6.7. Application of derivatives to evaluate the limits of a functions

A typical requirement in finding the limit value or the limit of a function is set, when $x \to c \in R$ or when $x \to \pm\infty$, when the function is given in the form of a product or a quotient of two functions *f* and *g*. However, the result is often a product that is not defined, and to which already known theorems on the limit values of functions cannot be applied. The extension of the theorems to undefined cases was given by G.F.A. de L`Hospital. Therefore, it is often said: to evaluate the limits of the indeterminate forms

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0$$

the L'Hospital's Rule is applied.

Note: L'Hospital's rule is used to evaluate the limits of the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, and other indeterminate forms are reduced to one of these two forms using appropriate transformations.

Theorem (L'Hospital's Rule):

If *f* and *g* are differentiable functions nearby the point $c \in R$ where $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$. If $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L \in R$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ and also

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

In particular, if f' and g' are continuous functions at point *c* and $g'(c) \neq 0$, then $L = \frac{f'(c)}{g'(c)}$.

Note:

i) L'Hospital's rule can be applied when $x \to \pm\infty$, or when calculating one-sided limits (when $x \to c^+$ or when $x \to c^-$).

ii) L'Hospital's rule can be applied or repeatedly if the resulting limit has one of the intedeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

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Example 1

Find limits:

(1.) $\lim\limits_{x\to 0}\dfrac{e^{7x}-1}{3x}$; (2.) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^2}$.

Solution:

$$(1.)\ \lim_{x\to 0}\frac{e^{7x}-1}{3x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(e^{7x}-1\right)'}{(3x)'}=\lim_{x\to 0}\frac{7e^{7x}}{3}=\frac{7}{3};$$

$$(2.)\ \lim_{x\to\infty}\frac{\ln x}{x^2}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{\left(x^2\right)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=\left[\frac{1}{\infty}\right]=0.$$

Example 2

Find limits:

(1.) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$; (2.) $\lim\limits_{x\to\infty}\dfrac{\ln^2 x}{x}$.

Solution:

$$(1.)\ \lim_{x\to 0}\frac{x-\sin x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(x-\sin x)'}{\left(x^3\right)'}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(1-\cos x)'}{\left(3x^2\right)'}=$$

$$=\lim_{x\to 0}\frac{\sin x}{6x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{(6x)'}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6};$$

$$(2.)\ \lim_{x\to\infty}\frac{\ln^2 x}{x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(\ln^2 x\right)'}{(x)'}=\lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}=\left[\frac{\infty}{\infty}\right]=$$

$$=2\lim_{x\to\infty}\frac{(\ln x)'}{(x)'}=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\frac{0}{1}=0.$$

- For the indeterminate form $\underline{0\cdot\infty}$ the following transformation is used

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$$\lim_{x\to c}\left(f(x)\cdot g(x)\right)=\lim_{x\to c}\frac{f(x)}{\frac{1}{g(x)}}\quad or\quad \lim_{x\to c}\left(f(x)\cdot g(x)\right)=\lim_{x\to c}\frac{g(x)}{\frac{1}{f(x)}}$$

It is reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 3

Find limits:

(1.) $\lim\limits_{x\to 0^+} x\ln x$; (2.). $\lim\limits_{x\to 0}\left(\ln(1-\sin x)\cdot\cot x\right)$

Solution:

(1.) $$\lim_{x\to 0^+}(x\ln x)=\left[0\cdot(-\infty)\right]=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\left[\frac{-\infty}{+\infty}\right]=\lim_{x\to 0^+}\frac{(\ln x)'}{\left(\frac{1}{x}\right)'}\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}(-x)=0;$$

(2.) $$\lim_{x\to 0}\left(\ln(1-\sin x)\cdot ctgx\right)=\left[0\cdot\infty\right]=\lim_{x\to 0}\frac{\ln(1-\sin x)}{\frac{1}{\cot x}}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(\ln(1-\sin x)\right)'}{\left(\frac{1}{\cot x}\right)'}=$$

$$=\lim_{x\to 0}\frac{\frac{1}{1-\sin x}\cdot(1-\sin x)'}{(\tan x)'}=\lim_{x\to 0}\frac{\frac{-\cos x}{1-\sin x}}{\frac{1}{\cos^2 x}}=\lim_{x\to 0}\frac{-\cos^3 x}{1-\sin x}=\frac{-1}{1}=-1.$$

- *For the indeterminate form* $\infty-\infty$ an appropriate transformation is used in the following way:

(i) in order to exclude one member, the form is reduced to the indeterminate form $0\cdot\infty$, and then to $\frac{0}{0}$ or $\frac{\infty}{\infty}$;

(ii) in order to reduce the form to the common denominator, it is directly reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

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Example 4

Find limits:

(1.) $\lim\limits_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)$; (2.) $\lim\limits_{x\to-2}\left[\frac{1}{x+2}-\frac{1}{\ln(x+3)}\right]$.

