## 6.8. Properties of continuous real function and graph sketching

Examining the properties of a continuous real function $f$ that is defined analytically, i.e., by a formula, consists of the following steps:

### 1. *Area of definition (natural domain), parity and periodicity*

In order to determine the natural domain $D_f$ of a given function $f$ it is necessary to know the basic elementary functions and procedures for solving equations or inequalities.

**Definition (even, odd, and periodic function):**
The function $f$ is:
a) even if

$$f(-x) = f(x) \text{ for all } x \in D_f,$$

b) odd if

$$f(-x) = -f(x) \text{ for all } x \in D_f,$$

c) periodic if for some $P \neq 0$ for all $x \in D_f$ is:

$$f(x+P) = f(x).$$

**Definition (fundamental period of a function):**
Suppose that the function $f$ is periodic and $q$ denotes the smallest positive number so that

$$f(x+q) = f(x) \text{ for all } x \in D_f.$$

Such a number $q$ is called the ***fundamental*** or ***basic period*** of the periodic function $f$.

If the domain $D_f$ is determined, we examine whether the function $f$ is even, odd, or periodic. This is useful to know because it can significantly help in further examination of a function. Namely, if $f$ is an even function, then its graph is symmetric with respect to the $y$ −axis therefore, it is sufficient to examine the function $f$ only on the set $D_f \cap [0, \infty)$;
if $f$ is an odd function, then its graph is centrally symmetric with respect to the origin, so again it is sufficient to examine the flow of the function $f$ only on the set $D_f \cap [0, \infty)$;
if $f$ is the periodic function with the fundamental period $q$, then it is sufficient to examine the function $f$ only on the set $D_f \cap \left[-\frac{q}{2}, \frac{q}{2}\right]$.

### 2. $x$ − intercept (s) and $y$ − intercept

The intersection points of a graph of a function with coordinate axes are important indicators of the behaviour of a function.

**Definition and finding procedure:** We solve the equation $f(x) = 0$.


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If $x = x_0$ is the solution of that equation, then the point $N(x_0, 0)$ is called **zero-point**, shorter **zero**, or $\boldsymbol{x}$ **− intercept** of the function $f$.
If the equation $f(x) = 0$ has no solution, then the function $f$ has no zeros.
The equation $f(x) = 0$ can have multiple solutions, i.e., the function $f$ can have multiple zeros (there may even be an infinite number of them).
If $0 \in D_f$, then the point $(0, f(0))$ is called **y-intercept**.
If $0 \notin D_f$, then such point does not exist.

Note: Function $y = f(x)$ can have one $y -$ intercept at most.

## 3. Asymptotes

An asymptote of a function $f$ is a line that comes arbitrarily close to the graph of $f$ as the graph recedes indefinitely away from the origin.

**Definition (asymptotes):**

If $a \in \mathbb{R}$, the line $x = a$ is the right vertical asymptote of the function $f$ if

$$\lim_{x \to a^{+0}} f(x) = \infty \text{ or } \lim_{x \to a^{+0}} f(x) = -\infty.$$

The line $x = a$ is the left vertical asymptote of the function $f$ if

$$\lim_{x \to a^{-0}} f(x) = \infty \text{ or } \lim_{x \to a^{-0}} f(x) = -\infty.$$

The line $y = b$ is the right horizontal asymptote of the function $f$ if

$$\lim_{x \to \infty} f(x) = b \in \mathbb{R}.$$

The line $y = c$ is the left horizontal asymptote of the function $f$ if

$$\lim_{x \to -\infty} f(x) = c \in \mathbb{R}.$$

If $\lim\limits_{x \to \infty} \frac{f(x)}{x} = k_1 \in \mathbb{R} \setminus \{0\}$ and $\lim\limits_{x \to \infty} [f(x) - k_1 x] = l_1 \in \mathbb{R}$, then

$$y = k_1 x + l_1$$

Is the **right oblique** or the **slant asymptote** of the function $f$.

If $\lim\limits_{x \to -\infty} \frac{f(x)}{x} = k_2 \in \mathbb{R} \setminus \{0\}$ and $\lim\limits_{x \to -\infty} [f(x) - k_2 x] = l_2 \in \mathbb{R} \setminus \{0\}$, then

$$y = k_2 x + l_2$$

is the **left oblique or slant asymptote** of the function $f$.

Important notes:

1) The function $f$ can have a ***vertical*** asymptote $x = a$ only if $a$ is a point on the edge of the domain $D_f$ where the function is not defined.
2) The function $f$ can have more (even infinitely many) vertical asymptotes.


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3) The function $f$ cannot have a ***horizontal*** and an ***oblique*** asymptote on the same side of the graph.

Therefore, the function $f$ cannot have

the right horizontal and right oblique

or

the left horizontal and left oblique asymptote.

However, the function $f$ can have

the left horizontal and right oblique

or

the right horizontal and left oblique asymptote.

4) The function $f$ does not have to have any asymptotes.

Finding procedure: Taking into account the previous notes, all meaningful limits should be determined, and the equations of the corresponding asymptotes should be written. When drawing a graph of the function $f$, asymptotes are usually drawn with dashed lines.

## 4. Intervals of monotonicity and points of local extrema

Definition (open interval):

*Open intervals* (in the set $\mathbb{R}$) are intervals of the form:

$$(a, b), (a, \infty) \text{ or } (-\infty, b),$$

where $a$ and $b$ are real numbers.

Definition (monotonic function on open interval, intervals of monotonicity):

Suppose that the function $f$ is defined on the open interval $J$.

a) The function $f$ is *increasing* on $J$ if for any two points $x_1, x_2 \in J$ such that $x_1 < x_2$ it holds

$$f(x_1) \leq f(x_2).$$

b) The function $f$ is *decreasing* on $J$ if for any two points $x_1, x_2 \in J$ such that $x_1 < x_2$ it holds

$$f(x_1) \geq f(x_2).$$

In both cases, if $f$ is only increasing or only decreasing on $J$, it is said that $f$ is **monotone** on the interval $J$, and the interval $J$ is called the **interval of monotonicity** of the function $f$.

Theorem (sufficient condition for monotonicity):

Suppose that $f$ is differentiable function on the open interval $J$.

If $f'(x) > 0$ for all $x \in J$, then $f$ is **increasing** on $J$.

If $f'(x) < 0$ for all $x \in J$, then $f$ is **decreasing** on $J$.

Definition (point of local extremum):

Suppose that the function $f$ is defined on the open interval $J$ and that $c \in J$.

a) If exists $\varepsilon > 0$ such that



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$$f(x) \leq f(c) \text{ for all } x \in (c-\varepsilon, c) \cup (c, c+\varepsilon),$$

then the function $f$ has a *local extremum* (namely a *local maximum*) at the point $c$, and $M(c, f(c))$ is the point of the local maximum of the function $f$.

b) If exists $\varepsilon > 0$ such that

$$f(x) \geq f(c) \text{ for all } x \in (c-\varepsilon, c) \cup (c, c+\varepsilon),$$

then the function $f$ has a *local extremum* (namely a *local minimum*) at the point $c$, and $m(c, f(c))$ is the point of the local minimum of the function $f$.

**Definition (stationary point and critical point):**

Let the function $f$ be defined on the open interval $J$ and let $c \in J$.
The point $(c, f(c))$ is a *stationary point* of the function $f$ if $f'(c) = 0$.
The point $(c, f(c))$ is a *critical point* of the function $f$ if $f'(c) = 0$ or $f'(c)$ does not exist (in a set $\mathbb{R}$).

**Theorem (sufficient condition for the existence of a point of local extremum):**

Let the function $f$ be defined on the open interval $J$ and differentiable on $J$ except eventually at the point $c \in J$.
If exists $\varepsilon > 0$ such that

$$f'(x) > 0 \text{ for all } x \in (c-\varepsilon, c) \text{ and } f'(x) < 0 \text{ for all } x \in (c, c+\varepsilon),$$

then the function $f$ has a ***local maximum*** at the point $c$, i.e., $M(c, f(c))$ is the point of the local maximum of the function $f$.
If exists $\varepsilon > 0$ such that

$$f'(x) < 0 \text{ for all } x \in (c-\varepsilon, c) \text{ and } f'(x) > 0 \text{ for all } x \in (c, c+\varepsilon),$$

then the function $f$ has a ***local minimum*** at the point $c$, i.e., $m(c, f(c))$ is the point of the local minimum of the function $f$.

