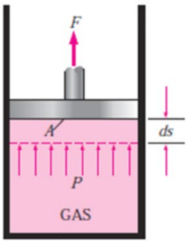
6 CALCULUS

Math in Thermodynamics

Partial derivation of ideal gas state equation



$$p \cdot V = n \cdot \mathfrak{R} \cdot T$$

$$f(p, V, t) = 0$$

$$p \cdot dV + v \cdot dp = dn \cdot \mathfrak{R} \cdot T + n \cdot d\mathfrak{R} \cdot T + n \cdot \mathfrak{R} \cdot dT$$

$$p \cdot dV + V \cdot dp = dn \cdot \mathfrak{R} \cdot T + n \cdot \mathfrak{R} \cdot dT$$

DETAILED DESCRIPTION:

Most first-year students find it hard to understand and acquire mathematical notions of differential calculus. This is often due to insufficient prior knowledge or because these notions are really difficult and require mathematical and logical maturity. Given the difficulties, this unit explains the matter gradually, starting with the targeted theoretical notions, which is followed by exercises and solved problems, with the aim of teaching the students how to solve tasks independently and how to apply the acquired knowledge in solving problem tasks in the area of maritime affairs.

Basic notions associated with the derivation of function are explained, along with the rules and techniques of derivatives. A particular attention is paid to the application of derivation in the problems of the tangent, the normal, the differential and the establishing the function limits. The application of derivations in the flow examination and function graph drawing are explained and followed by the application of derivations in maritime affairs.

AIM: Acquire knowledge and skills in those areas of differential calculus which are necessary to follow the curricula of other courses of the study programme, and are expected to be implemented in maritime practice.

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Learning Outcomes:

1. Define the notions of derivative, function limit and differential.
2. Apply simple and complex derivation rules when solving tasks.
3. Perform the derivation of the complex, parametrically or implicitly given function.
4. Explain the concept of the real variable of real functions and the geometric interpretation of the derivative at a point.
5. Apply the derivative in finding the local and global extremes of the function of a given variable, and the points of the function inflection.
6. Analyse the flow of an elementary function by using derivation, and sketch its graph.

Prior Knowledge: sets and functions, sequences and series, limits and continuity of the function

Relationship to real maritime problems: mechanics (problem of speed), meteorology (weather forecast – extreme sea states), electronics (graphic layouts), navigation (establishing the distance, navigability of the fairway)...

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CONTENT

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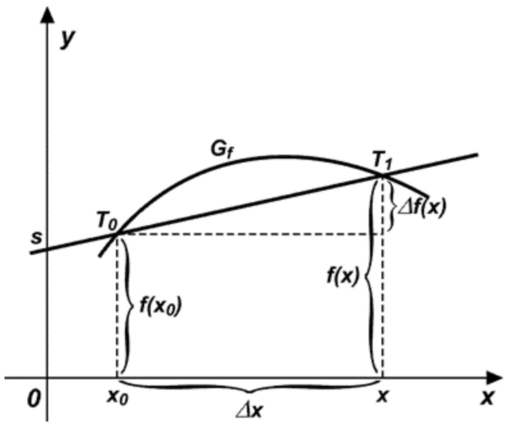
6.1. Derivative

The derivative is one of the essential notions in mathematics. It is necessary for making the so-called differential calculus. The notion was first introduced by the English mathematician, physicist and astronomer Isaac Newton (1642-1727). He described it before 1669, when solving the problem of motion of a body that is moving unevenly along the line, at any moment of its movement. The same discovery was attributed to Gottfried W. Leibniz (1646-1716), German mathematician and philosopher, who independently developed his foundations, while solving the problem of establishing the coefficient of the direction of the curve tangent.

Knowing that vessels move across water areas, we can only assume the actual importance of the application of derivatives in maritime affairs.

The notion of derivation becomes clear with the help of examples.

If $f : I \to R$ (or $y = f(x)$) is the given function on the interval $I \subseteq R$ and if $x_0 \in I$ is a point of the interval (see the figure).



If $x \neq x_0$, $x \in I$, we observe two values of the function: $f(x)$ i $f(x_0)$. The expression $\Delta x = x - x_0$ is called the ***growth (or differentiation) of the argument** x*, while $\Delta f(x_0) = f(x) - f(x_0)$ is called the ***change (or growth) of the function at the point** x_0*.

Let us now define the difference quotient:

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0. \tag{1}$$

The function $g(x)$ gives information on the rate of change of the function *f* from x_0 to *x*, that is, $g(x)$ measures the average change of the function from x_0 to *x*. The smaller the growth Δx is, the more accurate is the information on the function change of rate that $g(x)$ gives at the point x_0.

