CONIC SECTIONS IN POLAR CORDINATES

DETAILED DESCRIPTION:

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. Conic sections are curves that result from the intersection of a right circular cone and a plane.

OBJECTIVES AND OUTCOMES:

- Define conics in terms of a focus and directrix;
- Graph the polar equations of conics

CONTENTS

THE FOCUS-DIRECTRIX DEFINITIONS OF THE CONIC SECTIONS

The definition of a parabola is given in terms of a fixed point, the focus, and a fixed line, the directrix. By contrast, the definitions of an ellipse and hyperbola are given in terms of two fixed points, the foci. It is possible to define each of these conic sections in terms of a point and a line. **Figure 1** shows a conic section in the polar coordinate system. The fixed point, the focus, is at the pole. The fixed line, the directrix, is perpendicular to the polar axis.

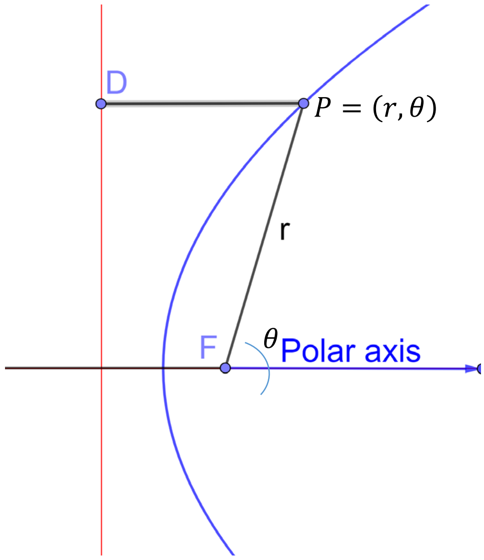


Figure 1 A conic in the polar coordinate system

Focus-directrix definitions of the conic sections

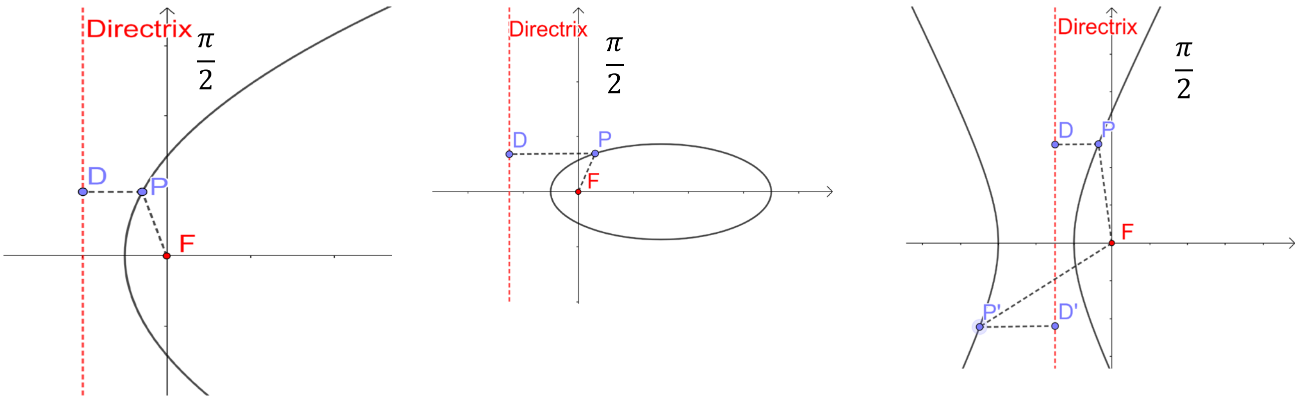
Let F be fixed point, the focus, and let D be a fixed line, the directrix, in a plane in **Figure 1**. A conic section, or conic, is the set of all points P in the plane such that

$$\frac{PF}{PD} = e,$$

where e is a fixed positive number, called the eccentricity.

- If $e = 1$, the conic is a parabola;
- If $e < 1$, the conic is an ellipse;
- If $e > 1$, the conic is a hyperbola.

Figure 2 illustrates the eccentricity for each type of conic. Notice that if $e = 1$, the definition of the parabola is the same as the focus-directrix definition with which you are familiar.



