6.10. Connections and applications

Examples of applications in the maritime domain.

Maritime affairs do not include only sailing or underway actions and navigation but it brings together terms from vessels, employees and companies over shipbuilding to trade, transport and management. Calculus can help us solve many types of real-world problems in maritime affairs.

We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.). Derivatives are met in many problems in maritime domain especially relating velocity and position and generally, rates of change related quantities.

6.10.1. Related Rates

Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 1:

Ship A is 50 miles west of ship B. The ship A is sailing east at 10 knots, and the ship B is sailing south at 15 knots. Find the rate of change of the distance between the ships after 5 hours.

Solution:

Step 1: Draw a picture placing the problem and introducing the variables.

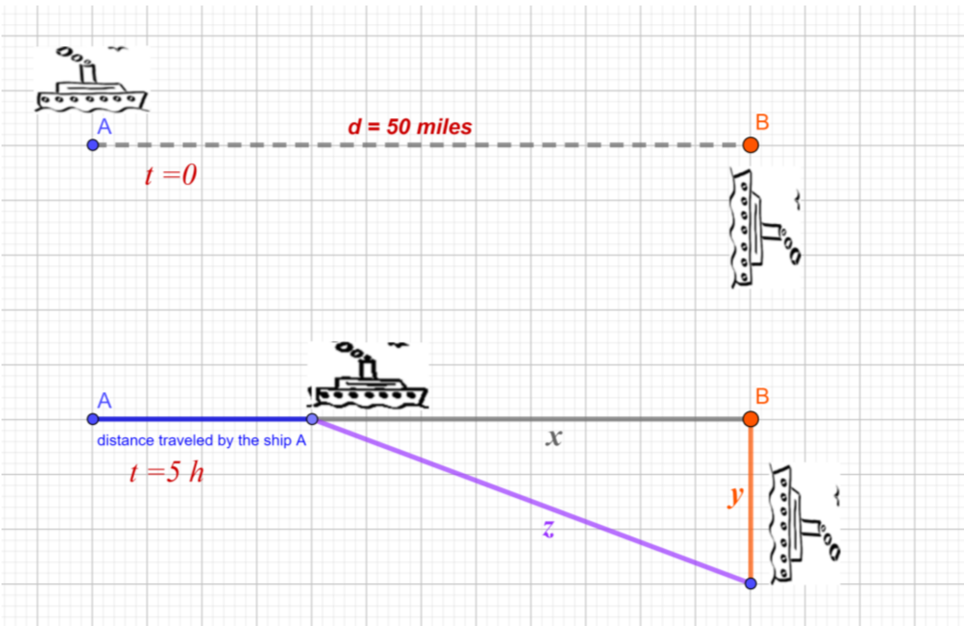


Figure 6.17

Co-funded by the Erasmus+ Programme of the European Union

Let y denote the distance sailed by the ship B, x denote the distance between the current position of the ship A and starting position of the ship B and z denote the ***current*** distance between the ships. Notice that x, y and z are functions of time and d does not depend on time - it is the initial distance between ships (***fixed number***).

Step 2: Since x denotes the horizontal distance between the current position of the ship ***A*** (at the time t) and the start point of the ship *B*, then $\frac{dx}{dt}$ represents the speed of the ship *A*.

It is told the speed of the ship ***A*** is 10 knots (mph) and it implies that the distance x decreases 10 miles every hour. Therefore, $\frac{dx}{dt} = -10\ mph$.

Similarly, *y* denotes the distance between the ***B*** ship position at the time *t* and its start point. $\frac{dy}{dt}$ represents the speed of the ship ***B***. It is told the speed of the ship ***B*** is 15 knots (mph) and it implies that the distance *y* increases 15 miles every hour. Therefore, $\frac{dy}{dt} = 15\ mph$.

Since, it is asked to find the **rate of change** in the distance *z* between the ships after 3 hours, we need to find $\frac{dz}{dt}$ when *t* = 3 *h*.

