6.5. Derivation of the parametrically given function

The trail of a material point T moving across the plane is the curve, for example: line, parabola, ellipse, hyperbola, cosine wave, etc. Each point T can be observed as a vessel sailing along a path (curve). When describing such a movement, it is necessary to know the point coordinates as the function of time, at each moment *t*. If we mark $x=\varphi(t)$ and $y=\psi(t)$ as the coordinates of the point T where φ and ψ are the real functions determined at the interval *I* during which the movement occurs, it is clear that φ and ψ are differentiable functions, because the speed is given by $v(t)=\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}$. Hence, when time *t* describes the interval $I\subseteq R$, then the point T$(\varphi(t),\psi(t))$ passes at least once through each point of the set, that is $K=\{(\varphi(t),\psi(t)):t\in I\}$.

Parametric equations of the curve (*t* is called the **parameter**) are expressed as:

$$\begin{cases} x=\varphi(t) \\ y=\psi(t), \quad t\in I \end{cases}$$

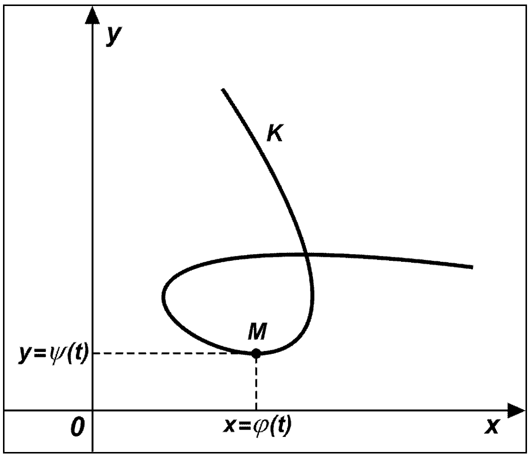

Figure 6.3

If φ is strictly monotone on the interval I (we know that there is an inverse function $t=\varphi^{-1}(x)$), then, by replacing the variable *t* by $\varphi^{-1}(x)$ we get $y=\psi(t)=\psi[\varphi^{-1}(x)]=f(x)$.

Applying the chain rule follows:

$$y'(x_0)=f'(x_0)=\psi'[\varphi^{-1}(x_0)]\cdot[\varphi^{-1}(x_0)]'=\psi'(t_0)\cdot\frac{1}{\varphi'(t_0)}, \text{ that is } f'(x_0)=\frac{\psi'(t_0)}{\varphi'(t_0)}.$$

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We can use a simpler way:

$$y'(x) = f'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

It is easily proven that:

$$f''(x) = \frac{\varphi'(t)\cdot\psi''(t) - \varphi''(t)\cdot\psi'(t)}{\left[\varphi'(t)\right]^3} \quad \Rightarrow$$

$$y''(x) = f''(x) = \frac{x'(t)\cdot y''(t) - x''(t)\cdot y'(t)}{\left[x'(t)\right]^3}.$$

Example 1

Find the derivative of the function *f*(*x, y*) at the point $x_0 = 3$, which is a parametrically given by formulas:

$$\begin{cases} x = 2t - 1 \\ y = t^3 \end{cases}$$

Solution:

First we calculate t_0 with the corresponding value $x_0 = 3$. From $x = 2t - 1 \Rightarrow t = \frac{x+1}{2}$, so that $x_0 = 3$; $t_0 = \frac{3+1}{2} = 2$.

Now, let us find $\varphi'(t)$ and $\psi'(t)$:

$\varphi'(t) = x'(t) = 2$ and $\psi'(t) = y'(t) = 3t^2$.

By using the formula, it follows that $f'(x_0) = \frac{\psi'(t_0)}{\varphi'(t_0)}$, that is: $f'(3) = \left.\frac{3{t_0}^2}{2}\right|_{t_0=2} = 6$.

Example 2

Find $f'(x)$ and derivative of the second order $f''(x)$ for the function $y = f(x)$, which is arametrically given by formulas:

$$\begin{cases} x = e^t \cos t, \\ y = e^t \sin t. \end{cases}$$

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Solution:

$$y'(x)=\frac{y'(t)}{x'(t)}=\frac{e^t\sin t+e^t\cos t}{e^t\cos t-e^t\sin t}=\frac{\sin t+\cos t}{\cos t-\sin t}.$$

$$y''(x)=\frac{x'(t)\cdot y''(t)-x''(t)\cdot y'(t)}{\left[x'(t)\right]^3}=$$

$$=\frac{(e^t\cos t-e^t\sin t)(e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t)}{(e^t\cos t-e^t\sin t)^3}-$$

$$-\frac{(e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t)(e^t\sin t+e^t\cos t)}{(e^t\cos t-e^t\sin t)^3}=$$

$$=\frac{e^t(\cos t-\sin t)2e^t\cos t+2e^t\sin t\cdot e^t(\sin t+\cos t)}{(e^t)^3(\cos t-\sin t)^3}=$$

$$=\frac{2e^{2t}\left[\cos^2 t-\sin t\cos t+\sin^2 t+\sin t\cos t\right]}{e^{2t}\cdot e^t(\cos t-\sin t)^3}=\frac{2}{e^t(\cos t-\sin t)^3}.$$

