6.6. The tangent and normal lines to the graph

Let a curve be defined by the formula $y = f(x)$, where f is a differentiable function. The curve secant passing through the points $(x_0, f(x_0))$ and $(x, f(x))$, where $x_0 \neq x$, is the line with a slope coefficient $tg\varphi = \frac{f(x) - f(x_0)}{x - x_0}$.

When $x \to x_0$, then the secant line tends tangent line to the curve at the point $(x_0, f(x_0))$, whose slope coefficient of a line is equal to $\tan\varphi_0$. Obviously, it is valid that: $x \to x_0 \Rightarrow tg\varphi \to tg\varphi_0 = f'(x_0)$.

Therefore:

*the **equation of the tangent*** line to the graph of function $y = f(x)$ at the point $T_0(x_0, y_0)$, where $y_0 = f(x_0)$, a $f'(x_0) = k \in R$, is:

$$y - f(x_0) = f'(x_0)(x - x_0);$$

while the ***equation of the normal line*** perpendicular to the tangent line at that point is:

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0).$$

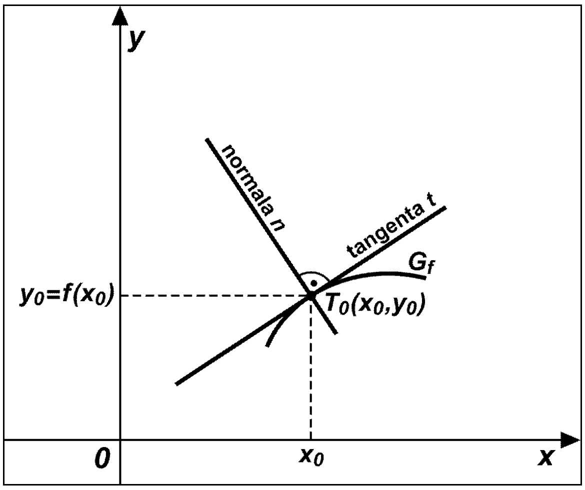

Figure 6.4

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The angle φ at which the graphs of the functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$ intersect at the point $T_0(x_0, y_0)$ is the **angle** between their tangents at that point, and is calculated according to the formula:

$$tg\varphi = tg(\varphi_2 - \varphi_1) = \frac{tg\varphi_2 - tg\varphi_1}{1 + tg\varphi_2 tg\varphi_2} = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)}.$$

The angle φ is an angle that we need to rotate the tangent t_1 of the function f_1 in the positive direction (counter-clockwise) around their mutual point, so that it could be aligned with the tangent t_2 of the function f_2 (see the figure).

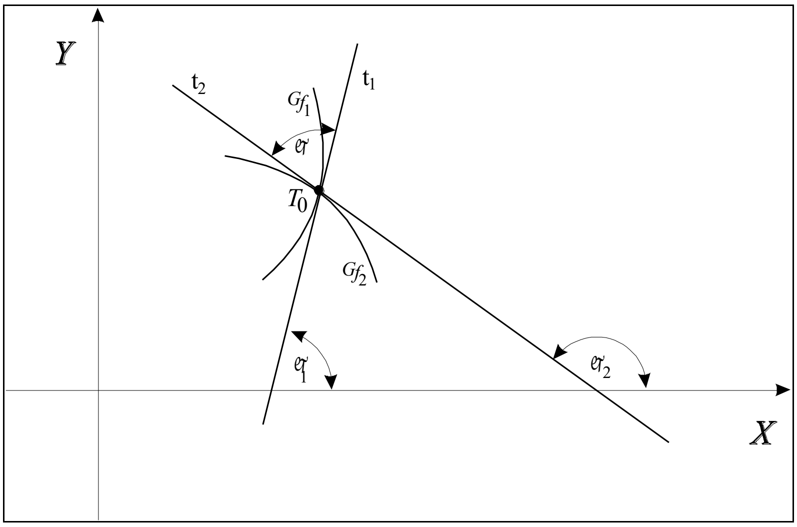

Figure 6.5

Example 1

Find the equations of the tangent and the normal lines to the graph of function $f(x) = x^3 - 3x + 2$ at a point whose abscissa is $x_0 = 2$.

Solution:

If we insert $x_0 = 2$ into the formula by which the function is given, we get the ordinate of the point T_0, that is $y_0 = f(x_0) = f(2) = 4$.

Now we look for the equations of the tangent and the normal at the point $T_0(2,4)$. We can find the derivative at the point $x_0 = 2$:

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$$f'(x)=3x^2-3 \quad \Rightarrow \quad f'(2)=9.$$

Equation of the tangent: $y-4=9(x-2)$ or

$9x-y-14=0$.