Definition 1: *It is said that the function $f: I \to R$ is **differentiable** (synonym: derivable) at the point $x_0 \in I = \langle a,b \rangle$ if there is a boundary value:*

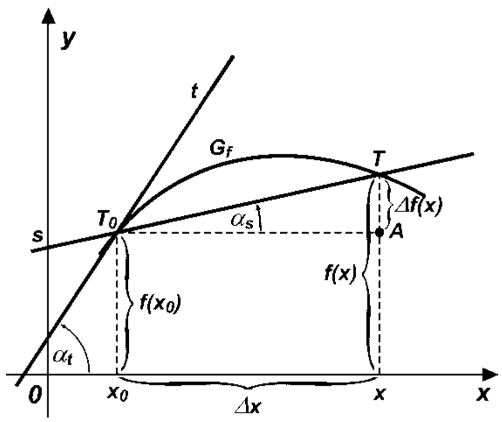
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in R. \tag{2}$$

The number $f'(x_0)$ is called the ***derivative f at the point** x_0.* It is said that *f* is differentiable at *I* if it is differentiable at any point $x \in I$.

If $f: I \to R$ is differentiable at *I*, then $x \to f'(x)$ defines a new function $f': I \to R$, the so-called derivative f at *I*.

From the philosophical standpoint, derivation is the ratio of yield to investment. In programming, derivation is the ratio of output to input. In physical world, derivation is the ratio of arbitrarily small travel through arbitrarily little time, i.e. speed.

In geometry, this can be explained by the following figure:



In the graph G_f of the function $f: I \to R$ there are points $T_0(x_0, f(x_0))$ i $T(x, f(x))$ to which is assigned the line s, called the ***secant graph of the function** G_f* on the interval [x_0, x]. The secant coefficient is achieved through formula:

$$\frac{f(x) - f(x_0)}{x - x_0} = k_s \tag{3}$$

If the point T "moves" across the graph G_f toward the point T_0, then the secant *s* turns around the point T_0. If, in this process, there is a limit line *t* with a position towards which the secant *s* streams, regardless of whether the point T streams toward T_0, right or left of the T_0, then *t* is the tangent of the graph G_f at the point T_0. If the function *f* has a derivation at the point $x_0 \in I$, then, according to the relation (3), it appears that

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}k_s=k_t\,, \tag{4}$$

That is, $f'(x_0)=k_t$. (5)

Therefore, the **numerical value** $f'(x_0)$ **of the derivative** f **at the point** x_0 **represents the coefficient of the direction** k_t **of the tangent** *t*, **drawn in the graph** G_f **at the point** $T_0(x_0,f(x_0))$.

Below are the examples of calculating the derivation by definition for some known elementary functions. They are followed by a table of their derivations, which can be proved in analogy with these examples.

Example 1

Let $f(x)=7x;\, x\in R$. Define $f'(x_0)$, where $x_0\in R$.

Solution:

$$f'(x_0)\;=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{5x-5x_0}{x-x_0}\lim_{x\to x_0}\frac{5(x-x_0)}{x-x_0}\;=\lim_{x\to x_0}5=5\,.$$

As we can see, $f'(x_0)$ does not depend on the point x_0, which is obvious as $f(x)=5x$ is a linear function whose average change rate is everywhere the same.

Example 2

Let $f(x)=x^2;\; x\in R$. Define $f'(x_0)$, where $x_0\in R$.

Solution:

$$f'(x_0)\;=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\;=\lim_{x\to x_0}\frac{x^2-x_0^{\,2}}{x-x_0}\;=\lim_{x\to x_0}\frac{(x-x_0)(x+x_0)}{x-x_0}=\lim_{x\to x_0}(x+x_0)=2x_0.$$

In this way, we have shown that $\left(x^2\right)'=2x$ ¸ $x\in R$.

Example 3

Let $f(x)=x^3;\; x\in R$. Define $f'(x_0)$, where $x_0\in R$.

Solution:

$$f'(x_0)\;=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\;=\lim_{x\to x_0}\frac{x^3-x_0^{\,3}}{x-x_0}\;=\lim_{x\to x_0}\frac{(x-x_0)\left(x^2+xx_0+x_0^{\,2}\right)}{x-x_0}=$$

$$=\lim_{x\to x_0}\left(x^2+xx_0+x_0^{\,2}\right)=3x_0^{\,2}.$$

In this way, we have shown that $\left(x^3\right)' = 3x^2$, $x \in R$.