Solution:

(1.)
$$\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)=[\infty-\infty]=\lim_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=[\infty\cdot 0]=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}=\left[\frac{0}{0}\right]=$$
$$=\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}}=\lim_{x\to\infty}e^{\frac{1}{x}}=e^{\lim\limits_{x\to\infty}\frac{1}{x}}=e^0=1;$$

(2.) $\lim\limits_{x\to-2}\left[\frac{1}{x+2}-\frac{1}{ln(x+3)}\right]=[\infty-\infty]=\left[\begin{matrix}\text{the expression is reduced here}\\ \text{to a common denominator}\end{matrix}\right]=$

$$=\lim_{x\to-2}\frac{ln(x+3)-x-2}{(x+2)ln(x+3)}=\left[\frac{0}{0}\right]=\lim_{x\to-2}\frac{\frac{1}{x+3}-1}{ln(x+3)+\frac{x+2}{x+3}}=$$

$$=-\lim_{x\to-2}\frac{x+2}{(x+3)ln(x+3)+(x+2)}=\left[\frac{0}{0}\right]=-\lim_{x\to-2}\frac{1}{ln(x+3)+\frac{x+3}{x+3}+1}=-\frac{1}{2}.$$

Indeterminate forms of type 1^∞, 0^0, ∞^0 can be transformed to the indeterminate form $0\cdot\infty$ taking the logarithms or using the identity:

$$\left[f(x)\right]^{g(x)}=e^{g(x)\ln f(x)},\ f(x)>0.$$

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Example 5

Find limits:

(1.) $\lim\limits_{x\to 0}\left(e^{2x}+x\right)^{\frac{1}{x}}$; (2.) $\lim\limits_{x\to 0^{+0}} x^x$; (3.) $\lim\limits_{x\to \frac{\pi}{2}^{-0}}\left(\tan x\right)^{\cot x}$.

Solution:

(1.) $\lim\limits_{x\to 0}(e^{2x}+x)^{\frac{1}{x}}=[1^\infty]=e^{\lim\limits_{x\to 0}\frac{1}{x}ln(e^{2x}+x)}=e^L$, where

$$L=\lim_{x\to 0}\frac{ln(e^{2x}+x)}{x}=\begin{bmatrix}0\\0\end{bmatrix}=\lim_{x\to 0}\frac{1}{e^{2x}+x}(2e^{2x}+1)=3. \quad \text{Now } \lim_{x\to 0}(e^{2x}+x)^{\frac{1}{x}}=e^L=e^3;$$

(2.) $\lim\limits_{x\to 0^{+0}} x^x=[0^0]=e^{\lim\limits_{x\to 0^{+0}}(x\,ln\,x)}=e^L$, where

$$L=\lim_{x\to 0^{+0}}(x\,ln\,x)=[0\cdot\infty]=\lim_{x\to 0^{+0}}\frac{ln\,x}{\frac{1}{x}}=\begin{bmatrix}\infty\\\infty\end{bmatrix}=\lim_{x\to 0^{+0}}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0. \quad \text{Now } \lim_{x\to 0^{+0}}x^x=e^L=e^0=1;$$

(3.) $\lim\limits_{x\to \frac{\pi}{2}^{-0}}(\tan x)^{\cot x}=[\infty^0]=e^{\lim\limits_{x\to \frac{\pi}{2}^{-0}}\cot x\cdot ln(\tan x)}=e^L$, where

$$L=\lim_{x\to \frac{\pi}{2}^{-0}}[\cot x\,ln(\tan x)]=[0\cdot\infty]=\lim_{x\to \frac{\pi}{2}^{-0}}\frac{ln(\tan x)}{\tan x}=\begin{bmatrix}\infty\\\infty\end{bmatrix}=$$

$$=\lim_{x\to \frac{\pi}{2}^{-0}}\frac{\frac{1}{\tan x}\cdot\frac{1}{cos^2 x}}{\frac{1}{cos^2 x}}=\lim_{x\to \frac{\pi}{2}^{-0}}\cot x=0. \text{ Now } \lim_{x\to \frac{\pi}{2}^{-0}}(\tan x)^{\cot x}=e^L=e^0=1.$$

