and $(0, -4)$ Foci: $(0, 5)$ and $(0, -5)$;
48. Line: Passes through $(-2, 4)$ and $(1, 7)$;
49. Line: Passes through $(3, -1)$and $(9, 12)$.

Find two different sets of parametric equations for each rectangular equation

50. $y = 4x - 3$;
51. $y = x^2 + 4$;
52. $y = 2x - 5$;
53. $y = x^2 - 3$.

The parametric equations of four plane curves are given. Graph each plane curve and determine how they differ from each other.

54. a) $x = t,\ y = t^2 - 4$; b) $x = t^2,\ y = t^4 - 4$; c) $x = \cos t\,,\ y = \cos^2 t - 4$; d) $x = e^t,\ y = e^{2t} - 4$;

a) $x = t,\ y = \sqrt{4 - t^2},\ -2 \le t \le 2$; b) $x = \sqrt{4 - t^2},\ y = t\ , -2 \le t \le 2$; c) $x = 2\sin t\,,\ y = 2\cos t\,,\ 0 \le t < 2\pi$; d) $x = 2\cos t\,,\ y = 2\sin t\,,\ 0 \le t$