6.9. Exercises

Task 1. Prove that:

(1.) $\left(\frac{a+bx}{c+dx}\right)' = \frac{bc-ad}{(c+dx)^2}$;

(2.) $\left(\frac{2x+3}{x^2-5x+5}\right)' = \frac{-2x^2-6x+25}{(x^2-5x+5)^2}$;

(3.) $\left(\frac{2}{2x-1}-\frac{1}{x}\right)' = \frac{1-4x}{x^2(2x-1)^2}$;

(4.) $[(x-1)\cdot e^x]' = xe^x$;

(5.) $\left(\frac{e^x}{x^2}\right)' = \frac{e^x(x-2)}{x^3}$;

(6.) $\left(\frac{1}{t^2+t+1}\right)' = -\frac{2t+1}{(t^2+t+1)^2}$.

Task 2. Find the derivatives:

(1.) $y=(1+3x+5x^2)^4$;

(2.) $y=(3-\sin x)^3$;

(3.) $y=\sqrt[3]{\sin^2 x}+\frac{1}{\cos^2 x}$;

(4.) $y=\sqrt[3]{2e^x+2^x+1}+\ln^5 x$;

(5.) $y=\sin 3x+\cos\frac{x}{5}+\tan\sqrt{x}$;

(6.) $y=\sin(x^2-5x+1)+\tan\frac{a}{x}$;

(7.) $y=\arctan(\ln x)+\ln(\arctan x)$;

(8.) $y=\ln^2\arctan\left(\frac{x}{3}\right)$;

(9.) $y=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$;

(10.) $y=\ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}+2\arctan\sqrt{\sin x}$;

(11.) $y=\frac{3}{4}\ln\frac{x^2+1}{x^2-1}+\frac{1}{4}\ln\frac{x-1}{x+1}+\frac{1}{2}\arctan x$;

(12.) $y=\frac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$;

(13.) $y=\frac{\sin t}{\cos^2 t}+\ln\frac{1+\sin t}{\cos t}$;

(14.) $y=e^x\arctan e^x-\ln\sqrt{1+e^{2x}}$.

(15.) $y=\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}$

(16.) $y=\arctan e^x-\ln\sqrt{\frac{e^{2x}}{e^{2x}+1}}$

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Answers:

(1.) $y' = 4(1+3x+5x^2)^3(3+10x)$; (2.) $y' = -3(3-\sin x)^2 \cos x$;

(3.) $y' = \frac{2\cos x}{3\cdot\sqrt[3]{\sin x}} + \frac{2\sin x}{\cos^3 x}$; (4.) $y' = \frac{2e^x + 2^x \ln 2}{3\cdot\sqrt[3]{(2e^x+2^x+1)^2}} + \frac{5\ln^4 x}{x}$;

(5.) $y' = 3\cos 3x - \frac{1}{5}\sin\frac{x}{5} + \frac{1}{2\sqrt{x}}\cdot\frac{1}{\cos^2\sqrt{x}}$; (6.) $y' = (2x-5)\cos(x^2-5x+1) - \frac{a}{x^2}\cdot\frac{1}{\cos^2\frac{a}{x}}$

;

(7.) $y' = \frac{1}{1+\ln^2 x}\cdot\frac{1}{x} + \frac{1}{\arctan x}\cdot\frac{1}{1+x^2}$; (8.) $y' = 2\ln\arctan\left(\frac{x}{3}\right)\cdot\frac{1}{\arctan\frac{x}{3}}\cdot\frac{3}{9+x^2}$;

(9.) $y' = \frac{\sqrt{x^2+1}}{x}$; (10.) $y' = \frac{2}{\cos x\sqrt{\sin x}}$;

(11.) $y' = \frac{x^2-3x}{x^4-1}$; (12.) $y' = \frac{\arcsin x}{(1-x^2)^{\frac{3}{2}}}$;

(13.) $y' = \frac{2}{\cos^3 t}$; (14.) $y' = e^x \arctan e^x$;

(15.) $y' = -\cos 2x$; (16.) $y' = \frac{e^x - 1}{e^{2x}+1}$.