Example 4

Let $f(x) = \sqrt{x}$; $x \in R$. Define $f'(x_0)$, where $x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \cdot \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}} =$$

$$= \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)\sqrt{x} + \sqrt{x_0}} = \lim_{x \to x_0} \frac{1}{2\sqrt{x_0}}.$$

In this way, we have shown that $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$, $x \in R$.

6.1 Table and rules of deriving elementary functions

If we continue to solve the examples similar to the ones presented in the previous unit, we can prove the validity of the following ***table of deriving*** elementary functions, which we can "take for granted":

1)	$C' = 0$	$(C \in R)$
2)	$x' = 1$	
3)	$\left(x^{\alpha}\right)' = \alpha \cdot x^{\alpha - 1}$	$\alpha \in R$
4)	$\left(e^x\right)' = e^x$	
5)	$\left(a^x\right)' = a^x \ln a$	$a > 0$
6)	$(ln\, x)' = \frac{1}{x}$	$x > 0$
7)	$(\log_a x)' = \frac{1}{x \ln a}$	$x > 0$, $a > 0$

8)	$(\sin x)' = \cos x$	
9)	$(\cos x)' = -\sin x$	
10)	$(tgx)' = \dfrac{1}{\cos^2 x}$	
11)	$(ctgx)' = -\dfrac{1}{\sin^2 x}$	
12)	$(Arc\sin x)' = \dfrac{1}{\sqrt{1-x^2}}$	$\lvert x\rvert < 1$
13)	$(Arc\cos x)' = -\dfrac{1}{\sqrt{1-x^2}}$.	$\lvert x\rvert < 1$
14)	$(Arctg\, x)' = \dfrac{1}{1+x^2}$	
15)	$(Arcctg\, x)' = -\dfrac{1}{1+x^2}$	
16)	$(shx)' = chx$	
17)	$(chx)' = shx$	
18)	$(thx)' = \dfrac{1}{ch^2 x}$	
19)	$(cthx)' = -\dfrac{1}{sh^2 x}$	
20)	$(Arshx)' = \dfrac{1}{\sqrt{1+x^2}}$	

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21)	$(Archx)' = \frac{1}{\sqrt{x^2-1}}$	$\lvert x\rvert > 1$
22)	$(Arthx)' = \frac{1}{1-x^2}$	$\lvert x\rvert < 1$
23)	$(Arcthx)' = -\frac{1}{x^2-1}$	$\lvert x\rvert > 1$

Derivation of any other function should be calculated by applying the ***rules of deriving the functions***, as follows:

$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$	derivation of the sum and the difference of functions
$[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$	derivation of the product of functions
$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$	derivation of the function quotient $(g(x) \neq 0)$
$\{g[f(x)]\}' = g'[f(x)]\cdot f'(x)$	derivation of the functions' composition

Example 1

Define the derivative $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution:

$$f'(x) = (6x^3 - 4x^2 + 3x - 2)' = 6\cdot(x^3)' - 4\cdot(x^2)' + 3\cdot x' - 2' =$$
$$= 6\cdot 3x^2 - 4\cdot 2x + 3\cdot 1 - 0 = 18x^2 - 8x + 3.$$

Example 2

Define $h'(x)$ if $h(x) = e^x \cdot \sin x$.

Solution:

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x) = (e^x)'\cdot \sin x + e^x\cdot(\sin x)' =$$
$$= e^x\cdot \sin x + e^x \cdot \cos x = e^x(\sin x + \cos x).$$

Example 3

Let $h(x)=\dfrac{x^2+3x+1}{x^2-3x+2}$. Define $h'(x)$.

Solution:

$$h'(x)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)}=\frac{\left(x^2+3x+1\right)'\left(x^2-3x+2\right)-\left(x^2+3x+1\right)\left(x^2-3x+2\right)'}{\left(x^2-3x+2\right)^2}=$$

$$=\frac{(2x+3)\left(x^2-3x+2\right)-\left(x^2+3x+1\right)(2x-3)}{\left(x^2-3x+2\right)^2}=\frac{-6x^2+2x+9}{\left(x^2-3x+2\right)^2}.$$

Example 4

Define the derivatives:

(1.) $h(x)=e^{3x^2-2x+1}$; (2.) $h(x)=\left(2x^2-3x\right)^3$;

(3.) $h(x)=\sin\left(3x^3-4\right)$;

Solution:

(1.) $h'(x)=\left(e^{3x^2-2x+1}\right)'=e^{3x^2-2x+1}\cdot\left(3x^2-2x+1\right)'=e^{3x^2-2x+1}\cdot(6x-2)$.