Step 3: Find the value of *x* and *y* after 3 hours of sailing. We will use input values directly in the formulas.

$$x = d - v_A \cdot t = 50 - 10 \cdot 3 = 30\ miles$$

$$y = v_B \cdot t = 15 \cdot 3 = 45\ miles$$

Reminder:
The formula for distance d, speed v and time t:
$v = \frac{d}{t}$ $or\ d = v \cdot t$ or

Note, *z* is the hypotenuse of the right triangle with side x and side y from above figure. Thus,

$z = \sqrt{x^2 + y^2} = \sqrt{30^2 + 45^2} = 54.08$ miles.

Reminder:

$c^2 = a^2 + b^2$

Step 4: Find the **rate of change** in the distance *z with the respect to time*. It will be done by determination z′ given that x′ = −35 and y′ = 50 .

We can again use the Pythagorean Theorem here. First, write it down and differentiate the equation using derivation of the implicitly given function:

$$z^2 = x^2 + y^2 \Rightarrow$$

$$2zz' = 2xx' + 2yy'$$

$$z' = \frac{2xx' + 2yy'}{2z} = \frac{2 \cdot 30 \cdot (-30) + 2 \cdot 45 \cdot 45}{2 \cdot 54.08} = 20.8 \text{ miles}$$

x, y, and z are all changing over time and so the equation is differentiated using Implicit Differentiation.

Therefore, after 3 hours the distance between ships is changing with the rate of 20.8 miles per hour.

Co-funded by the
Erasmus+ Programme
of the European Union

Example 2:

A ship sails according the law:

$$s = \left(1272.7 \cdot ln\frac{1 + 6 \cdot e^{0.055t}}{7} - 50t\right) \quad [m]$$

The start velocity of the ship according this voyage should be determined.

Solution:

Let *s* is the distance travelled by a ship and it is changing with time. So it can be denoted ***s(t)***.

Since velocity ***v*** is the instantaneous rate of change of travelled distance with respect to time *t* we need to find the value of the derivative s′(t).

$$v = \frac{ds}{dt} = 1272.7 \cdot \frac{7}{1 + 6 \cdot e^{0.055t}} \cdot \frac{6}{7} \cdot e^{0{,}055t} \cdot 0.055 - 50$$

$$v = \frac{420}{1 + 6 \cdot e^{0.055t}} - 50$$

To get the start velocity of the ship it is needed to calculate the s′(t) at $t = 0$.

$$v_0 = \frac{420}{1 + 6} - 50 = 10\ m/s$$

Example 3:

A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 1 m higher than the front of the boat. The rope is being pulled through the ring at the rate of 1 m/sec. How fast is the boat approaching the dock when 8 m of rope are out?

Solution:

Step 1: Draw a picture.

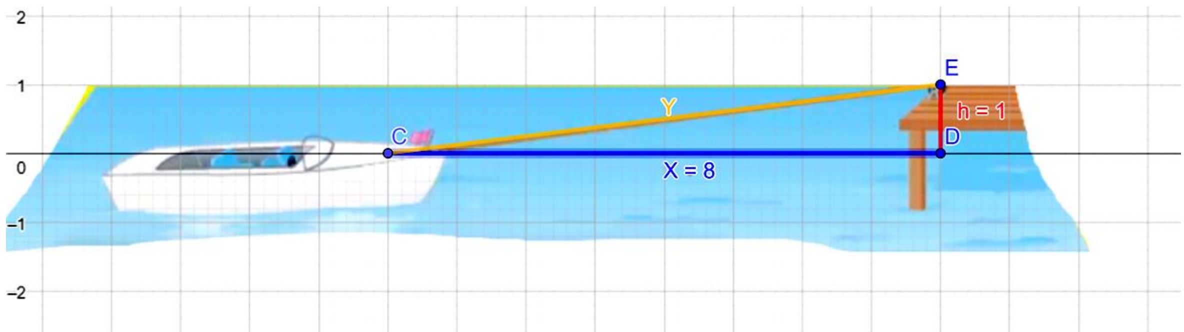


Figure 6.18

Co-funded by the
Erasmus+ Programme
of the European Union

The boat is approaching to the dock. This distance is unknown and let ***x*** denote that distance. It is known that the pulley is 1 meter higher than the front of the boat and let ***h*** denote this height. It is the constant value.

y denotes the length of the rope that the boat is pulled. From the figure above it can be noted that the angle $\sphericalangle CDE$ is the right angle.