Task 3. Prove that the function $y = xe^{-\frac{x^2}{2}}$ satisfies the equation $x\cdot y' = (1-x^2)y$.

Task 4. Prove that the function $y = \frac{1+\ln x}{x - x\ln x}$ satisfies the equation $2x^2\cdot y' - x^2y^2 - 1 = 0$.

Task 5. Find the derivatives:

(1.) $f(x) = 10^{x\tan x}$; (2.) $g(x) = \sqrt[x]{(1+x)^2}$;

(3.) $y = x^{\sin x}$; (4.) $y = \frac{(x-2)^9}{\sqrt{(x-1)^5\cdot(x-3)^{11}}}$;

(5.) $y = x^2\sqrt{\frac{2x-1}{x+1}}$.

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Answers:

(1.) $f'(x)=10^{x\tan x}\ln 10\left(\tan x+\frac{x}{\cos^2 x}\right)$;

(2.) $g'(x)=2\cdot\sqrt[x]{(1+x)^2}\left[\frac{1}{x(x+1)}-\frac{\ln(x+1)}{x^2}\right]$; $\forall x\in R\setminus\{0,-1\}$;

(3.) $y'=x^{\sin x}\left(\cos x\cdot\ln x+\frac{\sin x}{x}\right)$;

(4.) $y'=\frac{(x-2)^8\cdot(x^2-7x+1)}{\sqrt{(x-1)^7\cdot(x-3)^{13}}}$; $\forall x\in R\setminus\{1,3\}$;

(5.) $y'=\frac{x(8x^2+7x-4)}{2(x+1)(2x-1)}\cdot\sqrt{\frac{2x-1}{x+1}}$; $\forall x\in R\setminus\left\{\frac{1}{2},-1\right\}$.

Task 6. Find the derivative $y=f(x)$ of the implicitly given function:

(1.) $e^y=x+y$; (2.) $\ln y+\frac{x}{y}=C$; (3.) $\arctan\frac{y}{x}=\frac{1}{2}\ln\left(x^2+y^2\right)$;

(4.) $\sqrt{x^2+y^2}=C\cdot\arctan\frac{y}{x}$; (5.) $e^{\frac{y}{x}}=\arctan\sqrt{x^2+y^2}$.

Answers:

(1.) $y'=\frac{1}{x+y-1}$; (2.) $y'=\frac{y}{x-y}$; (3.) $y'=\frac{x+y}{x-y}$;

(4.) $y'=\frac{Cy+x\sqrt{x^2+y^2}}{Cx-y\sqrt{x^2+y^2}}$; (5.) $y'=\frac{x^3+y\cdot e^{\frac{y}{x}}(1+x^2+y^2)\sqrt{x^2+y^2}}{x\cdot e^{\frac{y}{x}}(1+x^2+y^2)\sqrt{x^2+y^2}-x^2y}$.

Task 7. Calculate the derivation y' of the given function at the stated points:

(1.) $(x+y)^3=27(x-y)$ at $T(2,1)$; (2.) $ye^y=e^{x+1}$ at $T(0,1)$;

(3.) $y^2=x+\ln\frac{y}{x}$ at $T(1,1)$.

Answers:

(1.) $y'\big|_{(2,1)}=\frac{27-3(x+y)^2}{3(x+y)^2+27}\Big|_{(2,1)}=0$; (2.) $y'\big|_{(0,1)}=\frac{e^{x+1}}{ye^y}\Big|_{(0,1)}=\frac{1}{2}$;

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(3.) $y'\big|_{(1,1)} = \dfrac{y(x-1)}{x(2y^2-1)}\bigg|_{(1,1)} = 0$.

Task 8. Find $y'(x);\, a,b,c \in R$ if:

(1.) $\begin{cases} x(t) = \dfrac{a\sin t}{1+b\cos t}, \\ y(t) = \dfrac{c\cdot\cos t}{1+b\cos t}. \end{cases}$

(2.) $\begin{cases} x(t) = \dfrac{3at}{1+t^3}, \\ y(t) = \dfrac{3at^2}{1+t^3}. \end{cases}$

(3.) $\begin{cases} x(t) = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}, \\ y(t) = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}, \end{cases}$

Answers:

(1.) $y'(x) = -\dfrac{c\sin t}{a(b+\cos t)}$;

(2.) $y'(x) = \dfrac{t(2-t^3)}{1-2t^3}$;

(3.) $y'(x) = -\tan 3t$.