Parabola

$$\frac{PF}{PD} = e = 1$$

Ellipse

$$\frac{PF}{PD} = e < 1$$

Hyperbola

$$\frac{PF}{PD} = \frac{P'F'}{P'D'} = e > 1$$

Figure 2 The eccentricity for each conic

POLAR EQUATION OF CONICS

By locating a focus at the pole, all conics can be represented by similar equations in the polar coordinate system. In each of these equations,

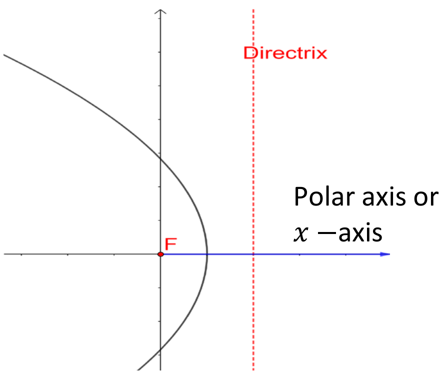
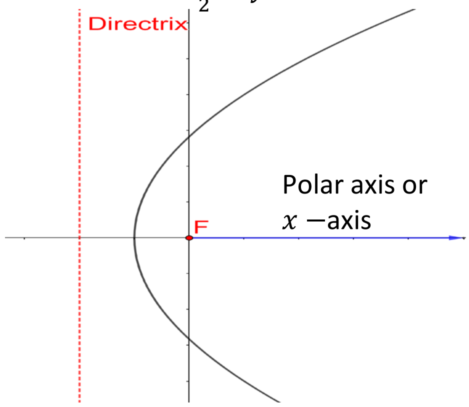
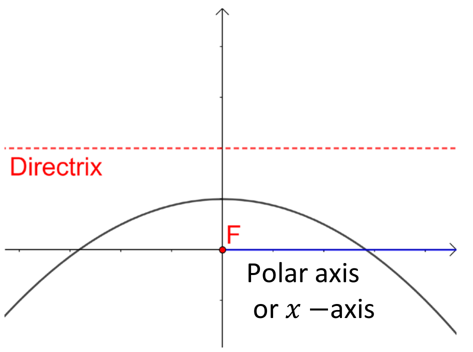
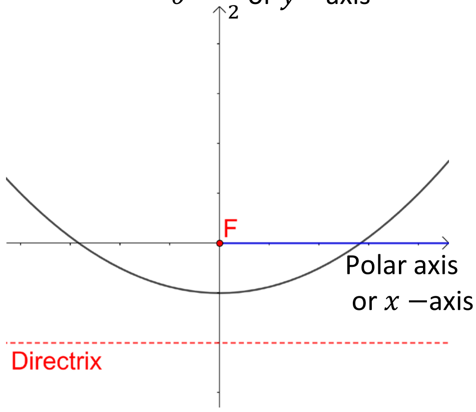
- (r, θ) is a point on the graph of the conic;
- e is the eccentricity (remember that $e > 0$);

- p is the distance between the focus (located at the pole) and the directrix.

STANDARD FORMS OF THE POLAR EQUATIONS OF CONICS

Let the pole be a focus of conic section of eccentricity e with the directrix p units from the focus. The equation of the conic is given by one of the four equations listed

Table 1

$r = \dfrac{e \cdot p}{1 + e\cos\theta}$ Directrix $x = p$ $\theta = \frac{\pi}{2}$ or y −axis  	$r = \dfrac{e \cdot p}{1 - e\cos\theta}$ Directrix $x = -p$ $\theta = \frac{\pi}{2}$ or y −axis 
$r = \dfrac{e \cdot p}{1 + e\sin\theta}$ Directrix $y = p$ $\theta = \frac{\pi}{2}$ or y −axis  	$r = \dfrac{e \cdot p}{1 - e\sin\theta}$ Directrix $y = -p$ $\theta = \frac{\pi}{2}$ or y −axis 

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The graphs above illustrate two kinds of symmetry – symmetry with respect to the y −axis. If the equation contains $\cos\theta$, the polar axis is an axis of symmetry. If the equation contains $\sin\theta$, the line $\theta = \frac{\pi}{2}$, or the y −axis, is an axis of symmetry.

