6.4. Derivation of the implicitly given function

Let F be the function of two independent variables *x* and *y*. Then, if the limits exist,

$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x, y) - F(x,y)}{\Delta x}$ is called the ***partial*** derivative *F* by *x* (here, *y* is considered a constant). Analogously,

$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{F(x, y+\Delta y) - F(x,y)}{\Delta y}$ is called the ***partial*** derivative *F* by *y* (here, *x* is considered a constant).

The rules for partial derivations (for all $\alpha, \beta \in \mathbf{R}$ and functions *F*(*x, y*) *and G*(*x, y*) for which the indicated derivations exist):

$$\frac{\partial}{\partial x}(\alpha F + \beta G) = \alpha \frac{\partial F}{\partial x} + \beta \frac{\partial G}{\partial x}; \qquad \frac{\partial}{\partial y}(\alpha F + \beta G) = \alpha \frac{\partial F}{\partial y} + \beta \frac{\partial G}{\partial y};$$

$$\frac{\partial}{\partial x}(F \cdot G) = \frac{\partial F}{\partial x} G + F \frac{\partial G}{\partial x}; \qquad \frac{\partial}{\partial y}(F \cdot G) = \frac{\partial F}{\partial y} G + F \frac{\partial G}{\partial y};$$

$$\frac{\partial}{\partial x}\left(\frac{F}{G}\right) = \frac{\frac{\partial F}{\partial x} G - F \frac{\partial G}{\partial x}}{G^2} \quad (G \neq 0); \qquad \frac{\partial}{\partial y}\left(\frac{F}{G}\right) = \frac{\frac{\partial F}{\partial y} G - F \frac{\partial G}{\partial y}}{G^2} \quad (G \neq 0).$$

There is an interval $I \subseteq R$ having the point x_0 and there is a unique function $f : I \to R$, as here:

(1.) $f(x_0) = y_0$;

(2.) $F(x, f(x)) = 0 \quad \forall x \in I$;

(3.) *f* has a derivative f' at every point $x \in I$. In addition, it is valid that:

$$f'(x_0) = -\frac{\frac{\partial F}{\partial x}(x_0, y_0)}{\frac{\partial F}{\partial y}(x_0, y_0)}.$$

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Note: If the function $y = f(x)$ is given in an implicit form, i.e. through the equation $F(x,y)=0$ and if *f* is a differentiable at the point *x*, then its derivation at that point can also be found in the following way:

1. differentiating both sides of the equation $F(x,y)=0$ (with respect to variable *x*) taking into consideration that y is the function of x and that y' is derivative of y with respect to variable x,
2. the obtained equation $\frac{d}{dx}F(x,y)=0$ is solved by y'.

Example 1

Find the values of partial derivatives at the point $T(-2,1)$ of the given functions:

(1.) $F(x,y)=7x^3-4x^2y^2+y^2$; (2.) $F(x,y)=\ln\sqrt{x^2+y^2}$;

Solution:

(1.) For $F(x,y)=7x^3-4x^2y^2+y^2$ it follows that

$$\frac{\partial F}{\partial x}=21x^2-8xy^2 \text{ (}y\text{ is considered a constant)} \Rightarrow \frac{\partial F}{\partial x}(-2,1)=21\cdot 4-8\cdot(-2)\cdot 1=100.$$

$$\frac{\partial F}{\partial y}=-8x^2y+2y \text{ (}x\text{ is considered a constant)} \Rightarrow \frac{\partial F}{\partial y}(-2,1)=-8\cdot 4\cdot 1+2\cdot 1=-30.$$

(2.) If $F(x,y)=\ln\sqrt{x^2+y^2}$, then

$$\frac{\partial F}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1\cdot 2x}{2\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2} \Rightarrow \frac{\partial F}{\partial x}(-2,1)=-\frac{2}{5}.$$

$$\frac{\partial F}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1\cdot 2y}{2\sqrt{x^2+y^2}}=\frac{y}{x^2+y^2} \Rightarrow \frac{\partial F}{\partial y}(-2,1)=\frac{1}{5}.$$

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Example 2

Find the derivative of the function $y = f(x)$ that is given implicitly $8x - 3y + 7 = 0$.