(2.) $h'(x)=\left(\left(2x^2-3x\right)^3\right)'=3\left(2x^2-3x\right)^2\left(2x^2-3x\right)'=3\left(2x^2-3x\right)^2\cdot(4x-3)$.

$$h'(x)=\left(\sin\left(3x^3-4\right)\right)'=\cos\left(3x^3-4\right)\cdot\left(3x^3-4\right)'=\cos\left(3x^3-4\right)\cdot\left(9x^2\right)=9x^2\cos\left(3x^3-4\right)$$

Exercises

Exercices 6.1

Find the derivative:

(1.) $f(x)=3\cdot\sqrt[6]{x^5}$; (2.) $g(x)=\dfrac{2x^2-3}{\sqrt[3]{x^2}}$.

Solution:

(1.) $f(x)=3\cdot\sqrt[6]{x^5}=3x^{\frac{5}{6}}$.

$$f'(x)=3\cdot\frac{5}{6}x^{\frac{5}{6}-1}=\frac{5}{2}x^{-\frac{1}{6}}=\frac{5}{2x^{\frac{1}{6}}}=\frac{5}{2\cdot\sqrt[6]{x}}.$$

(2.) $g(x)=\dfrac{2x^2-3}{\sqrt[3]{x^2}}=\dfrac{2x^2-3}{x^{\frac{2}{3}}}=2x^{\frac{4}{3}}-3x^{-\frac{2}{3}}$.

$$g'(x)=2\cdot\frac{4}{3}x^{\frac{1}{3}}-3\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}=\frac{8}{3}x^{\frac{1}{3}}+\frac{2}{x^{\frac{5}{3}}}=\frac{8x^2+6}{3\cdot\sqrt[3]{x^5}}.$$

Exercices 6.2

Find the derivative:

(1.) $y=ln\left(\sqrt{1+e^x}-1\right)-ln\left(\sqrt{1+e^x}+1\right)$; (2.) $y=\dfrac{1}{15}\cos^3 x\,(3\cos^2 x-5)$.

Solution:

(1.) $$y'=\frac{1}{\sqrt{1+e^x}-1}\cdot\frac{1\cdot e^x}{2\sqrt{1+e^x}}-\frac{1}{\sqrt{1+e^x}+1}\cdot\frac{1\cdot e^x}{2\sqrt{1+e^x}}=$$
$$=\frac{e^x}{2\sqrt{1+e^x}}\cdot\frac{\sqrt{1+e^x}+1-\left(\sqrt{1+e^x}-1\right)}{\left(\sqrt{1+e^x}-1\right)\left(\sqrt{1+e^x}+1\right)}=\frac{e^x}{2\sqrt{1+e^x}}\cdot\frac{2}{1+e^x-1}=\frac{1}{\sqrt{1+e^x}}.$$

(2.) $$y'=\frac{1}{15}\left[3cos^2 x\cdot(-sin\,x)\cdot\left(3cos^2 x-5\right)+cos^3 x\cdot 6cos\,x\cdot(-sin\,x)\right]=$$
$$=-\frac{3cos^2 x\cdot sin\,x}{15}\left(3cos^2 x-5+2cos^2 x\right)=-\frac{cos^2 x\cdot sin\,x}{5}\left(5cos^2 x-5\right)=$$
$$=\frac{5}{5}cos^2 x\cdot sin\,x\left(1-cos^2 x\right)=cos^2 x\cdot sin^3 x.$$

Exercices 6.3

Define the derivative:

(1.) $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$;

(2.) $f(x)=\dfrac{x^5}{e^x}$;

(3.) $f(\alpha)=\dfrac{2\alpha^2-\alpha+3}{2\alpha}$;

(4.) $f(t)=2t\cdot\sin t-(t^2-2)\cos t$.

Solution:

(1.)
$$f'(x)=\frac{(\sin x+\cos x)'(\sin x-\cos x)-(\sin x+\cos x)(\sin x-\cos x)'}{(\sin x-\cos x)^2}=$$
$$=\frac{(\cos x-\sin x)(\sin x-\cos x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}=$$
$$=\frac{\cos x\sin x-\sin^2 x-\cos^2 x+\sin x\cos x-(\sin x\cos x+\cos^2 x+\sin^2 x+\cos x\sin x)}{(\sin x-\cos x)^2}=$$
$$=\frac{2\cos x\sin x-1-2\cos x\sin x-1}{(\sin x-\cos x)^2}=-\frac{2}{(\sin x-\cos x)^2}.$$

(2.)
$$f'(x)=\frac{(x^5)'\cdot e^x-x^5\cdot(e^x)'}{(e^x)^2}=\frac{5x^4e^x-x^5e^x}{e^{2x}}=\frac{5x^4-x^5}{e^x}=\frac{x^4(5-x)}{e^x}.$$