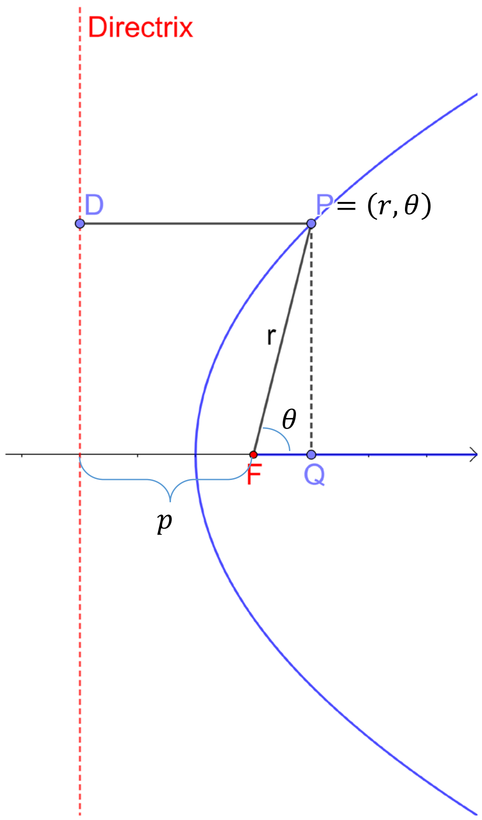


Figure 3

We will derive on of the equations displayed in the table above. The other three equations are obtained in a similar manner. In **Figure 3**, let $P = (r, \theta)$ be any point on a conic section.

By definition, the ratio of the distance between P and the directrix equals the positive constant e

$$\frac{PF}{PD} = e.$$

Figure 3 shows that the distance from P to the focus, lovated at the pole, is r: $PF = r$

$$\frac{r}{PD} = e.$$

Figure 3 shows that the distance from P to the directrix is $p + FQ$: $PD = p + FQ$

$$\frac{r}{p + FQ} = e.$$

Using the triangle in the figure, $\cos\theta = \frac{FQ}{r}$ and $FQ = r\cos\theta$

$$\frac{r}{p + r\cos\theta} = e.$$

By solving this equation for r, we will obtain the desired equation. Multiply both sides by $(p + r\cos\theta)$ and apply the distributive property

$$r = e(p + r\cos\theta) = e \cdot p + e \cdot r\cos\theta.$$

Subtract $e \cdot r\cos\theta$ from both sides to collect terms involving r on the same side

$$r - e \cdot r\cos\theta = e \cdot p.$$

Factor out r from the two terms on the left

$$r(1 - e\cos\theta) = e \cdot p.$$

Divide both sides by $(1 - e\cos\theta)$ and solve for r

$$r = \frac{e \cdot p}{1 - e\cos\theta}.$$

In summary, the standard forms of the polar equations of conic are

$$r = \frac{e \cdot p}{1 \pm e\cos\theta}, \quad r = \frac{e \cdot p}{1 \pm e\sin\theta}.$$

GRAPHING THE POLAR EQUATIONS OF A CONIC

1. If necessary, write the equation in one of the standard forms.
2. Use the standard form to determine values for e and p. Use the value of e to identify the conic.
3. Use the appropriate figure for the standard form of the equation shows in **Table 1** to help guide the graphing process.

EXAMPLE 1 GRAPHING THE POLAR EQUATION OF A CONICS

Graph the polar equation:

$$r = \frac{4}{2 + \cos\theta}.$$

Solution:

1. **Write the equation in the one of the standard forms**. The equation is not in standard form because the constant term in the denominator is not 1. Factor out 2 from the two terms in the denominator

$$r = \frac{4}{2\left(1 + \frac{1}{2}\cos\theta\right)}$$

and the equation in standard form is

$$r = \frac{2}{1 + \frac{1}{2}\cos\theta}.$$

2. **Use standard form to find e and p, and identify the conic**.

$$e = \frac{1}{2}$$

and

$$e \cdot p = 2, \ \frac{1}{2} \cdot p = 2.$$

Multiplying both sides by 2, we get

$$p = 4.$$

Because $e = \frac{1}{2} < 1$, the conic is an ellipse.

3. **Use the figure for the equation's standard form to guide the graphing process**. The figure for the conic's standard form is shown in Figure