**Definition (function that changes the sign at a point):**

Let the function $f$ be defined on the open interval $J$ except eventually at the point $c \in J$.
It is said that the function $f$ is a function that changes the sign at a point $c$ if exists $\varepsilon > 0$ such that

$$f(x) < 0 \text{ for all } x \in (c-\varepsilon, c) \text{ and } f(x) > 0 \text{ for all } x \in (c, c+\varepsilon)$$

or

$$f(x) > 0 \text{ for all } x \in (c-\varepsilon, c) \text{ and } f(x) < 0 \text{ for all } x \in (c, c+\varepsilon).$$


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Finding procedure:

Firstly, the function $f'$ should be determined, and then the set

$$S_1 = \{x \in D_f : f'(x) = 0 \text{ or } f'(x) \text{ does not exist}\}.$$

Therefore, the natural domain $D_{f'}$ of the function $f'$ should be determined, and then all possible solutions of the equation $f'(x) = 0$.

The elements of the set $S_1$ together with the edges of the domain $D_f$ of the function $f$ determine the edges of the interval of monotonicity of the function $f$.

On each interval of monotonicity, on which the function $f'$ is a continuous function and has no zeros in that interval, the same procedure is applied:

1) One point of that interval is chosen and the value of the function $f'$ is calculated at that point.
2) If this value is positive (**negative**), then the function $f$ is increasing (**decreasing**) on that interval.

If the function $f'$ is the function that changes sign at the point $c \in S_1$, then $(c, f(c))$ is the point of the local extremum of the function $f$.

The type of the extremum is determined, a point of the local maximum or local minimum, using the appropriate definition (definition of the point of local extremum).

## 5. Intervals of concavity and inflection points

Definition (concave function on open interval, interval of concavity):

Let us consider the function $f$ which is defined on an open interval $J$.

a) The function $f$ is *concave upward* on $J$ if for any two different points $x_1, x_2 \in J$ it holds

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

b) The function $f$ is *concave downward* on $J$ if for any two different points $x_1, x_2 \in J$ it holds

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2}.$$

In both cases, if $f$ is concave upward or concave downward on $J$, it is said that $f$ is **concave** on the interval $J$, and the interval $J$ is called the **interval of concavity** of the function $f$.

The function $f$ is concave upward (concave downward) on an open interval $J$ if and only if at each point of that interval the line tangent to the graph of the function $f$ is below (above) the graph of the function $f$.



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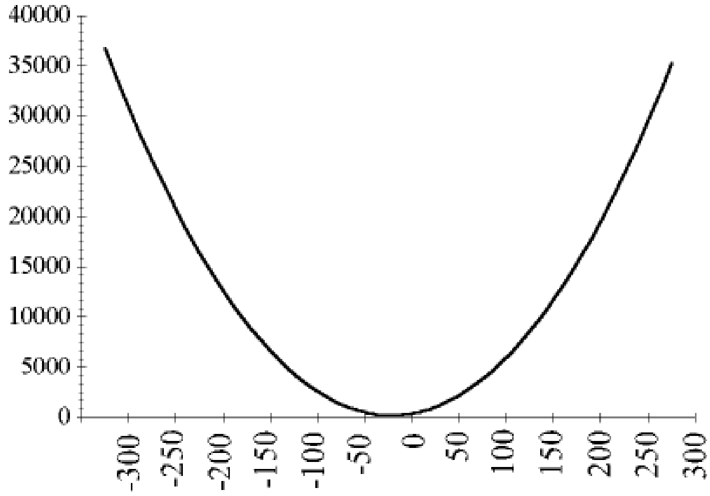


*Figure 6.7.* ***Graph of a concave upward function on the interval***

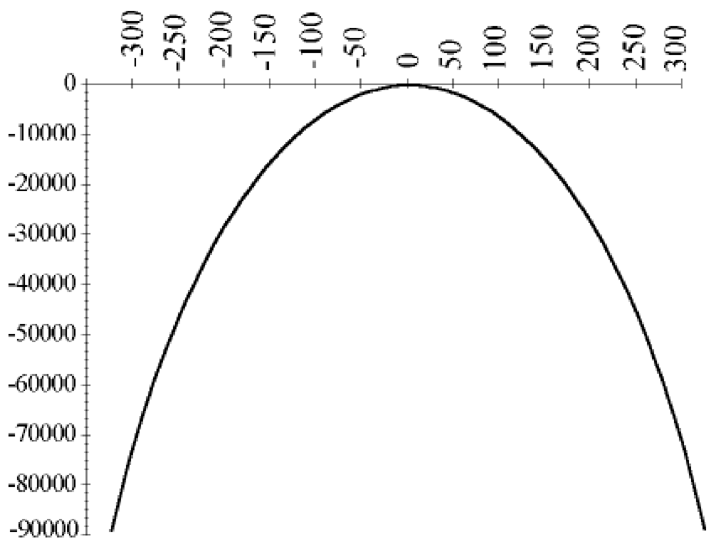


*Figure 6.8.* ***Graph of a concave downward function on the interval***

Theorem (sufficient condition for concavity):

Suppose that the function $f$ is twice differentiable on the open interval $J$.
If $f''(x) > 0$ for all $x \in J$, then $f$ is **concave upward** on $J$.
If $f''(x) < 0$ for all $x \in J$, then $f$ is **concave downward** on $J$.

Definition (inflection point):

Suppose that the function $f$ is defined on the open interval $J$ and that $c \in J$.
If exists $\varepsilon > 0$ such that $f$ is concave upward on the interval $(c-\varepsilon, c)$ and concave downward on the interval $(c, c+\varepsilon)$, or vice versa, then $f$ has an inflection at the point $c$, and $I\big(c, f(c)\big)$ is the inflection point of the function $f$.



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**Theorem (sufficient condition for the existence of an inflection point):**

Suppose that the function $f$ is defined on an open interval $J$ and twice differentiable on $J$ except eventually at the point $c \in J$. If $f''$ changes the sign at the point $c$, then the function $f$ has an inflection at the point $c$, and $I(c, f(c))$ is the inflection point of the function $f$.

**Finding procedure:**

Firstly, the function $f''$ is determined, and then the set

$$S_2 = \{x \in D_f : f''(x) = 0 \text{ or } f''(x) \text{ does not exist}\}.$$

Therefore, the natural domain $D_{f''}$ of the function $f''$ should be determined, and then all possible solutions of the equation $f''(x) = 0$.

The elements of the set $S_2$ together with the edges of the domain $D_f$ of the function $f$ determine the edges of the intervals of concavity of the function $f$.

On each interval of concavity, on which the function $f''$ is a continuous function (which has no zero points on that interval), the same procedure is applied:

1) One point of that interval is chosen and the value of the function $f''$ is calculated at that point.
2) If this value is positive (negative) then the function $f$ is concave upward (concave downward) on that interval.

If the function $f''$ changes sign at the point $c \in S_2$ then $I(c, f(c))$ is the inflection point of the function $f$.

## 6. Graph of the function

All obtained information about the function $f$ through steps 1-5 should be merged into a coherent image.

**Note:** When drawing a graph, it is possible to detect all inconsistencies, i.e., errors in the previous calculation and correct them.

In the following examples the characteristic properties of a non-periodic function $y = f(x)$ are examined for the purpose of drawing its graph.


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## Example 1

$$f(x)=\frac{16}{x^2(x-4)}.$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined). The same is true for its derivatives.

### 1. Area of definition (natural domain), parity and periodicity

$$D_f=\{x\in\mathbb{R}:\ x^2(x-4)\neq 0\}=\mathbb{R}\setminus\{0,4\}=(-\infty,0)\cup(0,4)\cup(4,\infty)$$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.

### 2. $x-$ intercept(s) and $y-$ intercept

$f(x)\neq 0$ for all $x\in D_f$ so the function has no zeros.

$f(0)$ does not exist because $0\notin D_f$. Therefore, the graph of the function $f$ neither intersects nor touches the $y-$ axis.

### 3. Asymptotes

Possible vertical asymptotes are lines $x=0$ and $x=4$. Namely, $0$ and $4$ are points on the edge of the domain $D_f$ where the function is not defined.

$\lim\limits_{x\to 0^{+0}}\frac{16}{x^2(x-4)}=-\infty\Rightarrow x=0$ is the right vertical asymptote.

$\lim\limits_{x\to 0^{-0}}\frac{16}{x^2(x-4)}=-\infty\Rightarrow x=0$ is the left vertical asymptote.

$\lim\limits_{x\to 4^{+0}}\frac{16}{x^2(x-4)}=\infty\Rightarrow x=4$ is the right vertical asymptote.

$\lim\limits_{x\to 4^{-0}}\frac{16}{x^2(x-4)}=-\infty\Rightarrow x=4$ is the left vertical asymptote.