Task 9. Find $y''(x)$

(1°) $\begin{cases} x(t) = \arctan t, \\ y(t) = \dfrac{1}{2}t^2. \end{cases}$

(2°) $\begin{cases} x(t) = \ln t, \\ y(t) = \dfrac{1}{1-t}. \end{cases}$

(3°) $\begin{cases} x(t) = \arcsin t, \\ y(t) = \sqrt{1-t^2}. \end{cases}$

(4°) $\begin{cases} x(t) = \ln\left(t+\sqrt{1+t^2}\right), \\ y(t) = \sqrt{1+t^2}. \end{cases}$

Answers:

(1.) $y''(x) = (1+t^2)(1+3t^2)$;

(2.) $y''(x) = \dfrac{(1+t)\cdot t}{(1-t)^3}$;

(3.) $y''(x) = -\sqrt{1-t^2}$;

(4.) $y''(x) = \sqrt{1+t^2}$.

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Task 10. Find the equation of the slope line tangent to the graph of the function $y = x^2 - 4x + 3$ at the left zero-point.

Answer:

Zero-points are $x_1 = 1$ and $x_2 = 3$; left $T_0(1,0)$; $t:\ y + 2x - 2 = 0$.

Task 11. Find the equation of the slope line tangent and the slope line normal to the graph of the functions:

(1.) $f(x) = ln(cos\, x)$ at the point $x_0 = 2\pi$.

(2.) $f(x) = \dfrac{8}{x^2 + 4}$ at the point $T(2, f(2))$.

Answers:

(1.) $x_0 = 2\pi$; $y_0 = 0$; $f'(x_0) = 0$;
$t:\ y = 0$; $n: x - 2\pi = 0$;

(2.) $x_0 = 2$; $y_0 = 1$; $k_t = f'(2) = -\dfrac{1}{2}$; $t: x + 2y - 4 = 0$;
$n: 2x - y - 3 = 0$.

Task 12. From the point $T(4,1)$ find the slope line tangent to the graph of the function $f(x) = \dfrac{x-1}{x}$.

Answer:

$D\left(2, \dfrac{1}{2}\right)$; $t: 4y - 4 = 0$.

Task 13. At which point of the parabola $y = x^2 + 2x + 1$ the tangent line makes identical angles on both sides of the coordinate axis?

Answer:

$T_1\left(-\dfrac{1}{2}, \dfrac{1}{4}\right)$; $T_2\left(-\dfrac{3}{2}, \dfrac{1}{4}\right)$.

Task 14. Find the equation of the slope line tangent and the slope line normal to the graph of the functions on the parametrically given curve:

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(1.) $\begin{cases} x(t) = \dfrac{1+t}{t^3}, \\ y(t) = \dfrac{3}{2t^2} + \dfrac{1}{2t}; \end{cases}$ at the point $T_0(2,2)$; (2.) $\begin{cases} x(t) = \dfrac{2t}{t+2}, \\ y(t) = \dfrac{t}{t-1}; \end{cases}$ for $t_0 = 2$;

(3.) $\begin{cases} x(t) = \sin t, \\ y(t) = a^t; \end{cases}$ at the point for which $t_0 = 0$.

Answers:

(1.) $t:\ 7x - 10y + 6 = 0,$ $n: 10x + 7y - 34 = 0;$ (2.) $t: 4x + y - 6 = 0,$ $n: x - 4y + 7 = 0;$

(3.) $t: y - x\ln a - 1 = 0,$ $n: y + \dfrac{1}{\ln a}x - 1 = 0.$

Task 15. Find the angle at which the parabolas intersect:

(1.) $y = 4 - \dfrac{x^2}{2}$ and $y = \dfrac{x^2}{2}$; (2.) $y = x^2$ and $y^2 = x$.

Answer:

(1.) $\varphi = 126°52'$; (2.) $\varphi_1 = 36°50'$ and $\varphi_2 = 90°$.

