6.2. Table of derivatives of elementary functions and basic rules of differentiation

If we continue to solve the examples similar to the ones presented in the previous unit, we can prove the validity of the following ***table of derivatives of elementary functions***, which we can "take for granted":

1)	$C' = 0$	$(C \in R)$
2)	$x' = 1$	
3)	$(x^{\alpha})' = \alpha \cdot x^{\alpha - 1}$	$\alpha \in R$
4)	$(e^x)' = e^x$	
5)	$(a^x)' = a^x \ln a$	$a > 0, a \neq 1$
6)	$(ln\, x)' = \frac{1}{x}$	$x > 0$
7)	$(\log_a x)' = \frac{1}{x \ln a}$	$x > 0,\ a > 0, a \neq 1$
8)	$(sin\, x)' = cos\, x$	
9)	$(cos\, x)' = -sin\, x$	
10)	$(\tan x)' = \frac{1}{\cos^2 x}$	
11)	$(\cot x)' = -\frac{1}{\sin^2 x}$	
12)	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
13)	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.	$\|x\| < 1$

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14)	$(\arctan x)' = \frac{1}{1+x^2}$	
15)	$(\text{arccot}\, x)' = -\frac{1}{1+x^2}$	
16)	$(\sinh x)' = \cosh x$	
17)	$(\cosh x)' = \sinh x$	
18)	$(\tanh x)' = \frac{1}{\cosh^2 x}$	
19)	$(\text{coth}x)' = -\frac{1}{\sinh^2 x}$	
20)	$(\text{arsinh}\, x)' = \frac{1}{\sqrt{1+x^2}}$	
21)	$(\text{arcosh}\, x)' = \frac{1}{\sqrt{x^2-1}}$	$\|x\| > 1$
22)	$(\text{artanh}x)' = \frac{1}{1-x^2}$	$\|x\| < 1$
23)	$(\text{arcoth}x)' = -\frac{1}{x^2-1}$	$\|x\| > 1$

Derivation of any other function should be calculated by applying the ***basic rules of differentiation rules*** as follows:

$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$	derivation of the sum or the difference of functions
$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$	derivation of the product rule
$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$	derivation of the quotient rule $(g(x) \neq 0)$
$\{g[f(x)]\}' = g'[f(x)] \cdot f'(x)$	derivation of the chain rule

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Example 1

Find the derivative of the given function $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution:

$$f'(x) = \left(6x^3 - 4x^2 + 3x - 2\right)' = 6 \cdot \left(x^3\right)' - 4 \cdot \left(x^2\right)' + 3 \cdot x' - 2' =$$
$$= 6 \cdot 3x^2 - 4 \cdot 2x + 3 \cdot 1 - 0 = 18x^2 - 8x + 3.$$

Example 2

Find the derivative of the given function $h'(x)$ if $h(x) = e^x \cdot \sin x$.

Solution:

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = \left(e^x\right)' \cdot \sin x + e^x \cdot (\sin x)' =$$
$$= e^x \cdot \sin x + e^x \cdot \cos x = e^x(\sin x + \cos x).$$

Example 3

Find the derivative of the given function $h'(x)$ if $h(x) = \dfrac{x^2 + 3x + 1}{x^2 - 3x + 2}$.

Solution:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} = \frac{\left(x^2 + 3x + 1\right)'\left(x^2 - 3x + 2\right) - \left(x^2 + 3x + 1\right)\left(x^2 - 3x + 2\right)'}{\left(x^2 - 3x + 2\right)^2} =$$
$$= \frac{(2x + 3)\left(x^2 - 3x + 2\right) - \left(x^2 + 3x + 1\right)(2x - 3)}{\left(x^2 - 3x + 2\right)^2} = \frac{-6x^2 + 2x + 9}{\left(x^2 - 3x + 2\right)^2}.$$

Example 4

Find the derivatives:

(1.) $h(x) = e^{3x^2 - 2x + 1}$;

(2.) $h(x) = \left(2x^2 - 3x\right)^3$;

(3.) $h(x) = \sin\left(3x^3 - 4\right)$

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Solution:

(1.) $h'(x)=\left(e^{3x^2-2x+1}\right)'=e^{3x^2-2x+1}\cdot\left(3x^2-2x+1\right)'=e^{3x^2-2x+1}\cdot(6x-2)$

(2.) $h'(x)=\left(\left(2x^2-3x\right)^3\right)'=3\left(2x^2-3x\right)^2\left(2x^2-3x\right)'=3\left(2x^2-3x\right)^2\cdot(4x-3)$

(3.) $h'(x)=\left(\sin\left(3x^3-4\right)\right)'=\cos\left(3x^3-4\right)\cdot\left(3x^3-4\right)'=\cos\left(3x^3-4\right)\cdot\left(9x^2\right)=9x^2\cos\left(3x^3-4\right)$

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Exercises 6.1

Find the derivatives of the following functions:

(1.) $f(x)=3\cdot\sqrt[6]{x^5}$; (2.) $g(x)=\dfrac{2x^2-3}{\sqrt[3]{x^2}}$.