Since the rope is being pulled at the rate of 1 m/sec, we know that $\frac{dy}{dt} = -1\ m/sec$. It is the negative value because the length of the rope is shorter and shorter by pulling the boat (it is shorter for 1 meter per second).

If the boat is apart 8 m from the dock, it is needed to find how fast the boat is approaching to the dock, i.e. the rate of change in the distance *d* between the boat and the dock per second. We need to find $\frac{dx}{dt} = ?\, m/s$ when x is 8m.

> Note that both ***x*** and ***y*** are functions of time, and the height ***h*** is the constant.

Step 2: From the right triangle CDE we can use the **Pythagorean Theorem** to write an equation relating ***x*** and ***y*** (***h*** =1m):

$$y^2 = x^2 + 1^2$$

> **Reminder**
> If h is a constant then $\frac{dh}{dt} = 0.$

Step 3: Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation:

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + 0$$

$$\frac{2y}{2x}\frac{dy}{dt} = \frac{2x}{2x}\frac{dx}{dt}$$

$$\frac{2 \cdot 8.06m}{2 \cdot 8\,m} \cdot (-1\ m/s) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{8.06}{8} m/s = -1.011 m/s$$

> We can use the Pythagorean theorem to determine the lenght ***y*** when
>
> *x* = 8m, and the height is 1 m . Solving the equation:
>
> $$y^2 = x^2 + h^2$$
>
> $$y = \sqrt{x^2 + h^2}$$

Co-funded by the Erasmus+ Programme of the European Union

Example 4:

The atmospheric pressure P varies with altitude above sea level ***x*** in accordance with the law:

$$P(x) = P_0 \cdot e^{-0.12104\, x}$$

where P_0 is the atmospheric pressure at sea level. If the atmospheric pressure is 1013 millibars at sea level, how fast the atmospheric pressure is changing with respect to altitude at an altitude of 20 km.

<u>Solution:</u>

The rate of pressure change is derivative of the function $P(x)$ with respect to the altitude x. Thus,

$P'(x) = 1013 \cdot (e^{-0.12104\, x})' = 1013 \cdot (-0.12104) \cdot (e^{-0.12104\, x}) = -122.61 \cdot (e^{-0.12104\, x})$

$P'(20) = -122.61 \cdot (e^{-0.12104 \cdot 20}) = 10.8939$ milibars per km

Example 5:

Determine analytically the output signal of an ideal operational amplifier differentiator and compare graphically the input and output signals when the following values are known.