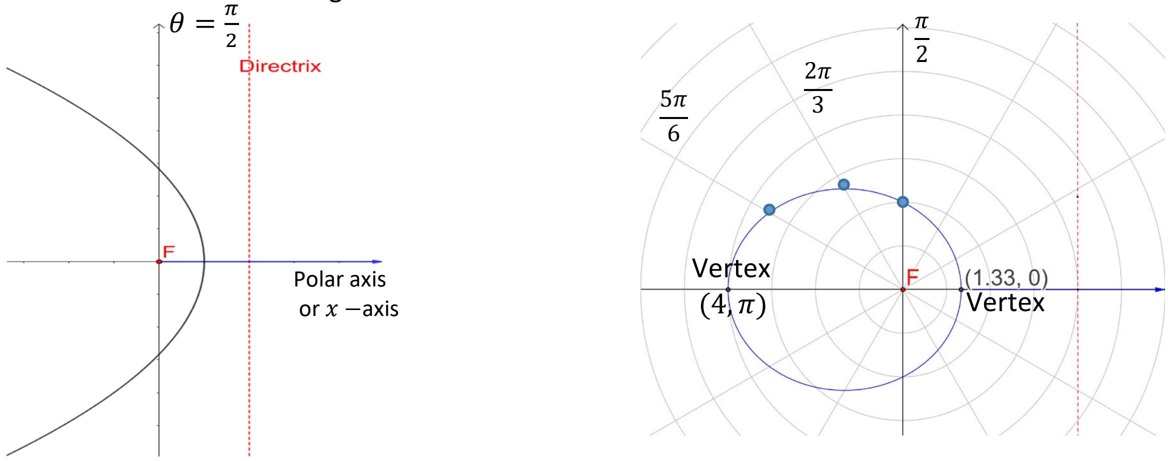


Figure 4 a) Using $\frac{e \cdot p}{1+e\cos\theta}$ *to graph* $r = \frac{2}{1+\frac{1}{2}\cos\theta}$ *b) The graph of* $r = \frac{2}{1+\frac{1}{2}\cos\theta}$

Figure 4a indicates that the major axis is on the polar axis. Thus, we find the vertices by selecting 0 and π for θ. The corresponding values for r are $\frac{4}{3} \approx 1.33$ and 4, respectively. **Figure 4b** shows the vertices, $(1.33,0)$ and $(4, \pi)$.

We can sketch the upper half of the ellipse by plotting some points from $\theta = 0$ to $\theta = \pi$.

$$r = \frac{4}{2 + \cos\theta}$$

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r	2	2.7	3.5

using symmetry with respect to the pola axis, we can sketch he lower half. The graph of the given equation is shown in **Figure 4b**.

EXAMPLE 2 GRAPHING THE POLAR EQUATION OF A CONICS

Graph the polar equation:

$$r = \frac{12}{3 + 3\sin\theta}.$$

Solution:

1. **Write the equation in the one of the standard forms**. The equation is not in standard form because the constant term in the denominator is not 1. Divide the numerator and denominator by 3 to write the standard form

$$r = \frac{4}{1 + \sin\theta}.$$

2. **Use standard form to find e and p, and identify the conic**.

$$e = 1$$

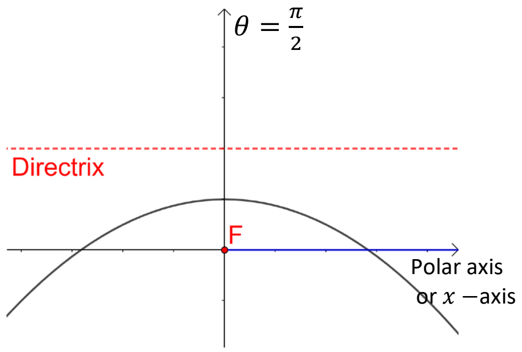
and

$$e \cdot p = 4,\ 1 \cdot p = 4,\ p = 4.$$

Because $e = 1$, the conic is a parabola.

3. **Use the figure for the equation's standard form to guide the graphing process**. **Figure 5a** indicates that we have symmetry with respect to $\theta = \frac{\pi}{2}$. The focus is at the pole, with $p = 4$, the directrix is $y = 4$, located four units above the pole.

   Figure 5b indicates that the vertex is on the line $\theta = \frac{\pi}{2}$, or the y −axis. Thus, we find the vertex by selecting $\frac{\pi}{2}$ for θ. The corresponding value for r is 2. **Figure 5b** shows the vertex $\left(2, \frac{\pi}{2}\right)$.