The function could have horizontal asymptotes because the limits $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$ make sense (in the domain $D_f$ is possible that $x\to\infty$ and $x\to-\infty$).

$$\lim_{x\to\infty}\frac{16}{x^2(x-4)}=0\in\mathbb{R}$$

$\Rightarrow y=0$ is the right horizontal asymptote.

$$\lim_{x\to-\infty}\frac{16}{x^2(x-4)}=0\in\mathbb{R}$$

$\Rightarrow y=0$ is the left horizontal asymptote.


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The function has right and left horizontal asymptotes so there are no oblique asymptotes.

## 4. Intervals of monotonicity and points of local extrema

$$f'(x) = \frac{-16[2x(x-4)+x^2]}{x^4(x-4)^2} = \frac{16x(8-3x)}{x^4(x-4)^2} = \frac{16(8-3x)}{x^3(x-4)^2}$$

$$D_{f'} = D_f$$

$$f'(x) = 0$$

$$\frac{16(8-3x)}{x^3(x-4)^2} = 0 \Leftrightarrow 8-3x = 0 \Leftrightarrow x = \frac{8}{3}$$

$$f\left(\frac{8}{3}\right) = \frac{16}{\left(\frac{8}{3}\right)^2\left(\frac{8}{3}-4\right)} = \frac{16}{\frac{64}{9}\cdot\frac{-4}{3}} = -\frac{27}{16}$$

Therefore, the only critical point of the given function is the stationary point $\left(\frac{8}{3}, -\frac{27}{16}\right)$.

$$S_1 = \left\{\frac{8}{3}\right\}$$

The edges of the domain $D_f$ of the function $f$ are:

$$-\infty, 0, 4, \infty$$

so, the intervals of monotonicity are:

$$(-\infty, 0), \left(0, \frac{8}{3}\right), \left(\frac{8}{3}, 4\right), (4, \infty).$$

$f'(-1) < 0 \Rightarrow f$ is decreasing on $(-\infty, 0)$;
$f'(1) > 0 \Rightarrow f$ is increasing on $\left(0, \frac{8}{3}\right)$;
$f'(3) < 0 \Rightarrow f$ is decreasing on $\left(\frac{8}{3}, 4\right)$;
$f'(5) < 0 \Rightarrow f$ is decreasing on $(4, \infty)$.

The point of the local extremum of the function $f$ can only be the critical point of that function.

$$f'(x) > 0 \text{ for all } x \in \left(0, \frac{8}{3}\right) \ \ (\text{because } f'(1) > 0)$$

and

$$f'(x) < 0 \text{ for all } x \in \left(\frac{8}{3}, 4\right) \ \ (\text{because } f'(3) < 0)$$

so $M\left(\frac{8}{3}, -\frac{27}{16}\right)$ is the point of the local maximum of the function $f$.



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## 5. Intervals of concavity and inflection points

$$
\begin{aligned}
f''(x) &= \frac{16}{x^6(x-4)^4}\{-3x^3(x-4)^2-[3x^2(x-4)^2+2x^3(x-4)](8-3x)\} \\
&= \frac{16x^2(x-4)}{x^6(x-4)^4}\{-3x(x-4)-[3(x-4)+2x](8-3x)\} \\
&= \frac{16}{x^4(x-4)^3}[-3x(x-4)-(5x-12)(8-3x)] \\
&= \frac{16}{x^4(x-4)^3}(-3x^2+12x-76x+15x^2+96) = \frac{16}{x^4(x-4)^3}(12x^2-64x+96) \\
&= \frac{64}{x^4(x-4)^3}(3x^2-16x+24)
\end{aligned}
$$

$$D_{f''} = D_f$$

$f''(x) \neq 0$ for all $x \in D_{f''}$ because the equation $3x^2-16x+24=0$ has no real solutions.

Therefore, $S_2 = \emptyset$ so the function has no inflection points.

The edges of the domain $D_f$ of the function $f$ are:

$$-\infty, 0, 4, \infty$$

so, the intervals of concavity are:

$$(-\infty, 0), (0,4), (4, \infty).$$

$f''(-1) < 0 \Rightarrow f$ is concave downward on $(-\infty, 0)$;
$f''(1) < 0 \Rightarrow f$ is concave downward on $(0,4)$;
$f''(5) > 0 \Rightarrow f$ is concave upward on $(4, \infty)$.



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## 6. Graph of the given function

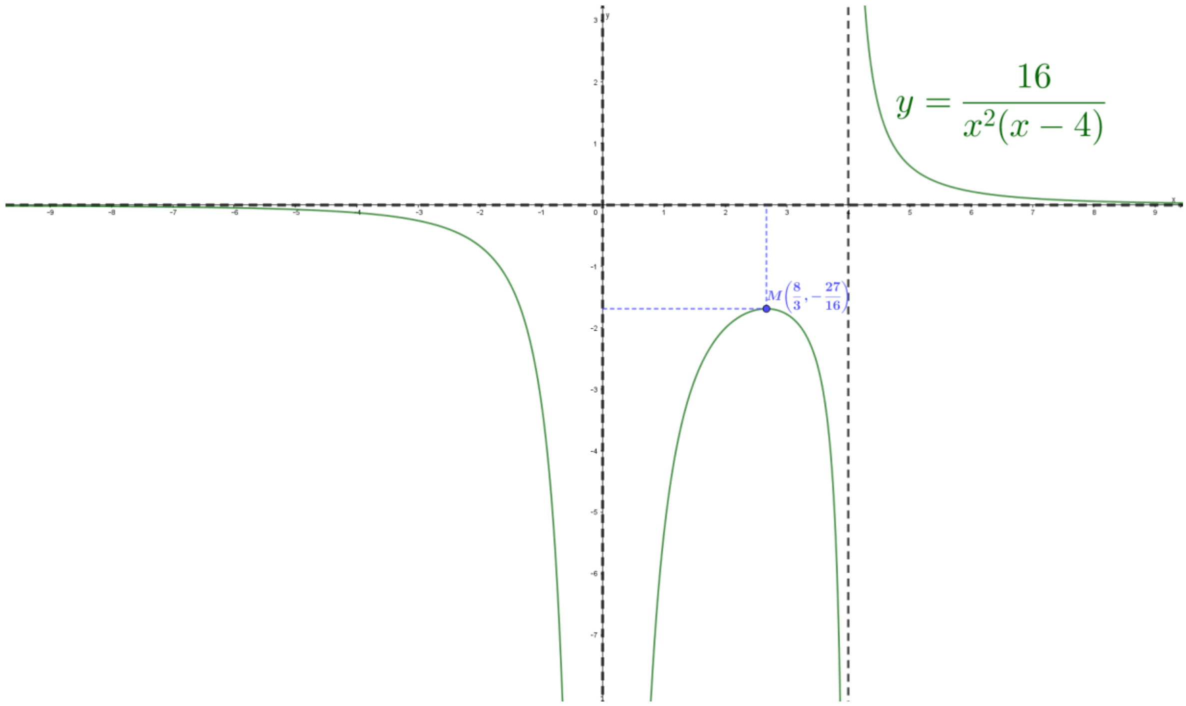


***Figure 6.9***

### Example 2

$$f(x) = x \cdot \ln^2 x$$

Solution:
The function is elementary and therefore continuous (at each point where it is defined). The same is true for its derivatives.

## 1. Area of definition (natural domain), parity and periodicity

$$D_f = \{x \in \mathbb{R}: \ln x \in \mathbb{R}\} = \mathbb{R}^+ = (0, \infty)$$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.

## 2. $x$ – intercept (s) and $y$ – intercept

$$f(x) = 0 \Leftrightarrow x \cdot \ln^2 x = 0 \Leftrightarrow \ln^2 x = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$$

Therefore, $N(1,0)$ is the zero of the function.


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$f(0)$ does not exist because $0 \notin D_f$. Therefore, the graph of the given function neither intersects nor touches the $y-$ axis.

## 3. Asymptotes

A possible right vertical asymptote is a line $x = 0$. Namely, zero is a point at the edge of the domain $D_f$ where the function is not defined, and only the right limit at that point makes sense because the function is not defined to the left of zero.

$$\lim_{x\to 0^{+0}} x \cdot \ln^2 x = [0 \cdot \infty] = \lim_{x\to 0^{+0}} \frac{\ln^2 x}{\frac{1}{x}} = \left[\frac{\infty}{\infty}\right] \overset{\text{Hr}}{=} \left[\frac{-\infty}{-\infty}\right] = \lim_{x\to 0^{+0}} \frac{\frac{2}{x}}{\frac{1}{x^2}} = 2 \lim_{x\to 0^{+0}} x = 0$$

**Note:** Equality $\overset{\text{Hr}}{=}$ is obtained by applying the l'Hospital's rule.

So, when $x \to 0^{+0}$ then $y \to 0^{+0}$ therefore $x = 0$ is not the right vertical asymptote of the given function.