Exercise 6.8

Find $y'(x)$ if $\begin{cases} x(t)=a\left(\ln\tan\frac{t}{2}+\cos t-\sin t\right), \\ y(t)=a(\sin t+\cos t). \end{cases}$

Solution:

$$y'(x)=\frac{y'(t)}{x'(t)}=\frac{a(\cos t-\sin t)}{a\left(\frac{1}{tg\frac{t}{2}}\cdot\frac{1}{\cos^2\frac{t}{2}}\cdot\frac{1}{2}-\sin t-\cos t\right)}=\frac{\cos t-\sin t}{\left(\frac{1}{\sin t}-\sin t-\cos t\right)}=$$

$$=\frac{\sin t(\cos t-\sin t)}{1-\sin^2 t-\sin t\cos t}=\frac{\sin t(\cos t-\sin t)}{\cos^2 t-\sin t\cos t}==\frac{\sin t(\cos t-\sin t)}{\cos t(\cos t-\sin t)}=tgt.$$

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Exercises 6.9

Find the coefficient of the tangent line to the graph of a function that is parametrically given:

$$\begin{cases} x(t) = t \ln t, \\ y(t) = \dfrac{\ln t}{t} \end{cases}$$ at the point $t_0 = 1$.

Solution:

As $y'(t) = \dfrac{\frac{1}{t} \cdot t - \ln t}{t^2} = \dfrac{1 - \ln t}{t^2}$, and $x'(t) = \ln t + 1$,

It follows that $y'(x) = \dfrac{y'(t)}{x'(t)} = \dfrac{1 - \ln t}{t^2(\ln t + 1)}$, so that

$$k_t = y'(x)\big|_{t_0 = 1} = \frac{1 - \ln 1}{1^2(\ln 1 + 1)} = 1.$$

Exercise 6.10

Find $y'(x)$ for the parametrically given function:

$$\begin{cases} x(t) = arccos \dfrac{1}{\sqrt{1+t^2}}, \\ y(t) = arcsin \dfrac{t}{\sqrt{1+t^2}}. \end{cases}$$

Solution:

Since:

$$y'(t) = \frac{1}{\sqrt{1 - \frac{t^2}{1+t^2}}} \cdot \frac{\sqrt{1+t^2} - t \cdot \frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{\sqrt{1+t^2}\left(1+t^2-t^2\right)}{\left(1+t^2\right)^{\frac{3}{2}}} = \frac{1}{1+t^2},$$

$$x'(t) = -\frac{1}{\sqrt{1 - \frac{1}{1+t^2}}} \cdot \frac{-\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2-1}} \cdot \frac{t}{\left(1+t^2\right)\sqrt{1+t^2}} = \frac{t}{|t| \cdot \left(1+t^2\right)}.$$

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We obtain $y'(x)=\dfrac{y'(t)}{x'(t)}=\dfrac{\frac{1}{1+t^2}}{\frac{t}{|t|}\frac{1}{1+t^2}}=\dfrac{|t|}{t}=\begin{cases}1, & t>0\\ -1, & t<0\end{cases}.$

Exercises 6.11

Find derivative of the second order $y''(x)$:

(1.) $\begin{cases}x(t)=a(\sin t-t\cos t),\\ y(t)=a(\cos t+t\sin t),\end{cases}$ (2.) $\begin{cases}x(t)=a\cos^3 t,\\ y(t)=a\sin^3 t.\end{cases}$

Solution:

(1.) $x'(t)=a[\cos t-\cos t-t(-\sin t)]=at\sin t$,

$y'(t)=a[-\sin t+\sin t+t\cos t]=at\cos t$.

So that $y'(x)=\dfrac{y'(t)}{x'(t)}=\dfrac{at\cos t}{at\sin t}=\cot t$. Cotangent t follows that:

$$y''(x)=\frac{\frac{d}{dt}[y'(x)]}{x'(t)}=\frac{-\frac{1}{\sin^2 t}}{at\sin t}=-\frac{1}{at\sin^3 t}.$$

(2.) $x'(t)=3a\cos^2 t(-\sin t)$,
$y'(t)=3a\sin^2 t(\cos t)$.

So that $y'(x)=\dfrac{y'(t)}{x'(t)}=-\dfrac{\sin t}{\cos t}=-tgt$.

Tangent t follows that

$$y''(x)=\frac{\frac{d}{dt}[y'(x)]}{x'(t)}=\frac{-\frac{1}{\cos^2 t}}{-3a\cos^2 t\sin t}=\frac{1}{3a\cos^4 t\sin t}.$$

Exercise 6.12

Prove that the function $y=f(x)$, given parametrically

$\begin{cases}x(t)=2t+3t^2\\ y(t)=t^2+2t^3\end{cases}$ satisfies the equation $2(y')^3+(y')^2-y=0$, $\left(y'=\dfrac{dy}{dx}\right)$.

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Solution:

$$y' = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t + 6t^2}{2 + 6t} = t\,.$$

If y' is inserted into the equation $2(y')^3 + (y')^2 - y = 0$, we obtain $2t^3 + t^2 - \left(t^2 + 2t^3\right) \equiv 0$

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