(3.)
$$f'(\alpha)=\frac{(2\alpha^2-\alpha+3)'\cdot 2\alpha-(2\alpha^2-\alpha+3)(2\alpha)'}{(2\alpha)^2}=\frac{(4\alpha-1)\cdot 2\alpha-(2\alpha^2-\alpha+3)\cdot 2}{4\alpha^2}=$$
$$=\frac{8\alpha^2-2\alpha-4\alpha^2+2\alpha-6}{4\alpha^2}=\frac{4\alpha^2-6}{4\alpha^2}=\frac{2\alpha^2-3}{2\alpha^2}.$$

(4.)
$$f'(t)=(2t)'\sin t+2t(\sin t)'-(t^2-2)'\cos t-(t^2-2)(\cos t)'=$$
$$=2\sin t+2t\cos t-2t\cos t-(t^2-2)(-\sin t)=2\sin t+t^2\sin t-2\sin t=t^2\sin t.$$

Exercices 6.4

Find the derivative:

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(1.) $y = sin^3 x$;

(2.) $y = ln(tgx)$;

(3.) $y = 5^{cos x}$;

(4.) $y = ln\, sin(x^3 + 1)$;

(5.) $y = arcsin\sqrt{1-x^2}$;

(6.) $y = ln^5(tg3x)$;

(7.) $y = sin^2\sqrt{\frac{1}{1-x}}$;

(8.) $y = arcsin\frac{2x}{1+x^2}$.

Solution:

(1.) $y' = \left(sin^3 x\right)' = 3\,sin^2 x\cdot(sin\,x)' = 3\,sin^2 x\cdot cos\,x$;

(2.) $y' = [ln(tgx)]' = \frac{1}{tgx}\cdot(tgx)' = \frac{cos\,x}{sin\,x}\cdot\frac{1}{cos^2 x} = \frac{1}{sin\,x\,cos\,x} = \frac{2}{sin\,2x}$;

(3.) $y' = \left[5^{cos\,x}\right]' = 5^{cos\,x}\cdot ln\,5\cdot(cos\,x)' = 5^{cos\,x}\cdot ln\,5\cdot(-sin\,x) = -5^{cos\,x}\cdot ln\,5\cdot sin\,x$;

(4.) $y' = \frac{1}{sin(x^3+1)}\cdot\left[sin(x^3+1)\right]' = \frac{1}{sin(x^3+1)}\cdot cos(x^3+1)(x^3+1)' = 3x^2 ctg(x^3+1)$;

(5.) $y' = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\left(\sqrt{1-x^2}\right)' = \frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{1\cdot(1-x^2)'}{2\sqrt{1-x^2}} =$

$= \frac{1}{\sqrt{x^2}}\cdot\frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{|x|\cdot\sqrt{1-x^2}}$ $(x \neq 0)$.

(6.) $y' = 5\,ln^4(tg3x)\cdot[ln(tg3x)]' = 5\,ln^4(tg3x)\cdot\frac{1}{tg3x}\cdot(tg3x)' =$

$= 5\,ln^4(tg3x)\cdot\frac{1}{tg3x}\cdot\frac{1}{cos^2 3x}\cdot(3x)' = 15\,ln^4(tg3x)\cdot\frac{1}{sin\,3x\cdot cos\,3x} = 30\frac{ln^4(tg3x)}{sin\,6x}$.

(7.) $y' = 2\,sin\sqrt{\frac{1}{1-x}}\cdot\left[sin\sqrt{\frac{1}{1-x}}\right]' = 2\,sin\sqrt{\frac{1}{1-x}}\cdot cos\sqrt{\frac{1}{1-x}}\cdot\left(\sqrt{\frac{1}{1-x}}\right)' =$

$= sin\frac{2}{\sqrt{1-x}}\cdot\frac{-\frac{1\cdot(1-x)'}{2\sqrt{1-x}}}{\left(\sqrt{1-x}\right)^2} = sin\frac{2}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{(1-x)^3}}$.

(8.) $y' = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\left(\frac{2x}{1+x^2}\right)' = \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\cdot\frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2} =$

$$=\frac{1}{\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}}\cdot\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\sqrt{\left(1-x^2\right)^2}\cdot\left(1+x^2\right)}=\frac{2\left(1-x^2\right)}{\left|1-x^2\right|\cdot\left(1+x^2\right)},$$

i.e. $y'=\begin{cases}\dfrac{2}{1+x^2} & \text{za } |x|<1\\ -\dfrac{2}{1+x^2} & \text{za } |x|>1\end{cases},$ (for $|x|$ = 1 derivation does not exist).