Equation of the normal: $y-4=-\frac{1}{9}(x-2)$ or

$x+9y-38=0$.

Example 2

Find the angle at which the functions $y=4-\frac{x^2}{2}$ and $y=4-x$ intersect.

Solution:

The angle φ at which the graphs of functions f_1 and f_2 intersect at the intersection point $M_1(x_1,y_1)$ is calculated through the well-known formula

$$\tan\phi=\frac{f_2'(x_1)-f_1'(x_1)}{1+f_1'(x_1)\,f_2'(x_1)}.$$

The intersection points of the given functions are obtained by solving the system of equations:

$$\left\{\begin{matrix} y=4-\frac{x^2}{2} \\ y=4-x \end{matrix}\right\} \Rightarrow \begin{cases} x_1=0, & y_1=4, \\ x_2=2, & y_2=2, \end{cases}$$

that is, $M_1(0,4);M_2(2,2)$. (see the figure)

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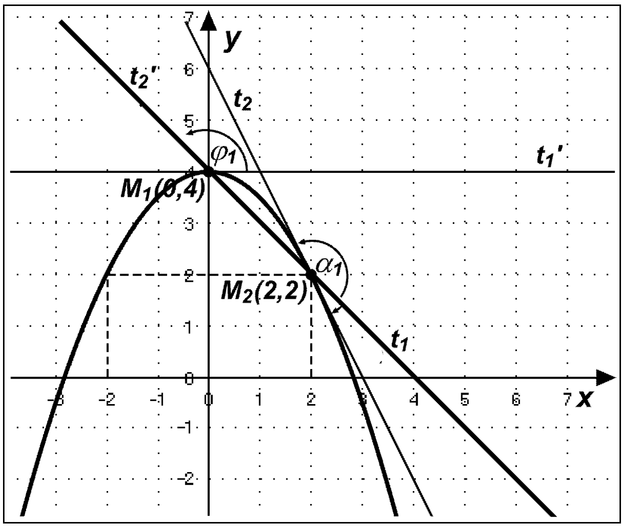

Figure 6.6

We obtain the angle at the point $M_1(0,4)$ by inserting the point into $f_1'(x)=-x$ and $f_2'(x)=-1$; so that $f_1'(0)=0$ and $f_2'(0)=-1$. Therefore, $\tan\phi_1=\frac{f_2'(0)-f_1'(0)}{1+f_1'(0)\cdot f_2'(0)}=-\frac{1}{1}$ $=-1 \Rightarrow \varphi_1=135°$.

Analogously, we obtain the angle at the point $M_2(2,2)$ by inserting into $f_1'(x)=-x$ and $f_2'(x)=-1$; so that $f_1'(2)=-2$ and $f_2'(2)=-1$. Therefore, $\tan(180^\circ-\alpha_1)=\frac{f_2'(2)-f_1'(2)}{1+f_1'(2)\cdot f_2'(2)}=\frac{-1+2}{1+(-2)(-1)}=\frac{1}{3} \Rightarrow 180^\circ-\alpha_1=18^\circ 26' 6'' \Rightarrow \alpha_1=161°33'54''$

Exercises 6.13

Find the equations of the tangent and normal lines to the functions at the given points:

(1.) $f(x)=2x^2-x+5$ at the point $T_0\left(-\frac{1}{2},6\right)$;

(2.) $f(x)=x^3+2x^2-4x-3$ at the point $T_0(-2,5)$;

(3.) $f(x)=\sqrt[3]{x-1}$ at the point with the abscissa $x_0=1$.

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Solution:

(1.) $f(x)=2x^2-x+5$;$T_0\left(-\frac{1}{2},6\right)$. Find the derivative at the point $x_0=-\frac{1}{2}$:

$f'(x)=4x-1 \Rightarrow f'\left(-\frac{1}{2}\right)=-3$.

The equation of the tangent through the point $T_0(x_0, f(x_0))$ has a standard form:

$y-f(x_0)=f'(x_0)(x-x_0)$, so that, for the given function, the equation of the tangent is:

$y-6=-3\left(x+\frac{1}{2}\right)$ or $2y+6x-9=0$.

The equation of the normal through the point $T_0(x_0, f(x_0))$ has the form:

$y-f(x_0)=-\frac{1}{f'(x_0)}(x-x_0)$, so that

$y-6=\frac{1}{3}\left(x+\frac{1}{2}\right)$ or $6y-2x-37=0$.

(2.) $f(x)=x^3+2x^2-4x-3$; $T_0(-2,5)$.