Solution:

$F(x,y) = 8x - 3y + 7$, so that $\frac{\partial F}{\partial x} = 8$, and $\frac{\partial F}{\partial y} = -3$,

$$y' = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-8}{-3} = \frac{8}{3}.$$

Example 3

Find the derivative of the implicitly given function $\ln x + e^{-\frac{y}{x}} = C$ in two ways.

Solution:

I. From $F(x,y) = \ln x + e^{-\frac{y}{x}} - C$ the result are

$\frac{\partial F}{\partial x} = \frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{y}{x^2}$; and $\frac{\partial F}{\partial y} = e^{-\frac{y}{x}} \cdot \left(-\frac{1}{x}\right)$, so that:

$$y'(x) = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{x + y \cdot e^{-\frac{y}{x}}}{x^2}}{-\frac{1}{x} \cdot e^{-\frac{y}{x}}} = \frac{x + ye^{-\frac{y}{x}}}{xe^{-\frac{y}{x}}} = e^{\frac{y}{x}} + \frac{y}{x}.$$

II. Given:

$$\ln x + e^{-\frac{y}{x}} = C, \qquad \Big| \frac{d}{dx}$$

$$\frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{d}{dx}\left(-\frac{y}{x}\right) = 0,$$

$$\frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{-y' \cdot x + y}{x^2} = 0, \qquad \Big| \cdot x^2$$

$$x + e^{-\frac{y}{x}}(y - y'x) = 0,$$

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$$x + ye^{-\frac{y}{x}} = y'x \cdot e^{-\frac{y}{x}},$$

$$y' = e^{\frac{y}{x}} + \frac{y}{x}.$$

Example 4

Find the derivative of the function $F(x,y) = \frac{xy}{x^2+y^2}$ at the point $T(-1,2)$.

Solution:

$$\frac{\partial F}{\partial x} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(x \cdot y) - xy\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2} =$$

$$= \frac{y(y^2-x^2)}{(x^2+y^2)^2} \quad \Rightarrow \quad \frac{\partial F}{\partial x}(-1,2) = \frac{2(2^2-(-1)^2)}{\left[(-1)^2+2^2\right]^2} = \frac{6}{25}.$$

$$\frac{\partial F}{\partial y} = \frac{x(x^2+y^2) - xy \cdot 2y}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2} \quad \Rightarrow \quad \frac{\partial F}{\partial y}(-1,2) = \frac{3}{25}.$$

$$y' = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{6}{25}}{\frac{3}{25}} = -2.$$

Exercises 6.7

Find the derivatives of the implicit functions:

(1.) $x^3 + x^2y + y^2 = 0$;

(2.) $y^3 = \frac{x-y}{x+y}$;

(3.) $xy = arctg\,\frac{x}{y}$

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Solution:

(1.) From $F(x,y)=x^3+x^2y+y^2$ it follows that $\frac{\partial F}{\partial x}=3x^2+2xy$; $\frac{\partial F}{\partial y}=x^2+2y$, so that:

$$y'(x)=f'(x)=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{3x^2+2xy}{x^2+2y}.$$

(2.) From $F(x,y)=y^3-\frac{x-y}{x+y}$ it follows that

$$\frac{\partial F}{\partial x}=\frac{-(x+y)+(x-y)}{(x+y)^2}=-\frac{2y}{(x+y)^2}$$ and

$$\frac{\partial F}{\partial y}=3y^2-\frac{-(x+y)-(x-y)}{(x+y)^2}=3y^2+\frac{2x}{(x+y)^2}=\frac{3y^2(x+y)^2+2x}{(x+y)^2},$$ so that

$$y'(x)=f'(x)=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{\frac{-2y}{(x+y)^2}}{\frac{3y^2(x+y)^2+2x}{(x+y)^2}}=\frac{2y}{3y^2(x+y)^2+2x}.$$

(3.) From $F(x,y)=xy-arctg\frac{x}{y}$ it follows that

$$\frac{\partial F}{\partial x}=y-\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=y-\frac{y}{y^2+x^2}=\frac{y^3+x^2y-y}{y^2+x^2}$$ and

$$\frac{\partial F}{\partial y}=x-\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^2}=x+\frac{x}{y^2+x^2}=\frac{xy^2+x^3+x}{y^2+x^2},$$ so that

$$y'(x)=f'(x)=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{y^3+x^2y-y}{xy^2+x^3+x}=\frac{y}{x}\cdot\frac{1-x^2-y^2}{1+x^2+y^2}.$$

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