The function could only have a right horizontal asymptote because only the limit $\lim_{x\to\infty} f(x)$ makes sense (in the domain $D_f$ is possible that $x \to \infty$, but not that $x \to -\infty$).

$$\lim_{x\to\infty} x \cdot \ln^2 x = \infty \notin \mathbb{R}$$

It can be concluded that the given function has no horizontal asymptotes.

The function could have only the right oblique asymptote because only the limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = k_1 \text{ and } \lim_{x\to\infty} [f(x) - k_1 x] = l_1$$

make sense.

$$k_1 = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x \cdot \ln^2 x}{x} = \infty \notin \mathbb{R}$$

so, the function has no oblique asymptotes.

## 4. Intervals of monotonicity and points of local extrema

$$f'(x) = \ln^2 x + x \cdot 2\ln x \cdot \frac{1}{x} = \ln x\,(\ln x + 2)$$

$$D_{f'} = D_f$$

$$f'(x) = 0$$

$$\ln x\,(\ln x + 2) = 0 \Leftrightarrow \ln x = 0 \text{ or } \ln x = -2 \Leftrightarrow x = 1 \text{ or } x = e^{-2}$$

$$f(1) = 1\ln^2 1 = 0, \;\; f(e^{-2}) = e^{-2}\ln^2 e^{-2} = e^{-2} \cdot (-2)^2 = 4e^{-2}$$



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Therefore, the critical points of the set function are stationary points $(1,0)$ and $(e^{-2}, 4e^{-2})$.

$$S_1 = \{e^{-2}, 1\}$$

The edges of the domain $D_f$ of the function $f$ are:

$$0, \infty$$

so, the intervals of monotonicity are:

$$(0, e^{-2}), (e^{-2}, 1), (1, \infty).$$

$f'(e^{-3}) > 0 \Rightarrow f$ is increasing on $(0, e^{-2})$;
$f'(e^{-1}) < 0 \Rightarrow f$ is decreasing on $(e^{-2}, 1)$;
$f'(e) > 0 \Rightarrow f$ is increasing on $(1, \infty)$.

The point of the local extremum of the function $f$ can only be the critical point of that function.

$$f'(x) > 0 \text{ for all } x \in (0, e^{-2}) \ \ (\text{because } f'(e^{-3}) > 0)$$

and

$$f'(x) < 0 \text{ for all } x \in (e^{-2}, 1) \ \ (\text{because } f'(e^{-1}) < 0)$$

so $M(e^{-2}, 4e^{-2})$ is the point of the local maximum of the function $f$.

$$f'(x) < 0 \text{ for all } x \in (e^{-2}, 1) \ \ (\text{because } f'(e^{-1}) < 0)$$

and

$$f'(x) > 0 \text{ for all } x \in (1, \infty) \ \ (\text{because } f'(e) > 0)$$

so $m(1,0)$ is the point of the local minimum of the function $f$.

## 5. Intervals of concavity and inflection points

$$f''(x) = 2\ln x \cdot \frac{1}{x} + 2 \cdot \frac{1}{x} = \frac{2}{x}(\ln x + 1)$$

$$D_{f''} = D_f$$

$$f''(x) = 0$$

$$\frac{2}{x}(\ln x + 1) = 0 \Leftrightarrow \ln x + 1 = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = e^{-1}$$

$$f(e^{-1}) = e^{-1}$$

$$S_2 = \{e^{-1}\}$$

The edges of the domain $D_f$ of the function $f$ are:

$$0, \infty$$

so, the intervals of concavity are:

$$(0, e^{-1}), (e^{-1}, \infty).$$


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$f''(e^{-2}) < 0 \Rightarrow f$ is concave downward on $(0, e^{-1})$;
$f''(1) > 0 \Rightarrow f$ is concave upward on $(e^{-1}, \infty)$.

The inflection point can only be the point $(e^{-1}, e^{-1})$.

$$f''(x) < 0 \text{ for all } x \in (0, e^{-1}) \ \ (\text{because } f''(e^{-2}) < 0)$$

and

$$f''(x) > 0 \text{ for all } x \in (e^{-1}, \infty) \ \ (\text{because } f''(1) > 0)$$

so $I(e^{-1}, e^{-1})$ is the inflection point of the function $f$.

## 6. Graph of the given function

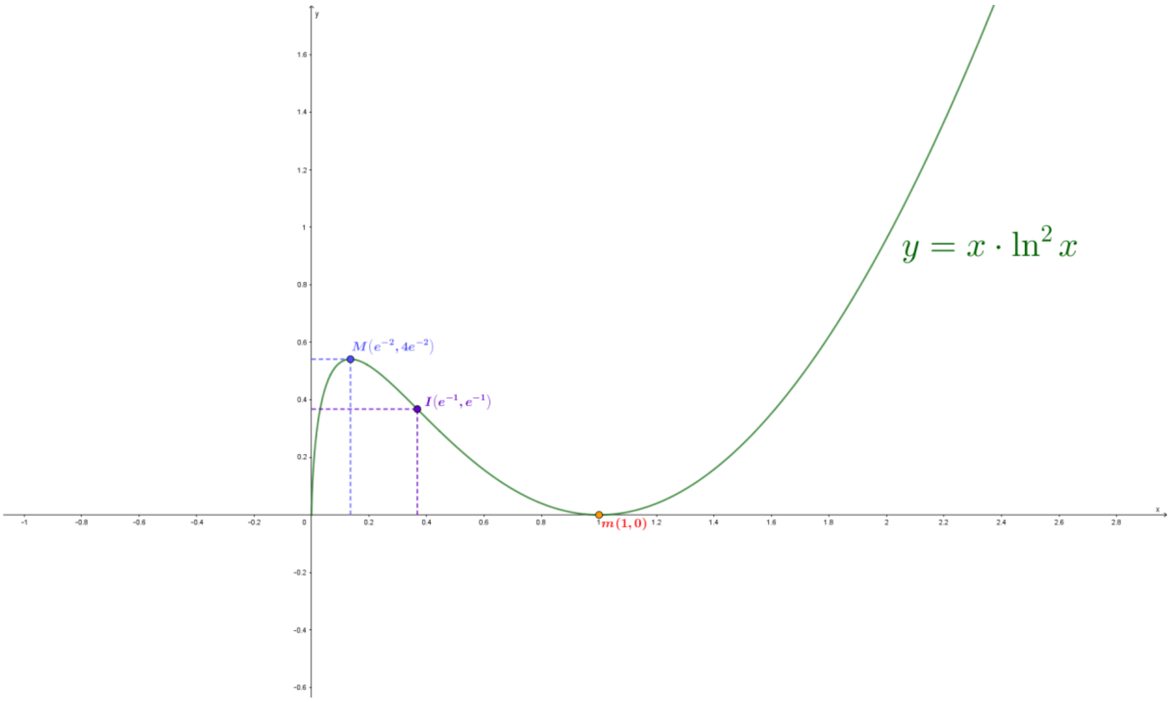


***Figure 6.10***

### Example 3

$$f(x) = xe^{-x^2}$$

Solution:
The function is elementary and therefore continuous (at each point where it is defined). The same is true for its derivatives.

## 1. Area of definition (natural domain), parity and periodicity

$$D_f = \mathbb{R} = (-\infty, \infty)$$


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For every $x \in D_f$

$$f(-x) = -xe^{-(-x)^2} = -xe^{-x^2} = -f(x)$$

therefore, the function $f$ is an odd function. So, the function is examined only at the set

$$D_f \cap [0,\infty) = \mathbb{R} \cap [0,\infty) = [0,\infty)$$

## 2. $x$ − intercept(s) and $y$ − intercept

$$f(x) = 0 \Leftrightarrow xe^{-x^2} = 0 \Leftrightarrow x = 0$$

$N(0,0)$ is the zero and $y$ − intercept.

## 3. Asymptotes

$D_f = \mathbb{R}$ so, the function has no vertical asymptotes.

The function could only have right horizontal asymptote because only the limit $\lim_{x\to\infty} f(x)$ make sense (in the domain $D_f$ is only possible that $x \to \infty$).

$$\lim_{x\to\infty} xe^{-x^2} = [\infty \cdot 0] = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \left[\frac{\infty}{\infty}\right] \overset{\text{Hr}}{=} \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$$

$\Rightarrow y = 0$ is the right horizontal asymptote.

The function does not have the right oblique asymptote because it has the right horizontal asymptote.