Exercices 6.5

Find the derivative $f(x)=\dfrac{e^{-x^2}\cdot arcsin\,e^{-x^2}}{\sqrt{1-e^{-2x^2}}}+\dfrac{1}{2}ln\left(1-e^{-2x^2}\right)$.

Solution:

$$f'(x)=\frac{\left[e^{-x^2}\cdot(-2x)arcsin\,e^{-x^2}+e^{-x^2}\cdot\dfrac{e^{-x^2}(-2x)}{\sqrt{1-e^{-2x^2}}}\right]\sqrt{1-e^{-2x^2}}}{1-e^{-2x^2}}-$$

$$-\frac{e^{-x^2}\cdot arcsin\,e^{-x^2}\cdot\dfrac{-e^{-2x^2}(-4x)}{2\sqrt{1-e^{-2x^2}}}}{1-e^{-2x^2}}-\frac{1}{2}\cdot\frac{e^{-2x^2}\cdot(-4x)}{1-e^{-2x^2}},$$

$$f'(x)=\frac{-2xe^{-x^2}\cdot arcsin\,e^{-x^2}\cdot\sqrt{1-e^{-2x^2}}-2xe^{-2x^2}}{1-e^{-2x^2}}+$$

$$+\frac{-2xe^{-3x^2}\cdot arcsin\,e^{-x^2}\left(1-e^{-2x^2}\right)^{-\frac{1}{2}}+2xe^{-2x^2}}{1-e^{-2x^2}}=$$

$$=\frac{-2xe^{-x^2}\cdot arcsin\,e^{-x^2}}{1-e^{-2x^2}}\left(\sqrt{1-e^{-2x^2}}+\frac{e^{-2x^2}}{\sqrt{1-e^{-2x^2}}}\right)=$$

$$\frac{-2xe^{-x^2}\cdot arcsin\,e^{-x^2}}{1-e^{-2x^2}}\cdot\frac{1-e^{-2x^2}+e^{-2x^2}}{\sqrt{1-e^{-2x^2}}},$$

That is,

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$$f'(x) = -\frac{2xe^{-x^2} \cdot \arcsin e^{-x^2}}{\left(1 - e^{-2x^2}\right)^{\frac{3}{2}}}.$$

Exercices 6.6

Prove that the function $y = \dfrac{1}{1 + x + \ln x}$ meets the equation $xy' = y(y \ln x - 1)$.

Solution:

As $y' = \dfrac{-\left(1 + \frac{1}{x}\right)}{(1 + x + \ln x)^2} = \dfrac{-(x+1)}{x(1 + x + \ln x)^2}$, this results in $x \cdot y' = \dfrac{-(x+1)}{(1 + x + \ln x)^2}$.

The right side of the given equation is

$$y(y \ln x - 1) = \frac{1}{1 + x + \ln x}\left(\frac{\ln x}{1 + x + \ln x} - 1\right) = \frac{\ln x - 1 - x - \ln x}{(1 + x + \ln x)^2} = \frac{-(x+1)}{(1 + x + \ln x)^2}.$$

As the equivalences on the right side are the same, this means that the function *y* meets the given equation.

6.2 Logarithmic derivative

The function having the form $y = [f(x)]^{g(x)}, \quad f(x) > 0$ has to be turned into a logarithm before derivation, i.e.

$\ln y = g(x) \cdot \ln f(x)$. We can now derivate it:

$(\ln y)' = g'(x) \cdot \ln f(x) + g(x) \cdot [\ln f(x)]'$ that is:

$\dfrac{1}{y} \cdot y' = g'(x) \cdot \ln f(x) + g(x) \cdot \dfrac{1}{f(x)} \cdot f'(x)$ so we get:

$$y' = y \cdot \left[g'(x) \cdot \ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)} \right] =$$

$$= \left[f(x) \right]^{g(x)} \cdot \left[g'(x) \cdot \ln f(x) + \frac{g(x) \cdot f'(x)}{f(x)} \right]$$

Note: The same formula is achieved by using the identity $[f(x)]^{g(x)} = e^{g(x) \cdot ln\, f(x)}$, $f(x) > 0$. Namely, through the process of derivation, we get:

$y' = \left\{ [f(x)]^{g(x)} \right\}' = e^{g(x) \cdot ln(x)} \cdot [g(x) \cdot ln\, f(x)]'$, so we have:

$$y' = \left\{ [f(x)]^{g(x)} \right\}' = [f(x)]^{g(x)} \cdot \left[g'(x) \cdot ln\, f(x) + \frac{g(x) \cdot f'(x)}{f(x)} \right].$$

Example 1

Define the derivative:

(1.) $f(x) = (cos\, x)^{sin\, x}$; (2.) $f(x) = \left(2 + \frac{1}{x} \right)^{3x}$.