$U_{input}(t) = 10\sin(2\pi \cdot 3000t)$

$R1 = 5k\Omega$

$C1 = 10nF$

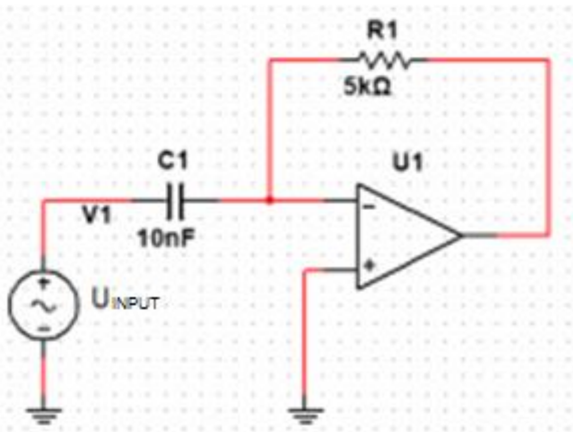


Figure 6.19

Co-funded by the Erasmus+ Programme of the European Union

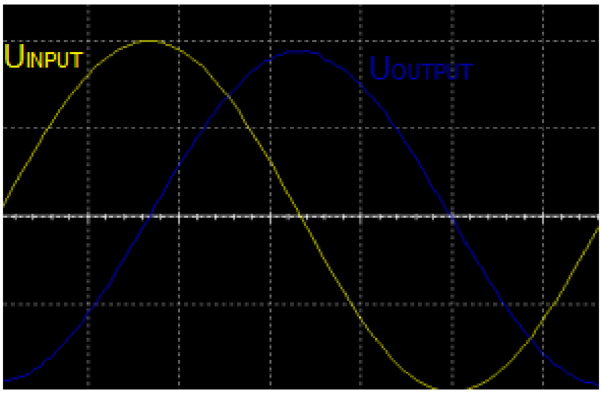
Solution:

The equation for output signal is:

$$U_{output} = -R1 \cdot C1 \cdot \frac{dU_{input}}{dt}$$

$$U_{output} = -5 \cdot 10^3 \cdot 10 \cdot 10^{-9} \cdot 10 \cdot \frac{d[\sin(2\pi \cdot 3000t)]}{dt}$$

$$U_{output} = -9.42 \cdot \cos\,(2\pi \cdot 3000t)$$



Example 6:

The law of rotational motion of the steam turbine during putting in operation should be determined. It is known that the increasing of angular velocity is proportional to third power of time and in the moment $t = 3\ s$ the velocity of rotation of the turbine's rotor is $n = 810\ min^{-1}$.

Solution:

From described problem, the law of rotation motion is proportioned third potential of time and can be expressed as:

$$\varphi = k \cdot t^3.$$

It can be said that the angular velocity is equal to change in angle over a change in time. So if we want to express it in calculus sense it would be the derivative of the angle with respect to time:

$$\omega = \frac{d\varphi}{dt} = 3 \cdot\ k \cdot t^2.$$

Known values can help us to get the proportionality constant *k* from previous equation.

Co-funded by the Erasmus+ Programme of the European Union

$$k = \frac{\omega}{3 \cdot t^2} = \frac{\pi \cdot n}{3 \cdot 30 \cdot t^2}$$

$$k = \frac{\pi \cdot 810}{3 \cdot 30 \cdot 9} = \pi.$$

The law of rotation motion of a steam turbine is:

$$\varphi = \pi \cdot t^3.$$

The angular velocity (ω) and angular acceleration (ε) are as follows:

$$\omega = 3 \cdot \pi \cdot t^2$$

$$\varepsilon = 6 \cdot \pi \cdot t.$$

Example 7:

The bending of the steel truss is given by the equation $f(x) = 10^{-4}(x^5 - 25x^2)$, where x denotes the distance from the girder. Calculate the second derivative (change in the slope coefficient of the tangent) for x = 3.

<u>Solution:</u>

$$y' = \frac{dy}{dx} = \frac{d(10^{-4}x^5)}{dx} - \frac{d\left(10^{-4}25x^2\right)}{dx} = 10^{-4}(5x^4 - 50x)$$

$$y'' = 10^{-4}\left(20x^3 - 50\right)$$

$$y''_{x=3} = 0.049$$

Co-funded by the
Erasmus+ Programme
of the European Union

6.10.2. Optimization problem (minimum, maximum)

Many important applied problems in maritime affairs involve finding the maximum or minimum value of some function like as the minimum time to rich the distance by a ship, the maximum profit, the minimum cost for doing a task, the maximum power of engines and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Example 8:

Two fishing boats sail in the same plane, in the direction, at the same speed, in knots. The sailing directions close an angle of 120°. At one point one of these boats is at the intersection of their directions, while the other boat is at p knots away from the intersection. Find the time when the distance between the boats will be the shortest and what it will be.