Task 16. Find the equation of the slope line tangent and the slope line normal on the curves that are given by an implicit equation:

(1.) $3x^5 - y - 2x + 1 = 0$, at the point $T(1,2)$;

(2.) $8x^2 - 9y^2 - 72 = 0$, at the point $T(-9,-8)$;

(3.) $x \cdot e^{-\frac{y}{2}} - y \cdot e^{-\frac{x}{2}} = 0$, at the point with the abscissa $x_0 = 0$.

Answers:

(1.) $t:\ 13x - y - 11 = 0,$ $n:\ x + 13y - 27 = 0;$ (2.) $t:\ x - y + 1 = 0,$ $n:\ x + y - 1 = 0;$

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(3.) $T_0(0,-2),\ k_t = e-1;$

$t: y=(e-1)x-2;\quad n: (e-1)y+x+2(e-1)=0.$

Task 17. Finding limits algebraically - calculus:

(1.) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$ (2.) $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\dfrac{1}{x}\right)\right]$ (3.) $\lim\limits_{x\to\infty}\dfrac{\pi-2\arctan x}{e^{\frac{2}{x}}-1}$

(4.) $\lim\limits_{x\to\infty}\left[\ln(x+1)-\ln x\right]$ (5.) $\lim\limits_{x\to 0}x^{\frac{3}{4+\ln x}}$ (6.) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$

(7.) $\lim\limits_{x\to 0}\dfrac{\sin\pi x}{\ln x}$ (8.) $\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}-1}{\frac{1}{x}}$ (9.) $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^2}$

Answers:

(1.) $\frac{2}{\pi}$ (2.) $\frac{1}{2}$ (3.) 1 (4.) 0 (5.) e^3

(6.) 0 (7.) $-\pi$ (8.) 1 (9.) 0

Task 18. Consider the function $f(x)=x^3-\frac{3}{2}x^2-18x$. The points x = 3 and x =-2 satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Answer:

f has a local maximum at -2 and a local minimum at 3 .

For tasks 19 - 23 determine:

a) Find the domain;
b) Find the x-intercept/zero and the y-intercept;
c) Find the asymptotes;
d) Find the derivative and the relative extrema;
e) Find the inflection points;
f) Sketch the curve.

Task 19.

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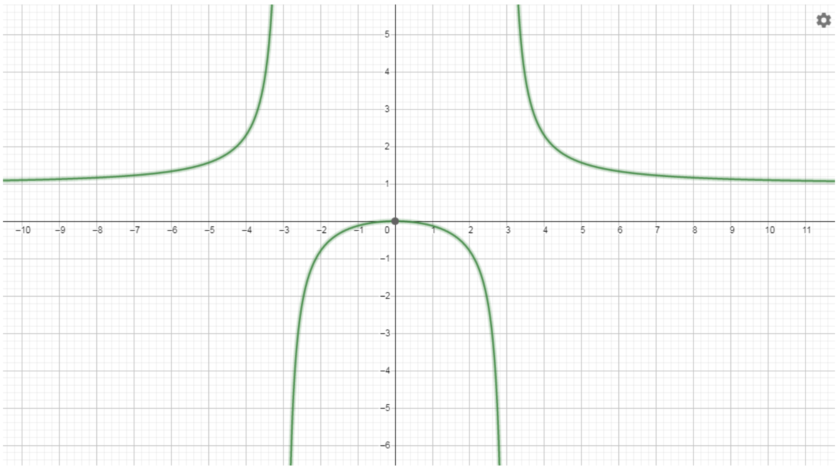
$$f(x) = -\frac{x^2}{9-x^2}$$

Answer:

a) $D_f = \mathbb{R} \setminus \{3, -3\}$;
b) $x = 0$, $y = f(0) = 0$;
c) Vertical asymptotes: $x = -3$, $x = 3$;
Horizontal asymptotes: $y = 1$; no slant asymptote;
d) $f'(x) = \frac{-18x}{(9-x^2)^2}$, the relative maximum is 0 at $x = 0$;
e) $f''(x) = -\frac{54x^2+162}{(9-x^2)^3}$, no inflection points;
f)



Task 20.