## 4. Intervals of monotonicity and points of local extrema

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$$

$$D_{f'} = D_f$$

$$f'(x) = 0$$

$$(1-2x^2)e^{-x^2} = 0 \Leftrightarrow 1-2x^2 = 0 \overset{x\geq 0}{\Rightarrow} x = \frac{1}{\sqrt{2}} \approx 0.707107$$

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$$

$$S_1 = \left\{\frac{1}{\sqrt{2}}\right\}$$

The edges of the interval $[0,\infty)$ are:

$$0, \infty$$

so, the intervals of monotonicity (on the interval $[0,\infty)$) are:

$$\left(0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \infty\right).$$


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$f'\left(\frac{1}{2}\right) > 0 \Rightarrow f$ is increasing on $\left(0, \frac{1}{\sqrt{2}}\right)$;
$f'(1) < 0 \Rightarrow f$ is decreasing on $\left(\frac{1}{\sqrt{2}}, \infty\right)$.

The point of the local extremum of the function $f$ can only be the critical point of that function.

$$f'(x) > 0 \text{ for all } x \in \left(0, \frac{1}{\sqrt{2}}\right) \quad \left(\text{because } f'\left(\frac{1}{2}\right) > 0\right)$$

and

$$f'(x) < 0 \text{ for all } x \in \left(\frac{1}{\sqrt{2}}, \infty\right) \quad (\text{because } f'(1) < 0)$$

so $M\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}\right)$ is the point of the local maximum of the function $f$.

## 5. Intervals of concavity and inflection points

$$f''(x) = -4xe^{-x^2} - 2x(1-2x^2)e^{-x^2} = 2x(2x^2-3)e^{-x^2}$$

$$D_{f''} = D_f$$

$$f''(x) = 0$$

$$2x(2x^2-3)e^{-x^2} = 0 \Leftrightarrow x(2x^2-3) = 0 \overset{x \geq 0}{\Rightarrow} x = 0 \text{ or } x = \sqrt{\frac{3}{2}} \approx 1.22474$$

$$f(0) = 0, \quad f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}} e^{-\frac{3}{2}}$$

$$S_2 = \left\{0, \sqrt{\frac{3}{2}}\right\}$$

The edges of the interval $[0, \infty)$ are:

$$0, \infty$$

so, the intervals of concavity (on the interval $[0, \infty)$) are:

$$\left(0, \sqrt{\frac{3}{2}}\right), \left(\sqrt{\frac{3}{2}}, \infty\right).$$

$f''(1) < 0 \Rightarrow f$ is concave downward on $\left(0, \sqrt{\frac{3}{2}}\right)$;


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$f''(2) > 0 \Rightarrow f$ is concave upward on $\left(\sqrt{\frac{3}{2}}, \infty\right)$.

$$f''(x) < 0 \text{ for all } x \in \left(0, \sqrt{\frac{3}{2}}\right) \quad (\text{because } f''(1) < 0)$$

and

$$f''(x) > 0 \text{ for all } x \in \left(\sqrt{\frac{3}{2}}, \infty\right) \quad (\text{because } f''(2) > 0)$$

so $I\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\right)$ is the inflection point of the function $f$.

## 6a. Graph of the given function on the interval $[\mathbf{0}, \infty\rangle$

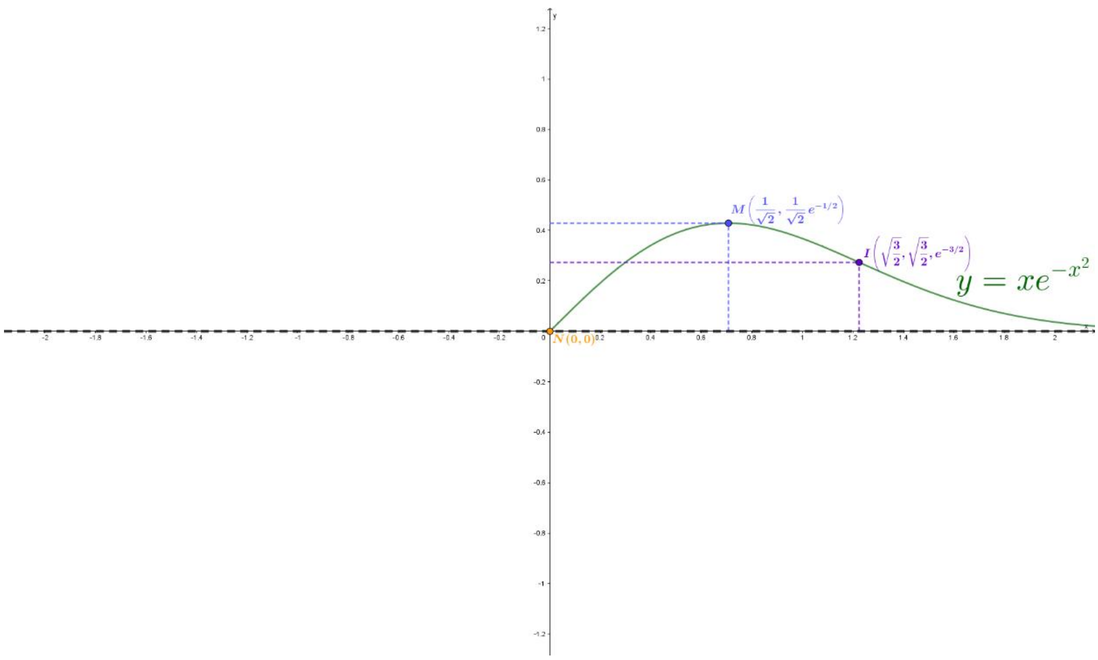


*Figure 6.11*



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## 6b. Graph of the given function

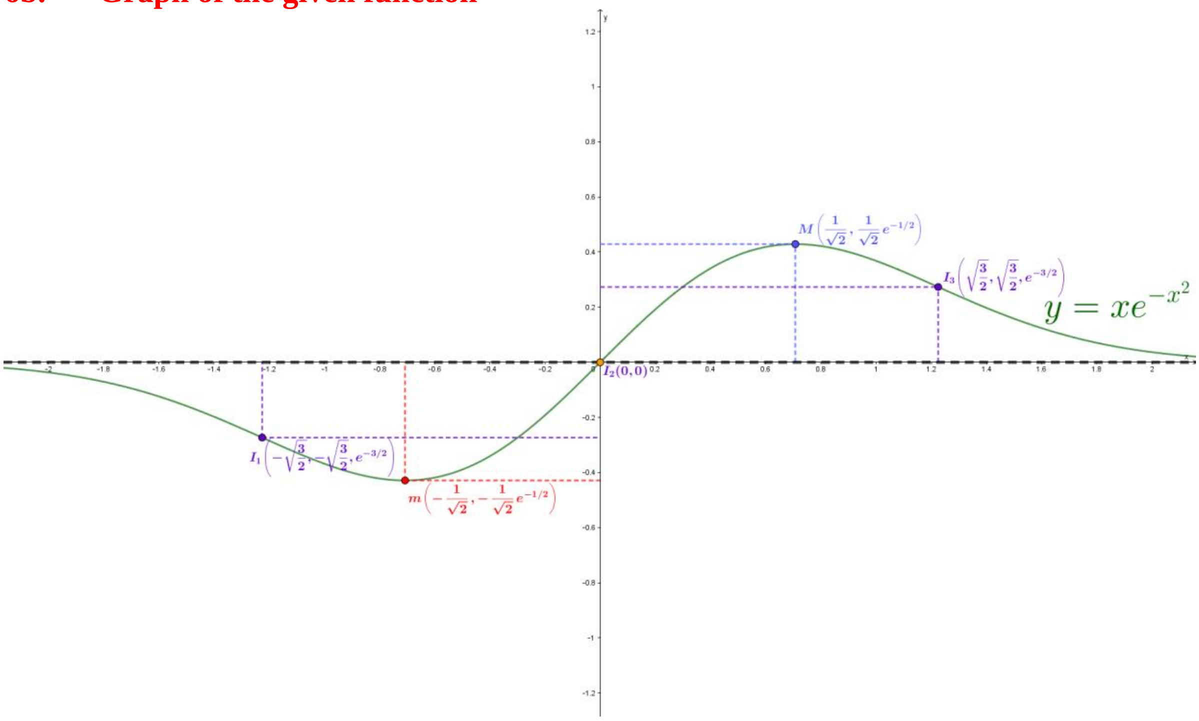


*Figure 6.12*

## Example 4

$$f(x) = xe^{\frac{1}{x-2}}$$

Solution:
The function is elementary and therefore continuous (at each point where it is defined). The same is true for its derivatives.

### 1. Area of definition (natural domain), parity and periodicity

$$D_f = \mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, \infty)$$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.

### 2. $x$ – intercept(s) and $y$ – intercept

$$f(x) = 0 \Leftrightarrow xe^{\frac{1}{x-2}} = 0 \Leftrightarrow x = 0$$

$N(0,0)$ is the zero and the $y$ – intercept.