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Exercises 6.27

Find limits:

(1.) $\lim\limits_{x\to 0}\frac{e^{5x}-1}{2x}$;

(2.) $\lim\limits_{x\to\infty}\frac{e^{x}}{x^{3}}$;

(3.) $\lim\limits_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{\frac{1}{3}\ln\frac{x+1}{x-1}}$;

(4.) $\lim\limits_{x\to 0^{+0}}\frac{\arcsin\sqrt{x}}{\sqrt{2x-x^2}}$.

The solution:

(1.) $$\lim_{x\to 0}\frac{e^{5x}-1}{2x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(e^{5x}-1\right)'}{(2x)'}=\lim_{x\to 0}\frac{5e^{5x}}{2}=\frac{5}{2};$$

(2.) $$\lim_{x\to\infty}\frac{e^{x}}{x^{3}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(e^{x}\right)'}{\left(x^{3}\right)'}=\lim_{x\to\infty}\frac{e^{x}}{3x^{2}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(e^{x}\right)'}{\left(3x^{2}\right)'}=\lim_{x\to\infty}\frac{e^{x}}{6x}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left(e^{x}\right)'}{(6x)'}=\lim_{x\to\infty}\frac{e^{x}}{6}=\infty;$$

(3.) $$\lim_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{\frac{1}{3}\ln\frac{x+1}{x-1}}=\left[\frac{0}{0}\right]=\lim_{x\to\infty}\frac{\left(\frac{\pi}{2}-\arctan x\right)'}{\left(\frac{1}{3}\ln\frac{x+1}{x-1}\right)'}=\lim_{x\to\infty}\frac{-\frac{1}{1+x^{2}}}{-\frac{2}{3\left(x^{2}-1\right)}}=\lim_{x\to\infty}\frac{3\left(x^{2}-1\right)}{2\left(x^{2}+1\right)}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left[3\left(x^{2}-1\right)\right]'}{\left[2\left(x^{2}-1\right)\right]'}=\lim_{x\to\infty}\frac{6x}{4x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(6x)'}{(4x)'}=\frac{6}{4}=\frac{3}{2};$$

(4.) $$\lim_{x\to 0^{+0}}\frac{\arcsin\sqrt{x}}{\sqrt{2x-x^{2}}}=\left[\frac{0}{0}\right]=\lim_{x\to 0^{+0}}\frac{\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}}{\frac{1(2-2x)}{2\sqrt{2x-x^{2}}}}=\lim_{x\to 0^{+0}}\frac{\frac{1}{2\sqrt{x}\cdot\sqrt{1-x}}}{\frac{1-x}{\sqrt{x}\cdot\sqrt{2-x}}}=$$

$$=\lim_{x\to 0^{+0}}\frac{\sqrt{2-x}}{2(1-x)\sqrt{1-x}}=\frac{\sqrt{2}}{2}.$$

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Exercises 6.228

Find limits:

(1.) $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]$; (2.) $\lim\limits_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot\cot(x-a)\right]$.

The solution:

(1.) $$\lim_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]=\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{2x}=\lim_{x\to 0}\frac{x}{2x(1+x)}=\frac{1}{2};$$

(2.) $$\lim_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot\cot(x-a)\right]=[0\cdot\infty]=\lim_{x\to a}\frac{\arcsin\frac{x-a}{a}}{\tan(x-a)}=$$

$$=\left[\frac{0}{0}\right]=\lim_{x\to \alpha}\frac{\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}}{\frac{1}{\cos^2(x-a)}}=\frac{1}{a}\cdot\lim_{x\to \alpha}\frac{\cos^2(x-a)}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}=\frac{1}{a}.$$

Exercises 6.239

Find limits:

(1.) $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{x}{\cot x}-\frac{\pi}{2\cos x}\right)$; (2.) $\lim\limits_{x\to 0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right)$;

(3.) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$.