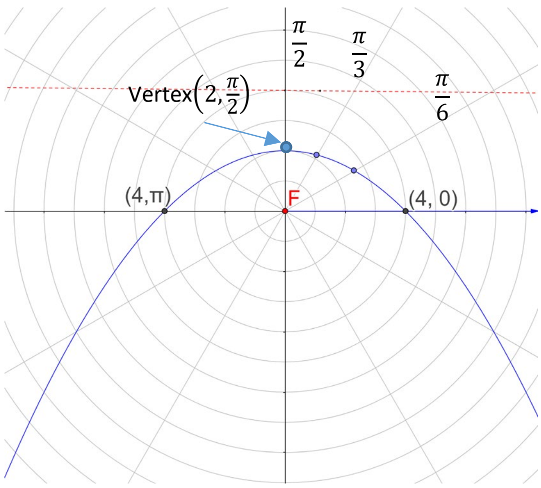


Figure 5 a) Using $\frac{e \cdot p}{1+e\sin\theta}$ *to graph* $r = \frac{4}{1+\sin\theta}$ *b)The graph of* $r = \frac{4}{1+\sin\theta}$

To find where the parabola crosses the polar axis, select $\theta = 0$ and $\theta = \pi$. The corresponding values for r are 4 and 4, respectively. **Figure 5b** shows the points $(4,0)$ and $(4,\pi)$ on the polar axis.

We can sketch the right half of the parabola by plotting some points from $\theta = 0$ and $\theta = \frac{\pi}{2}$.

$$r = \frac{12}{3 + 3\sin\theta}.$$

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$
r	2.7	2.1

Using symmetry with respect to $\theta = \frac{\pi}{2}$, we can sketch the left half. The graph of the given equation is shown in **Figure 5b**.

EXAMPLE 3 GRAPHING THE POLAR EQUATION OF A CONICS

Graph the polar equation:

$$r = \frac{9}{3 - 6\cos\theta}.$$

Solution:

1. **Write the equation in the one of the standard forms**. We can obtain a constant term of 1 in the denominator by dividing each term by 3.

$$r = \frac{3}{1 - 2\cos\theta}.$$

2. Use standard form to find e and p, and identify the conic.

$$e = 2$$

and

$$e \cdot p = 2 \cdot p = 3.$$

Thus, $e = 2$ and $p = \frac{3}{2}$. Because $e = 2 > 1$, the conic is a hyperbola.

3. Use the figure for the equation's standard form to guide the graphing process. Figure 6a indicates that we have symmetry with respect to the polar axis. One focus is at the pole and, with $p = \frac{3}{2}$, the corresponding directrix is $x = -\frac{3}{2}$, located 1.5 units to the left of the pole.

Figure 6a indicates that the transverse axis is horizontal and the vertices lie on the polar axis. Thus, we find the vertices be selecting 0 and π for θ. Figure 60b shows the vertices, $(-3,0)$ and $\left(3, \frac{3\pi}{2}\right)$ on the graph.

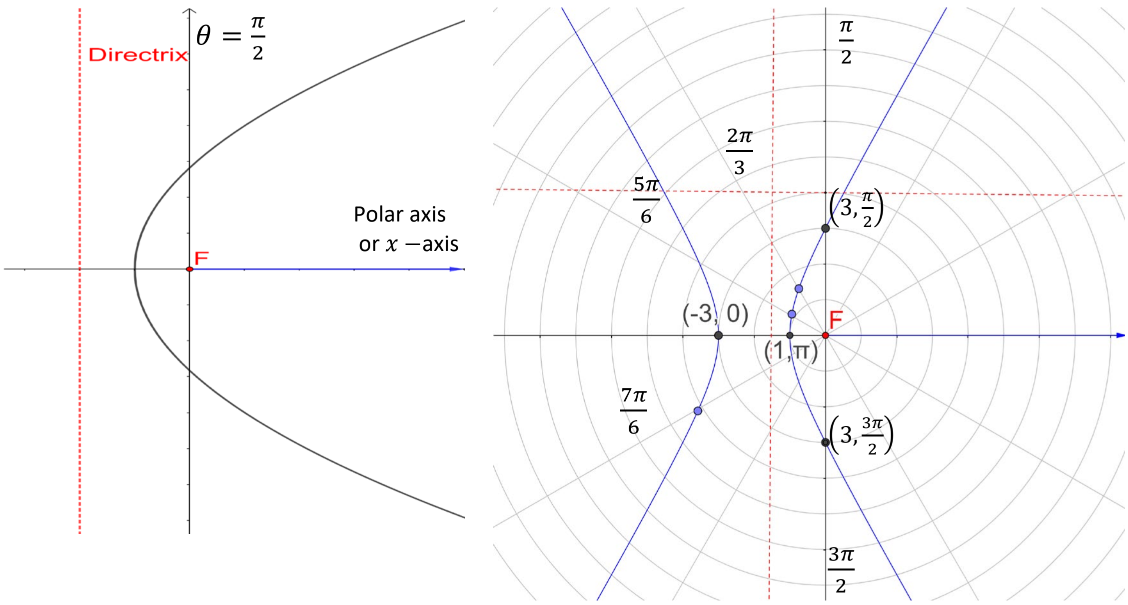


Figure 6 a) Using $r = \frac{e \cdot p}{1 - e\cos\theta}$ to graph $r = \frac{3}{1-2\cos\theta}$ *b) The graph of $r = \frac{3}{1-2\cos\theta}$*