Solution:

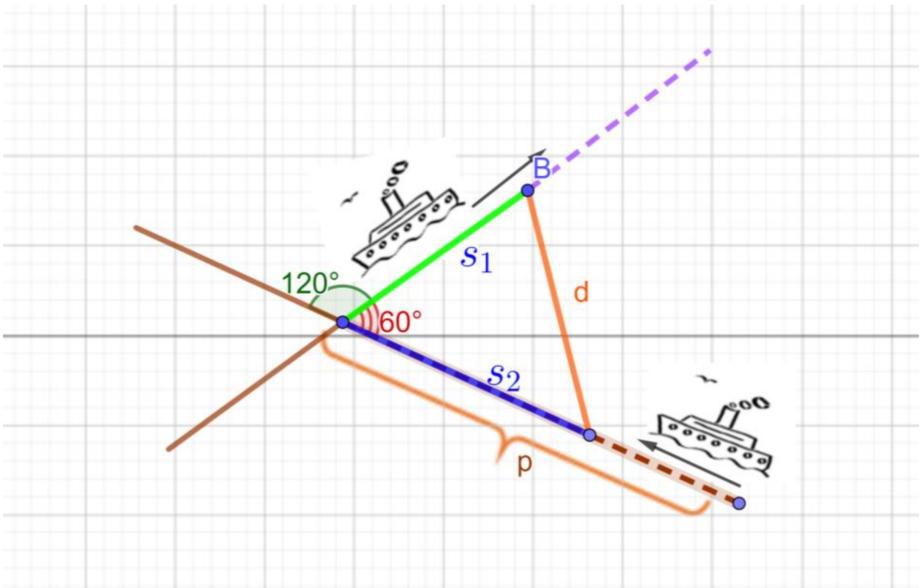


Figure 6.20

$$s = vt \Rightarrow \quad s_1 = vt, \qquad s_2 = p - vt$$

$$d^2 = {s_1}^2 + {s_2}^2 - 2\, s_1 s_2 \cos(60°) = \; v^2t^2 + \left(p^2 - 2pvt + v^2t^2\right) \; - 2vt(p - vt) \cdot \frac{1}{2}$$

$$d^2 = 3\, v^2t^2 - 3pvt + p^2$$

The distance ***d*** varies with time ***t*** and it is a function of time *d(t)*. To find the shortest distance we should solve the equation $d'(t) = 0$.

First we have to find $d'(t)$.

Co-funded by the
Erasmus+ Programme
of the European Union

$$d'(t) = \left(\sqrt{3\,v^2t^2 - 3pvt + p^2}\right)' = \frac{3\,v^2(2t) - 3pv \cdot 1 + 0}{2\sqrt{3\,v^2t^2 - 3pvt + p^2}} = \frac{3(2\,v^2t - pv)}{2\sqrt{3\,v^2t^2 - 3pvt + p^2}}$$

$d'(t) = 0$ if $2\,v^2t - pv = 0 \Rightarrow \; t = \frac{p}{2v}$

<u>Answer</u>: The shortest distance between boats will be at the time $t = \frac{p}{2v}$.

The length of the shortest distance will be as follows.

$$d = \sqrt{3\,v^2t^2 - 3pvt + p^2} = \sqrt{3\,v^2\frac{p^2}{4v^2} - 3pv\frac{p}{2v} + p^2} = \sqrt{\frac{3p^2}{4} - \frac{3p^2}{2} + p^2} = \sqrt{\frac{p^2}{4}}$$

$$d = \frac{p}{2}$$

Example 9:

A man is in a boat at 6 miles offshore, at the point S, and wants to get to a town T on the shore. Point S is $d1 = 6$ miles away from the closest point P on the shore, point T is at the distance $\boldsymbol{d} = 10$ miles down the shore from P.

If the man rows with a speed of $\boldsymbol{v_r} = 3$ miles per hour and walks with a speed of $\boldsymbol{v_w} = 4$ miles per hour at what point R should he land his boat in order to get from point S to point T in the shortest possible time?

<u>Solution:</u>

<u>Step 1:</u> Draw a picture introducing the variables.