The solution:

(1.) $$\lim_{x\to\frac{\pi}{2}}\left(\frac{x\sin x}{\cos x}-\frac{\pi}{2\cos x}\right)=[\infty-\infty]=\lim_{x\to\frac{\pi}{2}}\frac{2x\sin x-\pi}{2\cos x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\sin x+2x\cos x}{2(-\sin x)}=\frac{2}{-2}=-1.$$

(2.) $$\lim_{x\to 0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right)=[\infty-\infty]=\lim_{x\to 0}\frac{x-\sin x}{x^2\sin x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{1-\cos x}{2x\sin x+x^2\cos x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\sin x}{2\sin x+2x\cos x+2x\cos x+x^2(-\sin x)}=$$

$$=\lim_{x\to 0}\frac{\sin x}{2\sin x+4x\cos x-x^2\sin x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{\cos x}{2\cos x+4\cos x+4x(-\sin x)-2x\sin x-x^2\cos x}=\frac{1}{6}.$$

(3.) $$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=[\infty-\infty]\;=\lim_{x\to 0}\frac{1-x^2\cot^2 x}{x^2}=$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2\cos^2 x}{x^2\sin^2 x}\;=\lim_{x\to 0}\frac{(\sin x-x\cos x)(\sin x+x\cos x)}{x^2\sin x\cdot\sin x};$$

Since

$$\lim_{x\to 0}\frac{\sin x-x\cos x}{x^2\sin x}=\left[\frac{0}{0}\right]\;=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{2x\sin x+x^2\cos x}=$$

$$=\lim_{x\to 0}\frac{\sin x}{2\sin x+x\cos x}=\left[\frac{0}{0}\right]\;=\lim_{x\to 0}\frac{\cos x}{2\cos x+\cos x+x(-\sin x)}=\frac{1}{3}$$

and

$$\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}=2$$

it is

$$\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x^2\sin x}\right)\cdot\left(\frac{\sin x+x\cos x}{\sin x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x^2\sin x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x+x\cos x}{\sin x}\right)=\frac{2}{3}.$$

Exercises 6.24

Find limits:

(1.) $\lim\limits_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}}$;

(2.) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$;

(3.) $\lim\limits_{x\to 1}\left(\tan\frac{\pi\cdot x}{4}\right)^{\tan\frac{\pi\cdot x}{2}}$;

(4.) $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$;

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(5.) $\lim\limits_{x\to 0^{+0}} x^{\frac{3}{4+\ln x}}$;

(6.) $\lim\limits_{x\to 0^{+0}} (\cot x)^{\frac{1}{\ln x}}$;

(7.) $\lim\limits_{x\to 0} x^{\frac{1}{\ln(e^x-1)}}$;

(8.) $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}}$.

The solution:

(1.) $\lim\limits_{x\to 0} (\frac{2}{\pi} arccos)^{\frac{1}{x}} = [1^\infty] = e^{\lim\limits_{x\to 0}\frac{1}{x} ln\left(\frac{2}{\pi} arccos\, x\right)} = e^L$, where

$$L = \lim_{x\to 0} \frac{\ln\left(\frac{2}{\pi}\arccos x\right)}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\left[\left(\frac{\pi}{2\arccos x}\right)\cdot\left(-\frac{2}{\pi\sqrt{1-x^2}}\right)\right] = -\frac{2}{\pi};$$

$$\lim_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}} = e^{-\frac{2}{\pi}}$$

(2.) $\lim\limits_{x\to 0} \left(\frac{sin\, x}{x}\right)^{\frac{1}{x^2}} = [1^\infty] = e^{\lim\limits_{x\to 0}\frac{1}{x^2} ln\frac{sin\, x}{x}} = e^L$, where

$$L = \lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{2x} =$$

$$= \lim_{x\to 0}\frac{x}{\sin x} \cdot \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3} = 1\cdot\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0}\frac{cos\, x + x(-sin\, x) - cos\, x}{6x^2} = -\lim_{x\to 0}\frac{sin\, x}{6x} = -\frac{1}{6};$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$$