Solution:

(1.) From $f(x) = (cos\, x)^{sin\, x}$ by using logarithm, we get:

$ln\, f(x) = sin\, x \cdot ln(cos\, x)$, so that:

$\frac{1}{f(x)} \cdot f'(x) = (sin\, x)' \cdot ln(cos\, x) + sin\, x \cdot \frac{1}{cos\, x} \cdot (cos\, x)'$, that is:

$$f'(x) = (cos\, x)^{sin\, x} \cdot [cos\, x\, ln(cos\, x) - tgx \cdot sin\, x] .$$

(2.) Iz $f(x) = \left(2 + \frac{1}{x} \right)^{3x} \Rightarrow \ln f(x) = 3x \ln \left(2 + \frac{1}{x} \right)$;

$$\frac{1}{f(x)} \cdot f'(x) = 3 \ln \left(2 + \frac{1}{x} \right) + 3x \cdot \frac{1}{2 + \frac{1}{x}} \cdot \left(-\frac{1}{x^2} \right)$$

$$f'(x)=\left(2+\frac{1}{x}\right)^{3x}\left[3\ln\left(2+\frac{1}{x}\right)-\frac{3}{2x+1}\right].$$

Note: The use of logarithms can considerably facilitate the derivation of some rational functions, which can be observed in the following examples.

Example 2

Find the derivative:

(1.) $f(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}$; (2.) $g(x)=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\cdot\sqrt{(x+3)^3}}$.

Solution:

(1.) Through the logarithm $f(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}$ we achieve:

$ln\, f(x)=ln\, x+\frac{2}{3}ln\, x-\frac{1}{3}ln(x^2+1)$. So that now:

$\frac{1}{f(x)}\cdot f'(x)=\frac{1}{x}+\frac{2}{3x}-\frac{1\cdot 2x}{3(x^2+1)}$, that is:

$$f'(x)=x\cdot\sqrt[3]{\frac{x^2}{x^2+1}}\left[\frac{1}{x}+\frac{2}{3x}-\frac{2x}{3(x^2+1)}\right]=\sqrt[3]{\frac{x^2}{x^2+1}}\cdot\frac{3x^2+5}{3(x^2+1)}.$$

(2.) Through the logarithm $g(x)=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\cdot\sqrt{(x+3)^3}}$ we achieve:

$ln\, g(x)=\frac{1}{2}ln(x-1)-\frac{2}{3}ln(x+2)-\frac{3}{2}ln(x+3)$, so that:

$$\frac{1}{g(x)}\cdot g'(x)=\frac{1}{2(x-1)}-\frac{2}{3(x+2)}-\frac{3}{2(x+3)}$$

$$g'(x)=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^3}}\cdot\left[\frac{1}{2(x-1)}-\frac{2}{3(x+2)}-\frac{3}{2(x+3)}\right].$$

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6.3 Derivation of the implicitly given function

Let F be the function of two independent variables x and y. Then, if the limits exist,

$\frac{\partial F}{\partial x}=\lim_{\Delta x\to 0}\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}$ is called the ***partial*** derivative *F* by *x* (here, *y* is considered a constant). Analogously,

$\frac{\partial F}{\partial y}=\lim_{\Delta y\to 0}\frac{F(x,y+\Delta y)-F(x,y)}{\Delta y}$ is called the ***partial*** derivative *F* by *y* (here, *x* is considered a constant).

The rules for partial derivations (for all $\alpha,\beta\in\mathbf{R}$ and functions *F* and *G* for which the indicated derivations exist):

$$\frac{\partial}{\partial x}(\alpha F+\beta G)=\alpha\frac{\partial F}{\partial x}+\beta\frac{\partial G}{\partial x};\qquad \frac{\partial}{\partial y}(\alpha F+\beta G)=\alpha\frac{\partial F}{\partial y}+\beta\frac{\partial G}{\partial y};$$

$$\frac{\partial}{\partial x}(F\cdot G)=\frac{\partial F}{\partial x}G+F\frac{\partial G}{\partial x};\qquad \frac{\partial}{\partial y}(F\cdot G)=\frac{\partial F}{\partial y}G+F\frac{\partial G}{\partial y};$$

$$\frac{\partial}{\partial x}\left(\frac{F}{G}\right)=\frac{\frac{\partial F}{\partial x}G-F\frac{\partial G}{\partial x}}{G^2}\quad (G\neq 0);\qquad \frac{\partial}{\partial y}\left(\frac{F}{G}\right)=\frac{\frac{\partial F}{\partial y}G-F\frac{\partial G}{\partial y}}{G^2}\quad (G\neq 0).$$