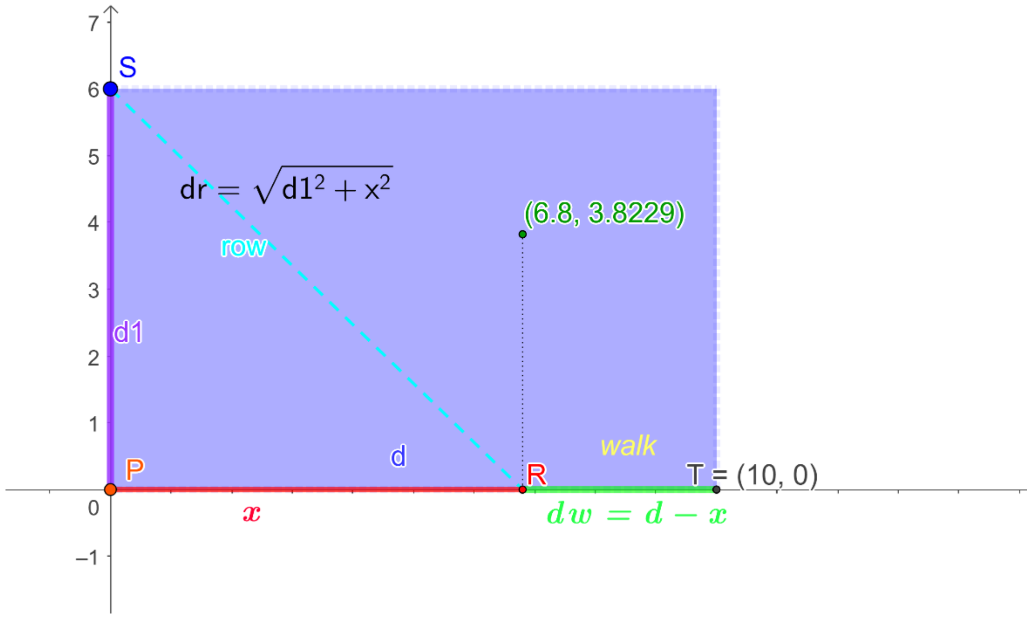


Figure 6.21

Co-funded by the
Erasmus+ Programme
of the European Union

Step 2:

Let is note x the distance down the shore where the boat is landed. On the figure x is the length of $\overline{PR}$. Then the length $\overline{RT}$ is $d - x = 10 - x\ [miles]$.

The question asks us to find **the point** ***R*** which **minimizes** rowing **time**.

$$t_r = \frac{d_r}{v_r}$$

$$t_r = \frac{\sqrt{36 + x^2}}{3}$$

The formula for distance d, speed (velocity) and time t
v = d/t

Step 3: We have to find the walking time.

Using Pythagoras' theorem for the right triangle ΔSPR

$$d_r = \sqrt{d_1^2 + x^2}$$

$$= \sqrt{6^2 + x^2} = \sqrt{36 + x^2}$$

$$t_w = \frac{d_w}{v_w}$$

$$t_w = \frac{10 - x}{4}$$

Step 4:

Since we want to minimize total time by setting the distance x, we should look for a function $t(x)$ representing the total time to rich the point Q from the point S when x is the distance down the shore where the boat is landed. Total time has to be converted into a function minimization problem:

$$t(x) = t_r(x) + t_w(x) = \frac{\sqrt{36 + x^2}}{3} + \frac{10 - x}{4}$$

Step 5. To solve this minimization problem (find the minimum of $t(x)$) we should determine the first derivative with respect to distance x.

$$t'(x) = t_r'(x) + t_w'(x) = \frac{2 \cdot x}{3 \cdot 2 \cdot \sqrt{36 + x^2}} + \frac{(-1)}{4} = \frac{x}{3 \cdot \sqrt{36 + x^2}} - \frac{1}{4}$$