(3.) $\lim\limits_{x\to 1} \left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}} = [1^\infty] = e^{\lim\limits_{x\to 1}\left[\tan\frac{\pi x}{2}\cdot ln\left(\tan\frac{\pi x}{4}\right)\right]} = e^L$, where

$$L=\lim_{x\to 1}\left[\tan\frac{\pi x}{2}\cdot\ln\left(\tan\frac{\pi x}{4}\right)\right]=[\infty\cdot 0]\quad=\lim_{x\to 1}\frac{\ln\left(\tan\frac{\pi x}{4}\right)}{\cot\left(\frac{\pi x}{2}\right)}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 1}\frac{\frac{1}{\tan\frac{\pi x}{4}}\cdot\frac{1}{\cos^2\frac{\pi x}{4}}\cdot\frac{\pi}{4}}{-\frac{1}{\sin^2\frac{\pi x}{2}}\cdot\frac{\pi}{2}}=\left(-\frac{1}{2}\right)\cdot\lim_{x\to 1}\frac{2\sin^2\frac{\pi x}{2}}{2\sin\frac{\pi x}{4}\cdot\cos\frac{\pi x}{4}}=$$

$$=-\lim_{x\to 1}\frac{sin^2\frac{\pi x}{2}}{sin\frac{\pi x}{2}}=-\lim_{x\to 1}\left(sin\frac{\pi x}{2}\right)=-1;$$

$$\lim_{x\to 1}\left(\tan\frac{\pi\cdot x}{4}\right)^{\tan\frac{\pi\cdot x}{2}}=e^{-1}$$

(4.) $\lim_{x\to\frac{\pi}{2}}(sin\,x)^{\tan x}=[1^\infty]=e^{\lim_{x\to\frac{\pi}{2}}[\tan x\, ln(sin\,x)]}=e^L,$ where

$$\text{L}=\lim_{x\to\frac{\pi}{2}}\left[\tan x\cdot\ln(\sin x)\right]=[\infty\cdot 0]\quad=\lim_{x\to\frac{\pi}{2}}\frac{\ln(\sin x)}{\cot x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{sin\,x}\cdot cos\,x}{-\frac{1}{sin^2 x}}=-\lim_{x\to\frac{\pi}{2}}\left[cos\,x\cdot sin\,x\right]=0;$$

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=e^0=1$$

(5.) $\lim_{x\to 0^{+0}}x^{\frac{3}{4+ln\,x}}=[0^0]=e^{\lim_{x\to 0^{+0}}\frac{3}{4+ln\,x}\cdot ln\,x}=e^L,$ where

$$\text{L}=\lim_{x\to 0^{+0}}\frac{3\ln x}{4+\ln x}=\left[\frac{\infty}{\infty}\right]\;=\lim_{x\to 0^{+0}}\frac{\frac{3}{x}}{\frac{1}{x}}=3;$$

$$\lim_{x\to 0^{+0}}x^{\frac{3}{4+\ln x}}=e^3$$

(6.) $\lim_{x\to 0^{+0}}(\cot x)^{\frac{1}{ln\,x}}=[\infty^0]=e^{\lim_{x\to 0^{+0}}\left[\frac{1}{ln\,x}ln(\cot x)\right]}=e^L,$ where

$$L=\lim_{x\to 0^{+0}}\frac{\ln(\cot x)}{\ln x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0^{+0}}\frac{\frac{1}{\cot x}\cdot\left(-\frac{1}{\sin^2 x}\right)}{\frac{1}{x}}=$$

$$=-\lim_{x\to 0^{+0}}\frac{x}{\cos x\sin x}=-\lim_{x\to 0^{+0}}\frac{2x}{\sin 2x}=\left[\frac{0}{0}\right]=-\lim_{x\to 0^{+0}}\frac{2}{\cos 2x\cdot 2}=-1;$$

$$\lim_{x\to 0^{+0}}(\cot x)^{\frac{1}{\ln x}}=e^{-1}$$

(7.) $\lim_{x\to 0} x^{\frac{1}{ln(e^x-1)}}=[0^0]=\lim_{x\to 0} e^{\frac{1}{ln(e^x-1)}\cdot ln\,x}=e^{\lim_{x\to 0}\left[\frac{1}{ln(e^x-1)}\cdot ln\,x\right]}=e^L$, where

$$\lim_{x\to 0}\frac{\ln x}{\ln(e^x-1)}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{e^x}{e^x-1}}=\lim_{x\to 0}\frac{e^x-1}{xe^x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{e^x}{e^x+xe^x}=1;$$

$$\lim_{x\to 0} x^{\frac{1}{\ln(e^x-1)}}=e^1=e$$

(8.) $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}=\left[1^\infty\right]=\lim_{x\to 0} e^{\frac{1}{x}\ln(1+\sin x)}=e^{\lim_{x\to 0}\left[\frac{ln(1+sin\,x)}{x}\right]}=e^{\lim_{x\to 0}\left[\frac{\frac{1}{1+sin\,x}\cdot cos\,x}{1}\right]}=e^1=e$.

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