$f'(x)=3x^2+4x-4 \Rightarrow f'(-2)=0$.

The equation of the tangent is $y-5=0$, while the equation of the normal is $x+2=0$.

***Note**: if $f'(x_0)=0$ then the equation of the tangent $y=f(x_0)$ (this is a line parallel to the x-axis passing through the point $T_0(x_0, f(x_0))$, and the equation of the normal is $x=x_0$.*

(3.) $f(x)=\sqrt[3]{x-1}$ at the point with the abscissa $x_0=1$.

Since $y_0=f(x_0)=0$, we should find the equation of the tangent and the normal passing through the point $T_0(1,0)$. The derivative of the function $f(x)=\sqrt[3]{x-1}$ at the point $x_0=1$ is

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$$f'(x)=\frac{1}{3\cdot\sqrt[3]{(x-1)^2}} \Rightarrow f'(x_0)=f'(1)$$ does not exist, that is

$$f'(1)=\lim_{x\to 1} f'(x)=\lim_{x\to 1}\frac{1}{\left[3\cdot\sqrt[3]{(x-1)^2}\right]}=\infty .$$

The equations of the tangent and the normal are $x-1=0$ and $y=0$ accordingly.

Note: if $f'(x_0)\to\infty$ as x tends to x index 0, $x\to 1$, then the equation of the normal is $y-f(x_0)=0$ and the tangent is $x-x_0=0$.

Exercise 6.14

Find the intersection of the tangents on the curve $y=\frac{1+3x^2}{3+x^2}$ at the points for which $y=1$.

Solution:

The conditions of the task imply that

$$\frac{1+3x^2}{3+x^2}=1\Rightarrow x^2=1\Rightarrow\begin{cases} x_1=1\\ x_2=-1\end{cases}$$, that is $T_1(-1,1)$ and $T_2(1,1)$.

Let us find the values of the derivative at the points T_1 and T_2. From $y'(x)=\frac{16x}{(3+x^2)^2}$ it follows that

$k_1=y'(-1)=-1;\ k_2=y'(1)=1$, so that

$t_1: y-1=-1(x+1)\Rightarrow y=-x$, and

$t_2: y-1=1(x-1)\Rightarrow y=x$.

Therefore, the intersection point of tangent lines t_1 and t_2 on the given curve is $S(0,0)$.

Exercise 6.25

Find the equation of the normal to the graph of function $f(x)=x\ln x$ which is parallel to the line $2x-3y+3=0$.

Solution:

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In order to achieve that the normal to the graph of the given function in x_0 is parallel to the line *p*, it must be valid that $k_p = -\frac{1}{f'(x_0)}$, where k_p is the slope coefficient of the line *p* direction. From the given derivative, it follows that

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1, \text{ and}$$

$$f'(x_0) = \ln x_0 + 1.$$

From $y = \frac{2}{3}x + 1 \Rightarrow k_p = \frac{2}{3}$: so that

$$-\frac{1}{\ln x_0 + 1} = \frac{2}{3} \Rightarrow 2\ln x_0 + 2 = -3 \Rightarrow \ln x_0 = -\frac{5}{2} \Rightarrow x_0 = e^{-\frac{5}{2}}.$$

Since $y_0 = f(x_0) = f\left(e^{-\frac{5}{2}}\right) = e^{-\frac{5}{2}} \cdot \ln e^{-\frac{5}{2}} = -\frac{5}{2}e^{-\frac{5}{2}}$, the graph of the given function has a normal parallel to the line $2x - 3y + 3 = 0$ at the point $T_0\left(e^{-\frac{5}{2}}, -\frac{5}{2e^{\frac{5}{2}}}\right)$, hence the equation of the required normal is:

$$y + \frac{5}{2e^{\frac{5}{2}}} = \frac{2}{3}\left(x - e^{-\frac{5}{2}}\right) \text{ or } y = \frac{2}{3}x - \frac{19}{6e^{\frac{5}{2}}}.$$

Exercise 6.36

On the graph of the function $f(x) = x^2 - 2x + 5$ find the point at which the tangent is vertical to the line $y = x$.

Solution:

In order to achieve that the tangent to the graph of the given function in x_0 is vertical to the line *p*, it must be valid that $f'(x_0) = -\frac{1}{k_p}$, where k_p is the coefficient of the line *p* direction.

From the derivative of the given function $f'(x) = 2x - 2$ it follows that $f'(x_0) = 2x_0 - 2$.

From $y = x$ it follows that $k_p = 1$, therefore $2x_0 - 2 = -1 \Rightarrow x_0 = \frac{1}{2}$.