We sketch the hyperbola by plotting some points from $\theta = 0$ to $\theta = \pi$.

$$r = \frac{3}{1 - 2\cos\theta}$$

θ	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r	-4.1	1.5	1.1

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Figure 6b shows the point $\left(\frac{\pi}{6}, -4.1\right)$, $\left(\frac{2\pi}{3}, 1.5\right)$, and $\left(\frac{5\pi}{6}, 1.1\right)$ on the graph. Observe that $\left(\frac{\pi}{6}, -4.1\right)$ is on the lower half. The graph of the given equation is shown in **Figure 6b**.

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PRACTICE EXERCISES

Identify the conic section that each polar equation represents and describe the location of a directrix from the focus located at the pole

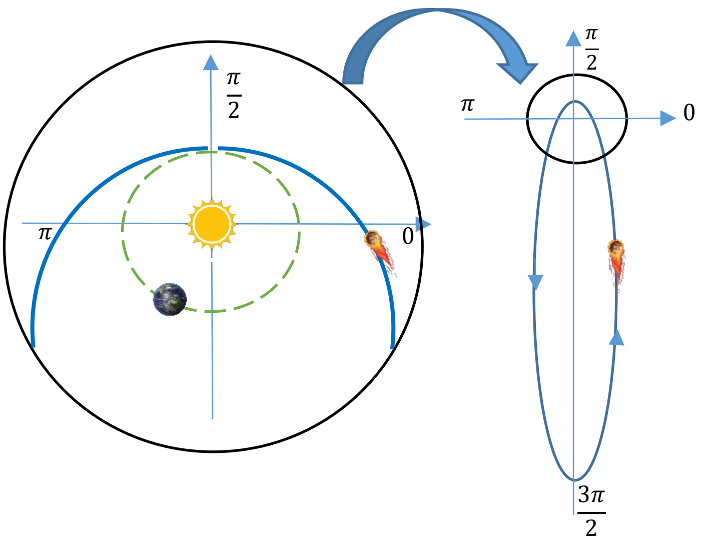
1. $r = \frac{3}{1+3\sin\theta}$;
2. $r = \frac{6}{3-2\cos\theta}$;
3. $r = \frac{8}{2+2\sin\theta}$;
4. $r = \frac{12}{2-4\cos\theta}$;
5. $r = \frac{3}{1+\cos\theta}$;
6. $r = \frac{6}{3+2\cos\theta}$;
7. $r = \frac{8}{2-2\sin\theta}$;
8. $r = \frac{12}{2+4\cos\theta}$.

Use the three steps to graph each polar equation

9. $r = \frac{1}{1+\sin\theta}$;
10. $r = \frac{1}{1+\cos\theta}$;
11. $r = \frac{2}{1-\cos\theta}$;
12. $r = \frac{2}{1-\sin\theta}$;
13. $r = \frac{12}{5+3\cos\theta}$;
14. $r = \frac{12}{5-3\cos\theta}$;
15. $r = \frac{6}{2-2\sin\theta}$;
16. $r = \frac{6}{2+2\sin\theta}$;
17. $r = \frac{8}{2-4\cos\theta}$;
18. $r = \frac{8}{2+4\cos\theta}$;
19. $r = \frac{12}{3-6\cos\theta}$;
20. $r = \frac{12}{3-3\cos\theta}$.

APPLICATION EXERCISES

Halley's Comet has an elliptical orbit with the sun at the focus. Its orbit, shown in the figure below, is given approximately by



$$r = \frac{1.069}{1 + 0.967\sin\theta}.$$

In the formula, r is measured in astronomical units (one astronomical unit is the average distance from Earth to the sun, approximately 93 million miles). Use the given formula and the figure to solve Exercises 21-22. Round to the nearest hundredth of an astronomical unit and the nearest million miles.

21. Find the distance from Halley’s Comet to the sun at its shortest distance from the sun.
22. Find the distance from Halley’s Comet to the sun at its greatest distance from the sun.