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## 3. Asymptotes

A possible right vertical asymptote is the line $x = 2$. Namely, $2$ is the point on the edge of the domain $D_f$ where the function is not defined. Obviously, both limits

$$\lim_{x\to 2^{+0}} f(x) \text{ and } \lim_{x\to 2^{-0}} f(x)$$

make sense.

$\lim\limits_{x\to 2^{+0}} xe^{\frac{1}{x-2}} = \infty$ so, $x = 2$ is the right vertical asymptote.

$$\lim_{x\to 2^{-0}} xe^{\frac{1}{x-2}} = 0$$

The function could have horizontal asymptotes because the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ make sense (in the domain $D_f$ is possible that $x \to \infty$ and $x \to -\infty$).

$$\left.\begin{matrix} \lim\limits_{x\to\infty} xe^{\frac{1}{x-2}} = \infty \notin \mathbb{R} \\ \lim\limits_{x\to-\infty} xe^{\frac{1}{x-2}} = -\infty \notin \mathbb{R} \end{matrix}\right\} \Rightarrow \text{the function has no horizontal asymptotes.}$$

The function could have oblique asymptotes because all the associated limits make sense (in the domain $D_f$ is possible that $x \to \infty$ and $x \to -\infty$).

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{xe^{\frac{1}{x-2}}}{x} = 1 = k_1 \in \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to\infty} [f(x) - k_1 x] = \lim_{x\to\infty} \left(xe^{\frac{1}{x-2}} - x\right) = [\infty - \infty] = \lim_{x\to\infty} x\left(e^{\frac{1}{x-2}} - 1\right) = [\infty \cdot 0]$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x-2}} - 1}{\frac{1}{x}} = \left[\frac{0}{0}\right] \overset{\text{Hr}}{=} \lim_{x\to\infty} \frac{-\frac{1}{(x-2)^2} e^{\frac{1}{x-2}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{(x-2)^2} e^{\frac{1}{x-2}}$$

$$= \underbrace{\lim_{x\to\infty} \frac{x^2}{x^2 - 4x + 4}}_{=1} \cdot \underbrace{\lim_{x\to\infty} e^{\frac{1}{x-2}}}_{=1} = 1 = l_1 \in \mathbb{R}$$

so, $y = x + 1$ is the right oblique asymptote.

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{xe^{\frac{1}{x-2}}}{x} = 1 = k_2 \in \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to-\infty} [f(x) - k_2 x] = \lim_{x\to-\infty} \left(xe^{\frac{1}{x-2}} - x\right) = [-\infty + \infty] = \lim_{x\to-\infty} x\left(e^{\frac{1}{x-2}} - 1\right) = [-\infty \cdot 0]$$


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$$= \lim_{x\to-\infty} \frac{e^{\frac{1}{x-2}}-1}{\frac{1}{x}} = \left[\frac{\mathbf{0}}{\mathbf{0}}\right] \overset{\text{Hr}}{=} \lim_{x\to-\infty} \frac{-\frac{1}{(x-2)^2}e^{\frac{1}{x-2}}}{-\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{x^2}{(x-2)^2}e^{\frac{1}{x-2}}$$

$$= \underbrace{\lim_{x\to-\infty} \frac{x^2}{x^2-4x+4}}_{=1} \cdot \underbrace{\lim_{x\to-\infty} e^{\frac{1}{x-2}}}_{=1} = 1 = l_2 \in \mathbb{R}$$

so, $y = x + 1$ is the left oblique asymptote.

## 4. Intervals of monotonicity and points of local extrema

$$f'(x) = e^{\frac{1}{x-2}} - \frac{x}{(x-2)^2}e^{\frac{1}{x-2}} = e^{\frac{1}{x-2}}\left[1 - \frac{x}{(x-2)^2}\right] = \frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}}$$

$$D_{f'} = D_f$$

$$f'(x) = 0$$

$$\frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}} = 0 \Leftrightarrow x^2-5x+4 = 0 \Leftrightarrow x = 1 \text{ or } x = 4$$

$$f(1) = e^{-1},\ \ f(4) = 4e^{\frac{1}{2}}$$

Therefore, the critical points of the given function are stationary points $(1, e^{-1})$ and $\left(4,4e^{\frac{1}{2}}\right)$.

$$S_1 = \{1,4\}$$

The edges of the domain $D_f$ of the function $f$ are:

$$-\infty, 2, \infty$$

so, the intervals of monotonicity are:

$$(-\infty,1), (1,2), (2,4), (4,\infty).$$

$f'(0) > 0 \Rightarrow f$ is increasing on $(-\infty, 1)$;
$f'\left(\frac{3}{2}\right) < 0 \Rightarrow f$ is decreasing on $(1,2)$;
$f'(3) < 0 \Rightarrow f$ is decreasing on $(2,4)$;
$f'(5) > 0 \Rightarrow f$ is increasing on $(4, \infty)$.

The points of the local extrema of the function $f$ can only be critical points of that function.

$$f'(x) > 0 \text{ for all } x \in (-\infty, 1)\ \ (\text{because } f'(0) > 0)$$

and


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$$f'(x) < 0 \text{ for all } x \in (1,2) \quad \left(\text{because } f'\left(\frac{3}{2}\right) < 0\right)$$

so $M(1, e^{-1})$ is the point of the local maximum of the function $f$.

$$f'(x) < 0 \text{ for all } x \in (2,4) \quad (\text{because } f'(3) < 0)$$

and

$$f'(x) > 0 \text{ for all } x \in (4,\infty) \quad (\text{because } f'(5) > 0)$$

so $m\left(4, 4e^{\frac{1}{2}}\right)$ is the point of the local minimum of the function $f$.

## 5. Intervals of concavity and inflection points

$$f''(x) = \frac{\left[(2x-5)e^{\frac{1}{x-2}} - \frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}}\right](x-2)^2 - 2(x-2)(x^2-5x+4)e^{\frac{1}{x-2}}}{(x-2)^4}$$

$$= \frac{(2x-5)(x-2) - \frac{x^2-5x+4}{x-2} - 2(x^2-5x+4)}{(x-2)^3}e^{\frac{1}{x-2}}$$

$$= \frac{\cancel{2x^2} - 9x + 10 - \frac{x^2-5x+4}{x-2} \cancel{-2x^2} + 10x - 8}{(x-2)^3}e^{\frac{1}{x-2}}$$

$$= \frac{x + 2 - \frac{x^2-5x+4}{x-2}}{(x-2)^3}e^{\frac{1}{x-2}} = \frac{(x+2)(x-2) - x^2 + 5x - 4}{(x-2)^4}e^{\frac{1}{x-2}}$$

$$= \frac{\cancel{x^2} - 4 \cancel{-x^2} + 5x - 4}{(x-2)^4}e^{\frac{1}{x-2}} = \frac{5x-8}{(x-2)^4}e^{\frac{1}{x-2}}$$

$$D_{f''} = D_f$$

$$f''(x) = 0$$

$$\frac{5x-8}{(x-2)^4}e^{\frac{1}{x-2}} = 0 \Leftrightarrow 5x - 8 = 0 \Leftrightarrow x = \frac{8}{5}$$

$$f\left(\frac{8}{5}\right) = \frac{8}{5}e^{-\frac{5}{2}}$$

$$S_2 = \left\{\frac{8}{5}\right\}$$

The edges of the domain $D_f$ of the function $f$ are:

$$-\infty, 2, \infty$$

so, the intervals of concavity are:

$$\left(-\infty, \frac{8}{5}\right), \left(\frac{8}{5}, 2\right), (2, \infty).$$



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$f''(1) < 0 \Rightarrow f$ is concave downward on $\left(-\infty, \frac{8}{5}\right)$;
$f''\left(\frac{9}{5}\right) > 0 \Rightarrow f$ is concave upward on $\left(\frac{8}{5}, 2\right)$;
$f''(3) > 0 \Rightarrow f$ is concave upward on $(2, \infty)$.

$$f''(x) < 0 \text{ for all } x \in \left(-\infty, \frac{8}{5}\right) \quad (\text{because } f''(1) < 0)$$

and

$$f''(x) > 0 \text{ for all } x \in \left(\frac{8}{5}, 2\right) \quad \left(\text{because } f''\left(\frac{9}{5}\right) > 0\right)$$

so $I\left(\frac{8}{5}, \frac{8}{5}e^{-\frac{5}{2}}\right)$ is the inflection point of the function $f$.