Since $y_0 = f(x_0) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 2\cdot\frac{1}{2} + 5 = \frac{17}{4}$, the graph of the given function has the tangent vertical to the line $y = x$ at the point $T_0\left(\frac{1}{2}, \frac{17}{4}\right)$.

Exercise 6.47

Find the angle at which the graph of the function $f(x) = \arctan\left(1 + \frac{1}{x}\right)$ intersects the *x*-axis.

Solution:

The angle at which the graph of function *f* intersects the *x*-axis is the angle the tangent at that point makes with the positive direction of the *x*-axis. The desired angle φ is obtained as a solution to the trigonometric equation $f'(x_0) = k_t = \tan\phi$, where x_0 is the zero of the function *f*.

x_0 is obtained by solving the equation

$$\arctan\left(1+\frac{1}{x}\right) = 0 \Rightarrow 1 + \frac{1}{x} = 0 \Rightarrow x_0 = -1.$$

We know that

$$f'(x) = \frac{1}{1+\left(1+\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+(x+1)^2},$$ so that

$$f'(x_0) = f'(-1) = -1.$$

It follows that $\tan\phi = -1 \Rightarrow \phi = \frac{3\pi}{4}$.

Exercise 6.58

Find the equations of the tangent and the normal at the point $T(1,1)$ on the graph of the function that is implicitly given by the equation $x^5 + y^5 - xy - 1 = 0$.

Solution:

If we denote $F(x,y) = x^5 + y^5 - xy - 1$, then $F(1,1) = 0$ and the partial derivative F by y

$$\frac{\partial F}{\partial y}\Big|_{(1,1)} = 5y^4 - x\Big|_{(1,1)} = 4 \neq 0,$$ so that

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$$f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{5x^4 - y}{5y^4 - x} \Rightarrow f'(1) = -1.$$

Therefore, the equation of the tangent is: $y - 1 = -1(x - 1)$ or $y + x - 2 = 0$.

The equation of the normal: $y - 1 = -\frac{1}{-1}(x - 1)$ or $y - x = 0$.

Exercise 6.69

Find the equations of the tangent and the normal on the parametrically given curve:

$$\begin{cases} x(t) = & \ln(\cos t) + 1, \\ y(t) = & \tan t + \cot t, \qquad \text{u} \quad t_0 = \frac{\pi}{4}. \end{cases}$$

Solution:

For the parameter value $t_0 = \frac{\pi}{4}$ it follows that

$$x_0 = x(t_0) = x\left(\frac{\pi}{4}\right) = \ln \cos\frac{\pi}{4} + 1 = \ln\frac{\sqrt{2}}{2} + 1, \text{ and}$$

$$y_0 = y(t_0) = y\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} + \cot\frac{\pi}{4} = 1 + 1 = 2.$$

We need to find the equation of the tangent and the normal at the point $T_0\left(ln\frac{\sqrt{2}}{2} + 1, 2\right)$.

Since:

$$y'(x) = \frac{y'(t_0)}{x'(t_0)} = \left.\frac{\frac{1}{cos^2 t} - \frac{1}{sin^2 t}}{-\frac{sin\, t}{cos\, t}}\right|_{t_0 = \frac{\pi}{4}} = 0, \text{ that is, } k_t = 0,$$

the tangent is parallel to the *x*-axis, while the normal is parallel to the *y*-axis at the point T_0, so that

tangent line t: $y - 2 = 0 \Rightarrow y = 2$, and

normal line n: $x - ln\frac{\sqrt{2}}{2} - 1 = 0 \Rightarrow x = ln\frac{\sqrt{2}}{2} + 1$.

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Exercises 6.20

Find the equation of the tangent and the normal at the point (0, 0) on the parametrically given curve:

$$\begin{cases} x(t) = t\,ln\,t, \\ y(t) = \dfrac{ln\,t}{t}, \end{cases}$$

Solution:

$$\left.\begin{aligned} x_0 = x(t_0) = 0 \Rightarrow \quad & t_0\,ln\,t_0 = 0 \\ y_0 = y(t_0) = 0 \Rightarrow \quad & \frac{ln\,t_0}{t_0} = 0 \end{aligned}\right\} \Rightarrow t_0 = 1$$

(parameter $t_1 = 0$ is not from the domain of the given functions and is not taken into consideration).

$$k_t = y'(x)\big|_{t_0=1} = \left.\frac{1 - ln\,t}{t^2(1 + ln\,t)}\right|_{t_0=1} = 1.$$

Now it is easy to obtain that the equation of the required tangent is $y = x$, while the equation of the normal is $y = -x$.