Solution:

(1.) $f(x)=3\cdot\sqrt[6]{x^5}=3x^{\frac{5}{6}}$.

$$f'(x)=3\cdot\frac{5}{6}x^{\frac{5}{6}-1}=\frac{5}{2}x^{-\frac{1}{6}}=\frac{5}{2x^{\frac{1}{6}}}=\frac{5}{2\cdot\sqrt[6]{x}}.$$

(2.) $g(x)=\dfrac{2x^2-3}{\sqrt[3]{x^2}}=\dfrac{2x^2-3}{x^{\frac{2}{3}}}=2x^{\frac{4}{3}}-3x^{-\frac{2}{3}}$.

$$g'(x)=2\cdot\frac{4}{3}x^{\frac{1}{3}}-3\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}=\frac{8}{3}x^{\frac{1}{3}}+\frac{2}{x^{\frac{5}{3}}}=\frac{8x^2+6}{3\cdot\sqrt[3]{x^5}}.$$

Exercises 6.2

Find the derivatives of the following functions:

(1.) $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$ (2.) $f(x)=\dfrac{x^5}{e^x}$

(3.) $f(\alpha)=\dfrac{2\alpha^2-\alpha+3}{2\alpha}$ (4.) $f(t)=2t\cdot\sin t-(t^2-2)\cos t$

Solution:

(1.) $$f'(x)=\frac{(\sin x+\cos x)'(\sin x-\cos x)-(\sin x+\cos x)(\sin x-\cos x)'}{(\sin x-\cos x)^2}=$$

$$=\frac{(\cos x-\sin x)(\sin x-\cos x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}=$$

$$=\frac{\cos x\sin x-\sin^2x-\cos^2x+\sin x\cos x-(\sin x\cos x+\cos^2x+\sin^2x+\cos x\sin x)}{(\sin x-\cos x)^2}=$$

$$=\frac{2\cos x\sin x-1-2\cos x\sin x-1}{(\sin x-\cos x)^2}=-\frac{2}{(\sin x-\cos x)^2}.$$

(2.) $f'(x)=\frac{(x^5)'\cdot e^x-x^5\cdot(e^x)'}{(e^x)^2}=\frac{5x^4e^x-x^5e^x}{e^{2x}}=\frac{5x^4-x^5}{e^x}=\frac{x^4(5-x)}{e^x}$.

(3.) $f'(\alpha)=\frac{(2\alpha^2-\alpha+3)'\cdot 2\alpha-(2\alpha^2-\alpha+3)(2\alpha)'}{(2\alpha)^2}=\frac{(4\alpha-1)\cdot 2\alpha-(2\alpha^2-\alpha+3)\cdot 2}{4\alpha^2}=$

$=\frac{8\alpha^2-2\alpha-4\alpha^2+2\alpha-6}{4\alpha^2}=\frac{4\alpha^2-6}{4\alpha^2}=\frac{2\alpha^2-3}{2\alpha^2}$.

(4.) $f'(t)=(2t)'\sin t+2t(\sin t)'-(t^2-2)'\cos t-(t^2-2)(\cos t)'=$
$=2\sin t+2t\cos t-2t\cos t-(t^2-2)(-\sin t)=2\sin t+t^2\sin t-2\sin t=t^2\sin t$.

Exercises 6.3

Find the derivatives of the following functions:

(1.) $y=\sin^3 x$;
(2.) $y=\ln(tgx)$;
(3.) $y=5^{\cos x}$;
(4.) $y=\ln\sin(x^3+1)$;
(5.) $y=\arcsin\sqrt{1-x^2}$;
(6.) $y=\ln^5(tg3x)$;

Solution:

(1.) $y'=(\sin^3 x)'=3\sin^2 x\cdot(\sin x)'=3\sin^2 x\cdot\cos x$;

(2.) $y'=\left[\ln(\tan x)\right]'=\frac{1}{\tan x}\cdot(\tan x)'=\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}=\frac{1}{\sin x\cos x}=\frac{2}{\sin 2x}$;

(3.) $y'=\left[5^{\cos x}\right]'=5^{\cos x}\cdot\ln 5\cdot(\cos x)'=5^{\cos x}\cdot\ln 5\cdot(-\sin x)=-5^{\cos x}\cdot\ln 5\cdot\sin x$;

(4.) $y'=\frac{1}{\sin(x^3+1)}\cdot\left[\sin(x^3+1)\right]'=\frac{1}{\sin(x^3+1)}\cdot\cos(x^3+1)(x^3+1)'=3x^2ctg(x^3+1)$;

(5.) $y'=\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\left(\sqrt{1-x^2}\right)'=\frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{1\cdot(1-x^2)'}{2\sqrt{1-x^2}}=$
$=\frac{1}{\sqrt{x^2}}\cdot\frac{-2x}{2\sqrt{1-x^2}}=-\frac{x}{|x|\cdot\sqrt{1-x^2}}$ $(x\neq 0)$.