Setting $t'(x) = 0 \Rightarrow$

$$\frac{4x - 3 \cdot \sqrt{36 + x^2}}{4 \cdot 3 \cdot \sqrt{36 + x^2}} = 0$$

$$4x - 3 \cdot \sqrt{36 + x^2} = 0$$

$$4x = 3 \cdot \sqrt{36 + x^2} \,/^2$$

$$16x^2 = 9 \cdot (36 + x^2)$$

$$7x^2 = 324$$

Co-funded by the
Erasmus+ Programme
of the European Union

$$x^2 = \frac{324}{7}$$

$$x = \frac{18}{\sqrt{7}} \approx 6.8$$

We get $x = \frac{18}{\sqrt{7}} \approx 6.8$ *as the only critical value and calculate*

$$t(6.8) = \frac{\sqrt{36 + 6.8^2}}{3} + \frac{10 - 6.8}{4} \approx 3.8229\ hours$$

Step 6:

We have to find a local minimum.

$$t''(x) = \left(\frac{x}{3 \cdot \sqrt{36 + x^2}} - \frac{1}{4}\right)' = \left(\frac{x}{3 \cdot \sqrt{36 + x^2}}\right)' - \left(\frac{1}{4}\right)' =$$

$$= \frac{3 \cdot \sqrt{36 + x^2} - x \cdot 3 \cdot 2x \cdot \dfrac{1}{2\sqrt{36 + x^2}}}{9 \cdot (36 + x^2)} = \frac{3 \cdot (36 + x^2) - 3x^2}{9 \cdot (36 + x^2) \cdot \sqrt{36 + x^2}}$$

$$t''(6.8) = \frac{3 \cdot (36 + 6.8^2) - 3 \cdot 6.8^2}{9 \cdot (36 + 6.8^2) \cdot \sqrt{36 + 6.8^2}} > 0$$

Since, $t''(6.8) > 0$*, there must be a local minimum at x = 6.8 , and since this is the only critical value it must be a global minimum as well.*

Example 10:

A rectangular storage container for bulk cargo with an open top, a square base and a volume of $5000\ m^3$ is to be constructed. What should the dimensions of the container be to minimize the surface area of the container? What is the minimum surface area?

Solution:

Let the variable x represent the length of each side of the square base; let y represent the height of the container and S denotes the surface area of the open-top box.

Co-funded by the
Erasmus+ Programme
of the European Union

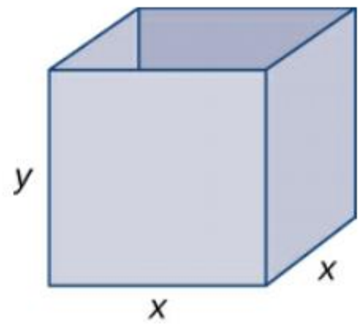


Figure 6.22

The surface area of the open-top container is calculated according following formula:

$$S = 4xy + x^2.$$

Volume of this container is:

$$V = x^2 y = 5000\,m^3$$

$$\Rightarrow y = \frac{5000}{x^2}$$

Therefore, we can write the surface area as a function of only one variable x:

$$S(x) = 4x \cdot \frac{5000}{x^2} + x^2$$

$$S(x) = \frac{20\,000}{x} + x^2, x > 0$$

Critical point:

$$S'(x) = -\frac{20\,000}{x^2} + 2x = 0 \Rightarrow x^3 = 10\,000 \Rightarrow x = 10\sqrt[3]{10}$$

$$\Rightarrow y = \frac{1}{2}\sqrt[3]{100}$$

$$S''(x) = 2 \cdot \frac{20\,000}{x^3} + 2$$

$$S''(10\sqrt[3]{10}) > 0$$

Therefore, $S(x)$ has the minimum at the critical point $x = 10\sqrt[3]{10}$. It implies that is the dimensions of the container should be $x = 10\sqrt[3]{10}, y = \frac{1}{2}\sqrt[3]{100}$.

$$S(x) = \frac{20\,000}{x} + x^2 = \frac{20\,000}{10\sqrt[3]{10}} + \left(10\sqrt[3]{10}\right)^2 = 300\sqrt[3]{10^2}$$

Co-funded by the
Erasmus+ Programme
of the European Union

Example 11:

Owners of a boat rental company have determined that if they charge customers $\boldsymbol{p}$ euros per day to rent a boat, where $(50 \leq p \leq 200)$, then the number of boats n they rent per day can be modelled by the linear function $n(p) = 1000 - 5p$. If they charge €50 per day or less, they will rent all their boats. If they charge €200 per day or more, they will not rent any boats. Assuming the owners plan to charge customers between €50 per day and €200 per day to rent a boat, how much should they charge to maximize their revenue?