## 6. Graph of the given function

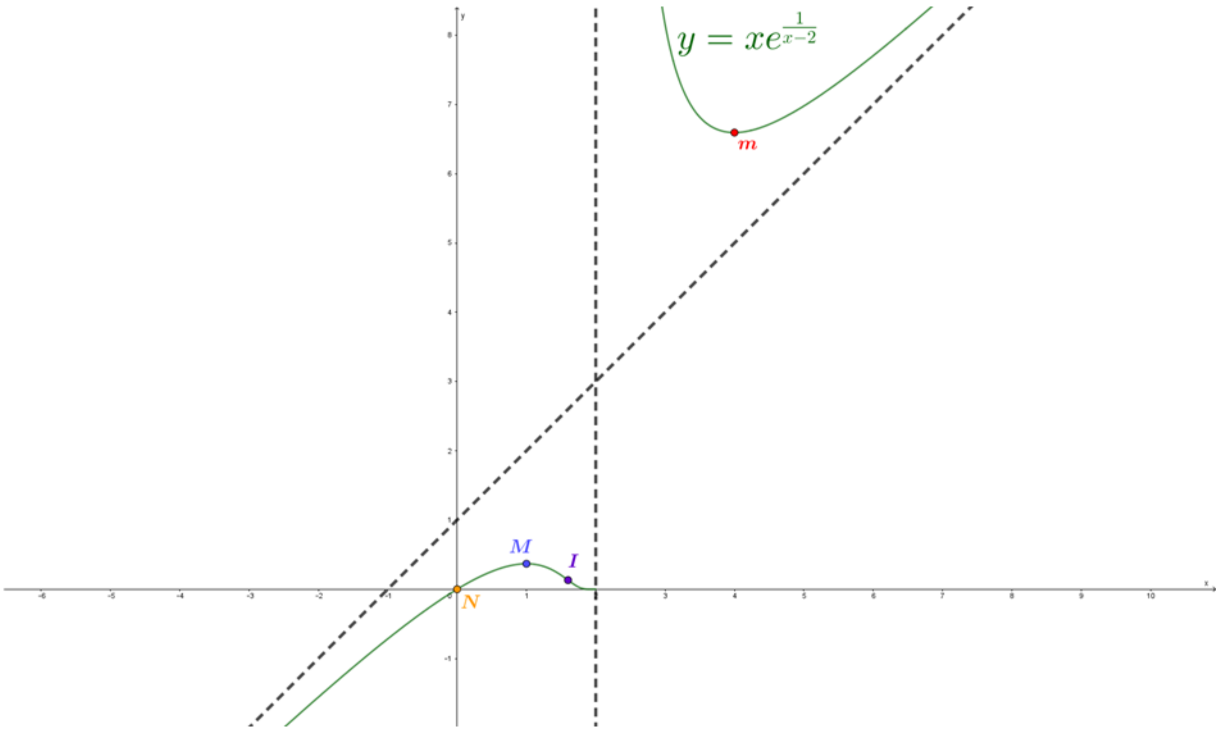


*Figure 6.13*

## Example 5

$$f(x) = \sqrt{|x|}(2 - \ln x^2)$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined). The same is true for its derivatives.

### 1. Area of definition (natural domain), parity and periodicity

$$D_f = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$$

For all $x \in D_f$:

$$f(-x) = \sqrt{|-x|}[2 - \ln(-x)^2] = \sqrt{|x|}(2 - \ln x^2) = f(x)$$

So, $f$ is an even function. Therefore, it is sufficient to examine the function only on the set

$$D_f \cap [0, \infty) = (0, \infty).$$

For every $x > 0$ is valid

$$f(x) = \sqrt{x}(2 - 2\ln x) = 2\sqrt{x}(1 - \ln x).$$

### 2. $x$ − intercept(s) and $y$ − intercept

$$f(x) = 0 \Leftrightarrow 2\sqrt{x}(1 - \ln x) = 0 \Leftrightarrow 1 - \ln x = 0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$$

Therefore, $N(e, 0)$ is the only zero of the function on the interval $(0, \infty)$.

$f(0)$ does not exist because $0 \notin D_f$. Therefore, the graph of the function neither intersects nor touches the $y$ − axis.

### 3. Asymptotes

$$\lim_{x \to 0^{+0}} f(x) = \lim_{x \to 0^{+0}} 2\sqrt{x}(1 - \ln x) = [0 \cdot \infty] = \lim_{x \to 0^{+0}} \frac{1 - \ln x}{\frac{1}{2\sqrt{x}}} = \left[\frac{\infty}{\infty}\right] \overset{\text{Hr}}{=} \lim_{x \to 0^{+0}} \frac{-\frac{1}{x}}{-\frac{1}{4x\sqrt{x}}}$$

$$= \lim_{x \to 0^{+0}} 4\sqrt{x} = 0$$

so, the function has no vertical asymptotes on the right side of the graph.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2\sqrt{x}(1 - \ln x) = -\infty \notin \mathbb{R}$$

so, the function has no right horizontal asymptote.



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$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{2(1-\ln x)}{\sqrt{x}}=\left[\frac{-\infty}{\infty}\right]\overset{\text{Hr}}{=}\lim_{x\to\infty}\frac{-\frac{2}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{-4}{\sqrt{x}}=0=k_1$$

$k_1 \notin \mathbb{R}\setminus\{0\}$ so, the function has no right oblique asymptote.

## 4. Intervals of monotonicity and points of local extrema

$$f'(x)=\frac{1}{\sqrt{x}}(1-\ln x)+2\sqrt{x}\cdot\frac{-1}{x}=\frac{1-\ln x-2}{\sqrt{x}}=-\frac{1+\ln x}{\sqrt{x}}$$

$$D_{f'}=D_f\cap[0,\infty)$$

$$f'(x)=0$$

$$-\frac{1+\ln x}{\sqrt{x}}=0\Leftrightarrow 1+\ln x=0\Leftrightarrow \ln x=-1\Leftrightarrow x=e^{-1}$$

$$f(e^{-1})=4\sqrt{e^{-1}}=4e^{-\frac{1}{2}}$$

$$S_1=\{e^{-1}\}$$

The edges of the interval $(0,\infty)$ are:

$$0,\infty$$

so, the intervals of monotonicity (on the interval $(0,\infty)$) are:

$$(0,e^{-1}),(e^{-1},\infty).$$

$f'(e^{-2})>0\Rightarrow f$ is increasing on $(0,e^{-1})$;
$f'(1)<0\Rightarrow f$ is decreasing on $(e^{-1},\infty)$.

The point of the local extremum of the function $f$ can only be the critical point of that function.

$$f'(x)>0\text{ for all }x\in(0,e^{-1})\ \ (\text{because }f'(e^{-2})>0)$$

and

$$f'(x)<0\text{ for all }x\in(e^{-1},\infty)\ \ (\text{because }f'(1)<0)$$

so $M\left(e^{-1},4e^{-\frac{1}{2}}\right)$ is the point of the local maximum of the function $f$.

## 5. Intervals of concavity and inflection points

$$f''(x)=-\frac{\frac{1}{x}\sqrt{x}-\frac{1}{2\sqrt{x}}(1+\ln x)}{x}=\frac{\frac{1}{2\sqrt{x}}(1+\ln x)-\frac{2}{2\sqrt{x}}}{x}=\frac{\ln x-1}{2x\sqrt{x}}$$



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$$D_{f''} = D_f \cap [0, \infty)$$

$$f''(x) = 0$$
$$\frac{\ln x - 1}{2x\sqrt{x}} = 0 \Leftrightarrow \ln x - 1 = 0 \Leftrightarrow x = e$$

$$f(e) = 0$$

$$S_2 = \{e\}$$

The edges of the interval $(0, \infty)$ are:
$$0, \infty$$
so, the intervals of concavity (on the interval $(0, \infty)$) are:
$$(0, e), (e, \infty).$$

$f''(1) < 0 \Rightarrow f$ is concave downward on $(0, e)$;
$f''(e^2) > 0 \Rightarrow f$ is concave upward on $(e, \infty)$.
$$f''(x) < 0 \text{ for all } x \in (0, e) \ \ (\text{because } f''(1) < 0)$$
and
$$f''(x) > 0 \text{ for all } x \in (e, \infty) \ \ (\text{because } f''(e^2) > 0)$$
so $I(e, 0)$ is the inflection point of the function $f$.