$$f(x) = \frac{4x-12}{(x-2)^2}$$

Answer:

a) $D_f = \mathbb{R} \setminus \{2\}$;

b) $x = 3$, $y = f(0) = -3$;

c) Vertical asymptotes: $x = 2$;

Horizontal asymptotes: $y = 0$; no slant asymptote;

d) $f'(x) = \frac{-4(x-4)}{(x-2)^3}$, the relative maximum is 1 at $x = 4$;

e) $f''(x) = \frac{8x-40}{(x-2)^4}$, the inflection points is $\left(5, \frac{8}{9}\right)$;

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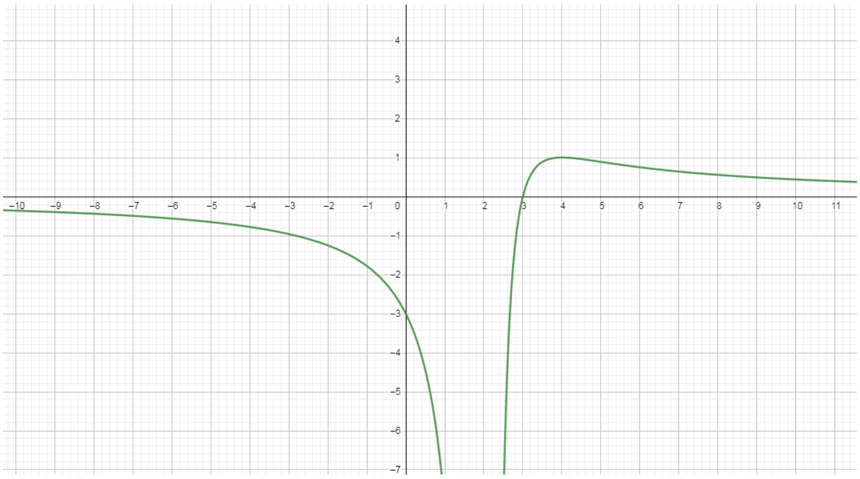
f) 

Task 21.

$$f(x)=\frac{x^2-4x+3}{x^2-2x}$$

Answer:

a) $D_f = \mathbb{R} \setminus \{0, 2\}$;

b) $x_1 = 1$, $x_2 = 3$; no y -intercept;

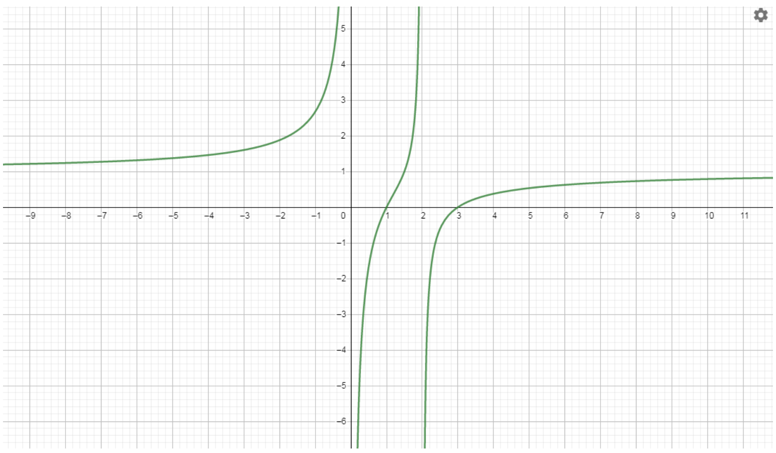
c) Vertical asymptotes: $x = 0$, $x = 2$,

Horizontal asymptotes: $y = 1$; no slant asymptote;

d) $f'(x) = \frac{2x^2-6x+6}{(x^2-2x)^2}$, no relative extrema;

e) $f''(x) = \frac{-4x^5+26x^4-72x^3+96x^2-48x}{(x^2-2x)^4}$;

f)



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Task 22.

$$f(x) = \frac{\ln(2x)}{x^2}$$

Answer:

a) $D_f = \langle 0, \infty \rangle$;

