6 CALCULUS

6.1. DERIVATIVE

The derivative is one of the essential notions in mathematics. It is necessary for making the so-called differential calculus. The notion was first introduced by the English mathematician, physicist and astronomer Isaac Newton (1642-1727). He described it before 1669, when solving the problem of motion of a body that is moving unevenly along the line, at any moment of its movement. The same discovery was attributed to Gottfried W. Leibniz (1646-1716), German mathematician and philosopher, who independently developed his foundations, while solving the problem of establishing the coefficient of the direction of the curve tangent.

Knowing that vessels move across water areas, we can only assume the actual importance of the application of derivatives in maritime affairs.

The notion of derivation becomes clear with the help of examples.

If $f: I \to R$ (or $y = f(x)$) is the given function on the interval $I \subseteq R$ and if $x_0 \in I$ is a point of the interval (see the figure).

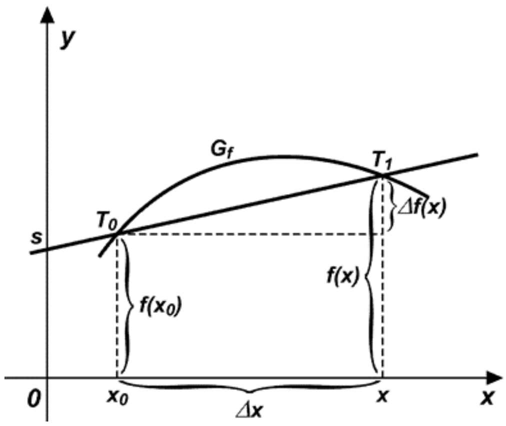

Figure 6.1

If $x \neq x_0$, $x \in I$, , we observe two values of the function: $f(x)$ and $f(x_0)$.The expression $\Delta x = x - x_0$ is called the *change (or differentiation) of the argument x*, while $\Delta f(x_0) = f(x) - f(x_0)$ is called the *change (or increase) of the function at the point x_0*.

Let us now define the difference quotient:

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0. \tag{1}$$

The function $g(x)$ gives information on the rate of change of the function *f* from x_0 to *x,* that is, $g(x)$ measures the average change of the function from x_0 to *x.* The smaller of the argument *x*

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is, the more accurate is the information on the function change of rate that $g(x)$ gives at the point x_0.

Definition: *Differentiable function*

*It is said that the function $f: I \to R$ is **differentiable** (synonym: derivable) at the point $x_0 \in I = (a, b)$ if the limit exists:*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in R. \tag{2}$$

The number $f'(x_0)$ is called the ***derivative f at the point*** x_0. It is said that *f* is differentiable at interval *I* if it is differentiable at any point $x \in I$.

If $f: I \to R$ is differentiable at interval *I*, then $y = f'(x)$ defines a new function $f': I \to R$, the so-called derivative of the function *f* at the interval I.

From the philosophical standpoint, derivation is the ratio of yield to investment. In programming, derivation is the ratio of output to input. In physical world, derivation is the ratio of arbitrarily small travel through arbitrarily little time, i.e. speed.

In geometry, this can be explained by the following figure:

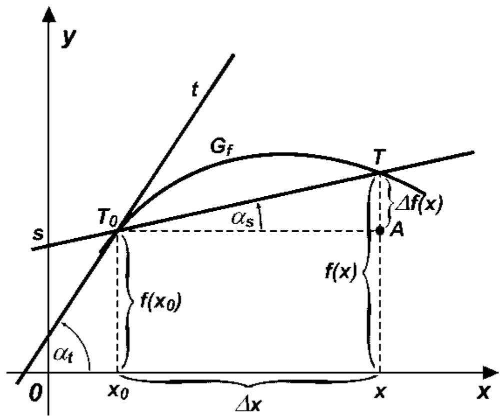

Figure 6.2

In the graph G_f of the function $f: I \to R$ there are points $T_0(x_0, f(x_0))$ and $T(x, f(x))$ to which is assigned the line s, called the ***secant line of the function*** G_f on the interval $[x_0, x]$. The coefficient of the secant line is achieved through formula:

$$\frac{f(x) - f(x_0)}{x - x_0} = k_s \tag{3}$$

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If the point T "moves" across the graph G_f toward the point T_0, then the secant ***s*** turns around the point T_0. If, in this process, there is a limit line t with a position towards which the secant s secant tends to, regardless of whether the point T secant tends to toward T_0, right or left of the T_0, then t is the tangent of the graph G_f at the point T_0.

If the function *f* is differentiable at at the point $x_0 \in I$, then, according to the relation (3), it appears that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} k_s = k_t, \tag{4}$$

Geometric Meaning of the Derivative of a Function,

The **numerical value** $f'(x_0)$ **of the derivative of the function *f* at the point** x_0 **represents the slope coefficient of the tangent line, drawn in the graph** G_f **at the point** $T_0(x_0, f(x_0))$.

Below are the examples of calculating the derivation by definition for some known elementary functions. They are followed by a table of their derivations, which can be proved in analogy with these examples.

Example 1

Let $f(x) = 5x;\ x \in R.$ Find $f'(x_0)$, where $x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{5x - 5x_0}{x - x_0} \lim_{x \to x_0} \frac{5(x - x_0)}{x - x_0} = \lim_{x \to x_0} 5 = 5.$$

As we can see, $f'(x_0)$ does not depend on the point x_0, which is obvious as $f(x) = 5x$ is a linear function whose average change rate is everywhere the same.

Example 2

Let $f(x) = x^2;\ x \in R$. Calculate $f'(x_0)$, where $x_0 \in R$.

Solution:

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$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^2-x_0{}^2}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x+x_0)}{x-x_0} = \lim_{x\to x_0}(x+x_0) = 2x_0.$$

In this way, we have shown that $(x^2)' = 2x$ ¸ $x \in R$.

Example 3

Let $f(x) = x^3$; $x \in R$. Calculate $f'(x_0)$, where $x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^3-x_0{}^3}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x^2+xx_0+x_0{}^2)}{x-x_0} =$$
$$= \lim_{x\to x_0}(x^2+xx_0+x_0{}^2) = 3x_0{}^2.$$

In this way, we have shown that $(x^3)' = 3x^2$ ¸ $x \in R$.

Example 4

Let $f(x) = \sqrt{x}$; $x \in R$. Calculate $f'(x_0)$, where $x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} = \lim_{x\to x_0}\frac{\sqrt{x}-\sqrt{x_0}}{x-x_0}\cdot\frac{\sqrt{x}+\sqrt{x_0}}{\sqrt{x}+\sqrt{x_0}} =$$
$$= \lim_{x\to x_0}\frac{x-x_0}{(x-x_0)\left(\sqrt{x}+\sqrt{x_0}\right)} = \frac{1}{2\sqrt{x_0}}.$$

In this way, we have shown that $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$ ¸ $x \in R$.