## 6a. Graph of the given function on the interval $\langle \mathbf{0}, \infty \rangle$

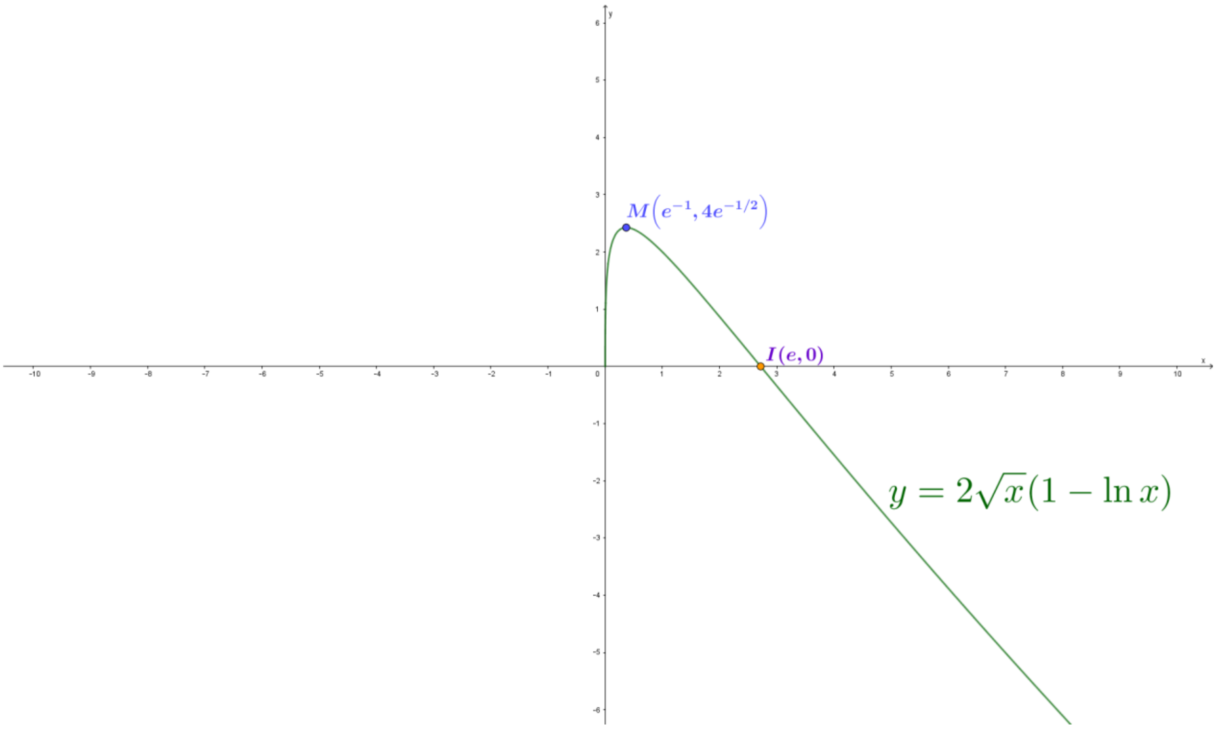


*Figure 6.14*


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## 6b. Graph of the given function

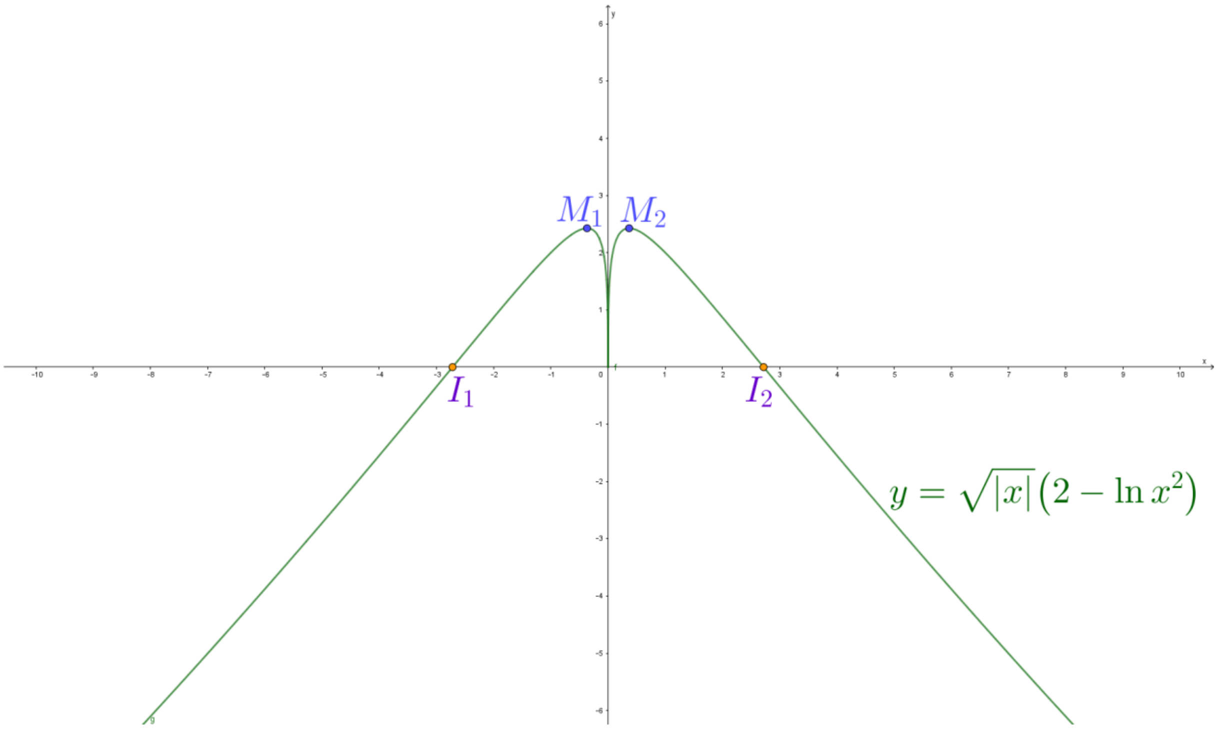


*Figure 6.15*

### Example 6

Example of a function where the point of the local extremum is a critical point that is not stationary:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$D_f = \mathbb{R} = (-\infty, \infty)$$

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The right derivation $f_r'$ and the left derivation $f_l'$ are not equal for $x = 0$.

$$\left.\begin{array}{l} f_r'(0) = \lim\limits_{t \to 0^{+0}} \frac{f(0+t)-f(0)}{t-0} = \lim\limits_{t \to 0^{+0}} \frac{t}{t} = 1 \\ f_l'(0) = \lim\limits_{t \to 0^{-0}} \frac{f(0+t)-f(0)}{t-0} = \lim\limits_{t \to 0^{-0}} \frac{-t}{t} = -1 \end{array}\right\} \Rightarrow f'(0) = \lim\limits_{t \to 0} \frac{f(0+t)-f(0)}{t-0} \text{ does not exist.}$$

$$f(0) = 0$$


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Therefore, $(0,0)$ is a critical point that is not stationary.

$$S_1 = \{0\}$$

The edges of the domain of the function $f$ are:

$$-\infty, \infty$$

so, the intervals of monotonicity are:

$$(-\infty, 0), (0, \infty).$$

$$f'(x) = -1 < 0 \text{ for all } x < 0$$

and

$$f'(x) = 1 > 0 \text{ for all } x > 0$$

so $m(0,0)$ is the point of the local minimum of the function $f$.

Graph of the function $y = |x|$:

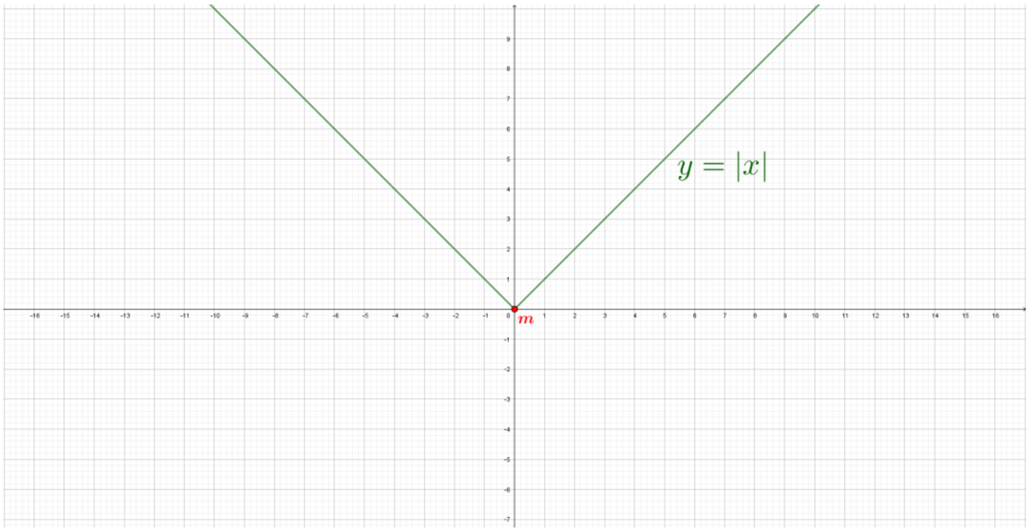


*Figure 6.16*



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