(6.) $y'=5\ln^4(tg3x)\cdot\left[\ln(tg3x)\right]'=5\ln^4(tg3x)\cdot\frac{1}{tg3x}\cdot(tg3x)'=$

$$=5ln^4(tg3x)\cdot\frac{1}{tg3x}\cdot\frac{1}{cos^2 3x}\cdot(3x)' =15ln^4(tg3x)\cdot\frac{1}{sin 3x\cdot cos 3x}=30\frac{ln^4(tg3x)}{sin 6x}.$$

Exercises 6.4

Find the derivatives of the following functions:

(1.) $y=ln\left(\sqrt{1+e^x}-1\right)-ln\left(\sqrt{1+e^x}+1\right)$; (2.) $y=\frac{1}{15}\cos^3 x\left(3\cos^2 x-5\right)$.

Solution:

$$(1.)\ y'=\frac{1}{\sqrt{1+e^x}-1}\cdot\frac{1\cdot e^x}{2\sqrt{1+e^x}}-\frac{1}{\sqrt{1+e^x}+1}\cdot\frac{1\cdot e^x}{2\sqrt{1+e^x}}=$$

$$=\frac{e^x}{2\sqrt{1+e^x}}\cdot\frac{\sqrt{1+e^x}+1-\left(\sqrt{1+e^x}-1\right)}{\left(\sqrt{1+e^x}-1\right)\left(\sqrt{1+e^x}+1\right)}=\frac{e^x}{2\sqrt{1+e^x}}\cdot\frac{2}{1+e^x-1}=\frac{1}{\sqrt{1+e^x}}.$$

$$(2.)\ y'=\frac{1}{15}\left[3cos^2 x\cdot(-sin x)\cdot\left(3cos^2 x-5\right)+cos^3 x\cdot 6cos x\cdot(-sin x)\right]=$$

$$=-\frac{3cos^2 x\cdot sin x}{15}\left(3cos^2 x-5+2cos^2 x\right)=-\frac{cos^2 x\cdot sin x}{5}\left(5cos^2 x-5\right)=$$

$$=\frac{5}{5}cos^2 x\cdot sin x\left(1-cos^2 x\right)=cos^2 x\cdot sin^3 x.$$

Exercise 6.5

Find the derivative of the function $f(x)=\frac{e^{-x^2}\cdot arcsin\, e^{-x^2}}{\sqrt{1-e^{-2x^2}}}+\frac{1}{2}ln\left(1-e^{-2x^2}\right)$.

Solution:

$$f'(x)=\frac{\left[e^{-x^2}\cdot(-2x)arcsin\, e^{-x^2}+e^{-x^2}\cdot\frac{e^{-x^2}(-2x)}{\sqrt{1-e^{-2x^2}}}\right]\sqrt{1-e^{-2x^2}}}{1-e^{-2x^2}}-$$

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$$-\frac{e^{-x^2}\cdot \arcsin e^{-x^2}\cdot\frac{-e^{-2x^2}(-4x)}{2\sqrt{1-e^{-2x^2}}}}{1-e^{-2x^2}}-\frac{1}{2}\cdot\frac{e^{-2x^2}\cdot(-4x)}{1-e^{-2x^2}},$$

$$f'(x)=\frac{-2xe^{-x^2}\cdot \arcsin e^{-x^2}\cdot\sqrt{1-e^{-2x^2}}-2xe^{-2x^2}}{1-e^{-2x^2}}+$$

$$+\frac{-2xe^{-3x^2}\cdot \arcsin e^{-x^2}\left(1-e^{-2x^2}\right)^{-\frac{1}{2}}+2xe^{-2x^2}}{1-e^{-2x^2}}=$$

$$=\frac{-2xe^{-x^2}\cdot \arcsin e^{-x^2}}{1-e^{-2x^2}}\left(\sqrt{1-e^{-2x^2}}+\frac{e^{-2x^2}}{\sqrt{1-e^{-2x^2}}}\right)=$$

$$\frac{-2xe^{-x^2}\cdot \arcsin e^{-x^2}}{1-e^{-2x^2}}\cdot\frac{1-e^{-2x^2}+e^{-2x^2}}{\sqrt{1-e^{-2x^2}}},$$

That is,

$$f'(x)=-\frac{2xe^{-x^2}\cdot \arcsin e^{-x^2}}{\left(1-e^{-2x^2}\right)^{\frac{3}{2}}}.$$

Exercise 6.6

Prove that the function $y=\frac{1}{1+x+\ln x}$ satisfies the equation $xy'=y(y\ln x-1)$.

Solution:

As $y'=\frac{-\left(1+\frac{1}{x}\right)}{(1+x+\ln x)^2}=\frac{-(x+1)}{x(1+x+\ln x)^2}$, this results in $x\cdot y'=\frac{-(x+1)}{(1+x+\ln x)^2}$.

The right side of the given equation is

$$y(y\ln x-1)=\frac{1}{1+x+\ln x}\left(\frac{\ln x}{1+x+\ln x}-1\right)=\frac{\ln x-1-x-\ln x}{(1+x+\ln x)^2}=\frac{-(x+1)}{(1+x+\ln x)^2}.$$

As the equivalences on the right side are the same, this means that the function y satisfies the given equation.

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