<u>Solution:</u>

From described problem, *p* denotes the price charged per boat per day, *n* the number of rented boats per day and *R* revenue per day. We have to find the maximum revenue *R*.

The revenue per day is determined with the number of boats rented per day times the price charged per boat per day. Thus,

$$R = n \cdot p = (1000 - 5p) \cdot p = -5p^2 + 1000p$$

According with the constraint that owners plan to charge between 50 and 200 euro per boat per day, the problem is to find the maximum revenue $R(p)$ (it must be satisfied $p \in [50, 200]$).

R is a continuous function over the closed, bounded interval [50, 200] and it has an absolute maximum in that interval.

$$R'(p) = -10p + 1000 = 0 \Rightarrow p = 100$$

$$R(100) = 50\,000$$

$$p = 50 \Rightarrow R(50) = 37\,500$$

$$p = 200 \Rightarrow R(200) = 0$$

The maximum revenue is reached for $p = 100$.

As conclusion: owners should charge 100 euro per boat per day to maximize their revenue.

Example 12:

The cargo *G* lowers according the law; $x = 80 \cdot t^{2,5}$, where $x[m]$ and $t[s]$. By lowering the cargo, the drum on which the rope holding the load G is wound is rotated. The angular velocity and angular acceleration of the drum must be determined.

Co-funded by the
Erasmus+ Programme
of the European Union

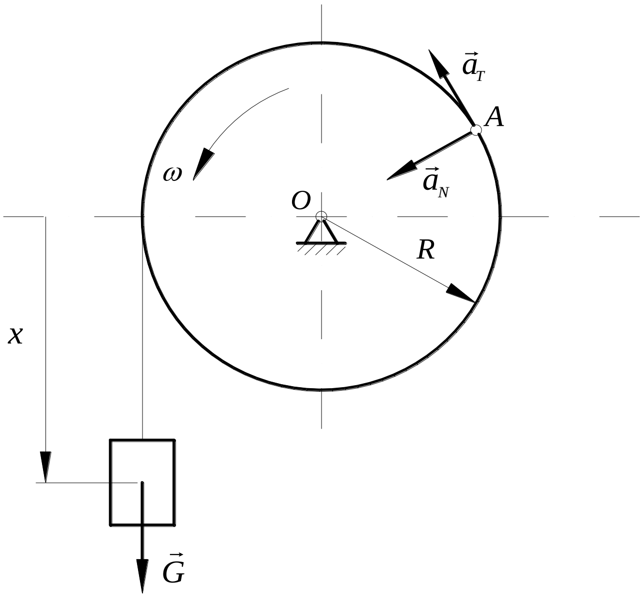


Figure 6.23

Solution:

The translational motion of the load brings the drum into rotational motion. With this movement, the speed of lowering the load is equal to the circumferential speed of rotational movement.

The speed of lowering the cargo is

$$v = \frac{dx}{dt} = \frac{d(80 \cdot t^{2,5})}{dt} = 200 \cdot t^{1,5} \left[\frac{m}{s}\right]$$

The angular velocity of the drum is obtained using the circumferential speed:

$$v = R \cdot \omega, \omega = \frac{v}{R} = \frac{200 \cdot t^{1,5}}{R}$$

$$\omega = 1000 \cdot t^{1,5} \, [s^{-1}]$$

The angular acceleration of the drum is

$$\varepsilon = \frac{d\omega}{dt} = \frac{d(1000 \cdot t^{1,5})}{dt} = 1500 \cdot t^{0,5} \quad [s^{-2}]$$

Co-funded by the
Erasmus+ Programme
of the European Union