6.3. Logarithmic differentiation

The function having the form $y=[f(x)]^{g(x)}, \quad f(x)>0$ has to be turned into a logarithm before derivation, i.e.

$ln\, y = g(x)\cdot ln\, f(x)$. We can now derivate it:

$$(ln\, y)' = g'(x)\cdot ln\, f(x)+g(x)\cdot [ln\, f(x)]' \text{ that is:}$$

$$\frac{1}{y}\cdot y' = g'(x)\cdot ln\, f(x)+g(x)\cdot\frac{1}{f(x)}\cdot f'(x) \text{ so we get:}$$

$$y' = y\cdot\left[g'(x)\cdot \ln f(x)+\frac{g(x)\cdot f'(x)}{f(x)}\right]=$$

$$=[f(x)]^{g(x)}\cdot\left[g'(x)\cdot \ln f(x)+\frac{g(x)\cdot f'(x)}{f(x)}\right]$$

Note: The same formula is achieved by using the identity $[f(x)]^{g(x)}=e^{g(x)\cdot ln\, f(x)}$, $f(x)>0$. Namely, through the process of derivation, we get:

$$y'=\left\{[f(x)]^{g(x)}\right\}'=e^{g(x)\cdot ln(x)}\cdot[g(x)\cdot ln\, f(x)]' \text{, so we have:}$$

$$y'=\left\{[f(x)]^{g(x)}\right\}'=[f(x)]^{g(x)}\cdot\left[g'(x)\cdot ln\, f(x)+\frac{g(x)\cdot f'(x)}{f(x)}\right].$$

Example 1

Find the derivatives of the following functions the derivative:

(1.) $f(x)=(cos\, x)^{sin\, x}$; (2.) $f(x)=\left(2+\frac{1}{x}\right)^{3x}$.

Solution:

(1.) From $f(x)=(cos\, x)^{sin\, x}$ by using logarithms, we get:

$ln\, f(x)=sin\, x\cdot ln(cos\, x)$, pa je

$$\frac{1}{f(x)}\cdot f'(x)=(sin\, x)'\cdot ln(cos\, x)+sin\, x\cdot\frac{1}{cos\, x}\cdot(cos\, x)', \text{ odnosno:}$$

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$$f'(x)=(\cos x)^{\sin x}\cdot[\cos x\ln(\cos x)-tgx\cdot\sin x]\ .$$

(2.) From $f(x)=\left(2+\frac{1}{x}\right)^{3x}$ $\Rightarrow$ $\ln f(x)=3x\ln\left(2+\frac{1}{x}\right)$;

$$\frac{1}{f(x)}\cdot f'(x)=3\ln\left(2+\frac{1}{x}\right)+3x\cdot\frac{1}{2+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)$$

$$f'(x)=\left(2+\frac{1}{x}\right)^{3x}\left[3\ln\left(2+\frac{1}{x}\right)-\frac{3}{2x+1}\right].$$

Note: The use of logarithms can considerably facilitate the derivation of some rational functions, which can be observed in the following examples.

Example 2

Find the derivatives of the following functions:

(1.) $f(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}$; (2.) $g(x)=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\cdot\sqrt{(x+3)^3}}$.

Solution:

(1.) Through the logarithm $f(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}$ we get:

$\ln f(x)=\ln x+\frac{2}{3}\ln x-\frac{1}{3}\ln(x^2+1)$. So that now:

$\frac{1}{f(x)}\cdot f'(x)=\frac{1}{x}+\frac{2}{3x}-\frac{1\cdot 2x}{3(x^2+1)}$, that is:

$$f'(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}\left[\frac{1}{x}+\frac{2}{3x}-\frac{2x}{3(x^2+1)}\right]=\sqrt[3]{\frac{x^2}{x^2+1}}\cdot\frac{3x^2+5}{3(x^2+1)}.$$

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(2.) Through the logarithm $g(x)=\dfrac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\cdot\sqrt{(x+3)^3}}$ we get:

$ln\, g(x)=\frac{1}{2}ln(x-1)-\frac{2}{3}ln(x+2)-\frac{3}{2}ln(x+3)$, so that:

$$\frac{1}{g(x)}\cdot g'(x)=\frac{1}{2(x-1)}-\frac{2}{3(x+2)}-\frac{3}{2(x+3)}$$

$$g'(x)=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^3}}\cdot\left[\frac{1}{2(x-1)}-\frac{2}{3(x+2)}-\frac{3}{2(x+3)}\right].$$

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