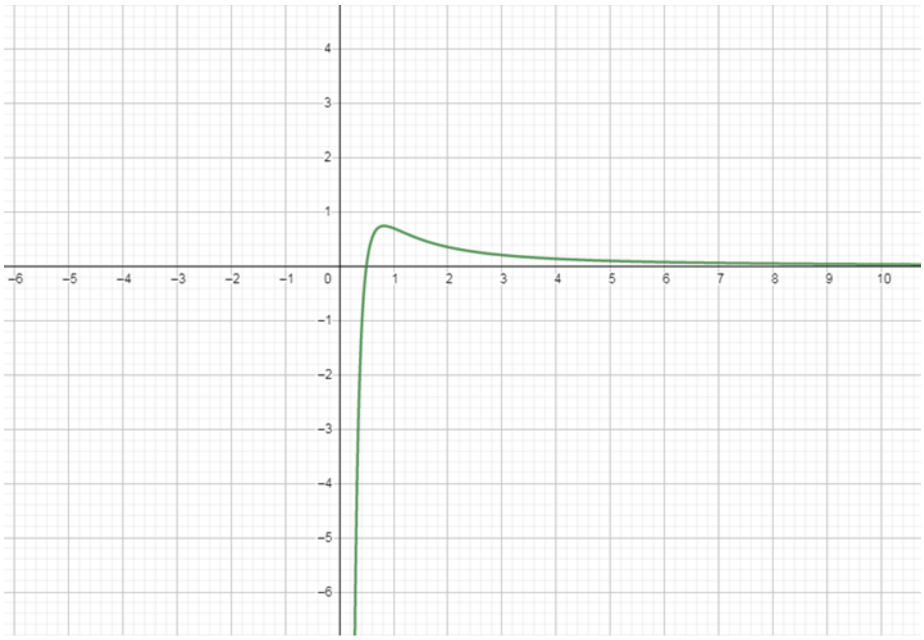
b) $x_1 = 0.5$; no y -intercept;

c) Vertical asymptot is $x = 0$;

d) $f'(x) = \frac{1-2\ln(2)-2ln(x)}{x^3}$, the relative maximum is $\frac{2}{e}$ at $x = \frac{\sqrt{e}}{2}$;

e) $f''(x) = \frac{6\ln(2)+6ln(x)-5}{x^4}$, the inflection points is $\left(\frac{\sqrt[6]{e^5}}{2}, \frac{10\sqrt[3]{e}}{3e^2}\right)$;

f)



Task 23.

$$f(x) = \frac{x}{1+x^2} e^x$$

Answer:

a) $D_f = \mathbb{R}$,

b) $x_1 = 0$, $y = f(0) = 0$;

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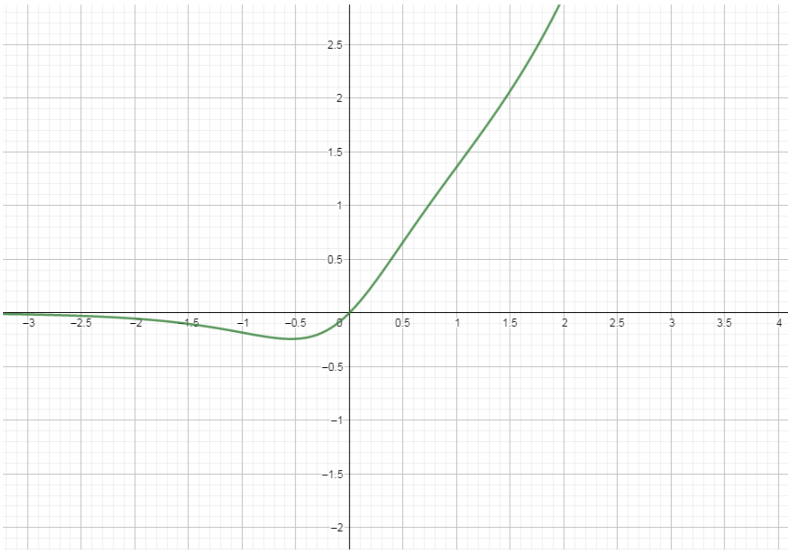
c) no asymptotes;

d) $f'(x) = \frac{e^x - x^2 e^x + x e^x + x^3 e^x}{(1+x)^2}$, no relative extrema;

f)



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