There is an interval $I\subseteq R$ having the point x_0 and there is a unique function $f:I\to R$, as here:

(1.) $f(x_0)=y_0$;

(2.) $F(x,f(x))=0 \quad \forall x\in I$;

(3.) f has a derivative f' at every point $x\in I$. In addition, it is valid that:

$$f'(x_0)=-\frac{\frac{\partial F}{\partial x}(x_0,y_0)}{\frac{\partial F}{\partial y}(x_0,y_0)}.$$

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Note: If the function $y=f(x)$ is given in an implicit form, i.e. through the equation $F(x,y)=0$ and if f is a derivable at the point x, then its derivation at that point can also be found in the following way:

1. deriving both sides of the equation $F(x,y)=0$ (by variable x) taking into consideration that y is the function of x and that y' is derivation of y by x (because y is a function of x),
2. the obtained equation $\frac{d}{dx}F(x,y)=0$ is solved by y'.

Example 1

Define the values of partial derivations at the point $T(-2,1)$ of the given functions:

(1.) $F(x,y)=7x^3-4x^2y^2+y^2$; (2.) $F(x,y)=\ln\sqrt{x^2+y^2}$;

Solution:

(1.) For $F(x,y)=3x^3+x^2y^2-2y^2$ it follows that

$\frac{\partial F}{\partial x}=21x^2-8xy^2$ (y is considered a constant),

$\frac{\partial F}{\partial y}=-8x^2y+2y$ (x is considered a constant).

By inserting:

$\frac{\partial F}{\partial x}(-2,1)=21\cdot 4-8\cdot(-2)\cdot 1=100=$ and $\frac{\partial F}{\partial y}(-2,1)=-8\cdot 4\cdot 1+2\cdot 1=-30=$.

(2.) If $F(x,y)=\ln\sqrt{x^2+y^2}$, then

$$\frac{\partial F}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1\cdot 2x}{2\sqrt{x^2+y^2}}=\frac{x}{x^2+y^2}\quad\Rightarrow\quad\frac{\partial F}{\partial x}(-2,1)=-\frac{2}{5}.$$

$$\frac{\partial F}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1\cdot 2y}{2\sqrt{x^2+y^2}}=\frac{y}{x^2+y^2}\quad\Rightarrow\quad\frac{\partial F}{\partial y}(-2,1)=\frac{1}{5}.$$

Example 2

Find the derivative $y = f(x)$ that is given implicitly $8x - 3y + 7 = 0$.

Solution:

Therefore, $F(x, y) = 8x - 3y + 7$, so that:

$\frac{\partial F}{\partial x} = 8$, and $\frac{\partial F}{\partial y} = -3$,

$$y' = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-8}{-3} = \frac{8}{3}.$$

Example 3

Find the derivation of the implicitly given function $ln\, x + e^{-\frac{y}{x}} = C$ in two ways.

Solution:

I. From $F(x, y) = \ln x + e^{-\frac{y}{x}} - C$ the result is:

$\frac{\partial F}{\partial x} = \frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{y}{x^2}$; and $\frac{\partial F}{\partial y} = e^{-\frac{y}{x}} \cdot \left(-\frac{1}{x}\right)$, so that:

$$y'(x) = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{x + y \cdot e^{-\frac{y}{x}}}{x^2}}{-\frac{1}{x} \cdot e^{-\frac{y}{x}}} = \frac{x + ye^{-\frac{y}{x}}}{xe^{-\frac{y}{x}}} = e^{\frac{y}{x}} + \frac{y}{x}.$$

II. Given:

$$\ln x + e^{-\frac{y}{x}} = C, \qquad \left| \frac{d}{dx} \right.$$

$$\frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{d}{dx}\left(-\frac{y}{x}\right) = 0,$$

$$\frac{1}{x} + e^{-\frac{y}{x}} \cdot \frac{-y' \cdot x + y}{x^2} = 0, \qquad \left| \cdot x^2 \right.$$

$$x + e^{-\frac{y}{x}}(y - y'x) = 0,$$

$$x + ye^{-\frac{y}{x}} = y'x \cdot e^{-\frac{y}{x}},$$

$$y' = e^{\frac{y}{x}} + \frac{y}{x}.$$

Example 4

Find the derivative $F(x,y) = \dfrac{xy}{x^2+y^2}$ at the point $T(-1,2)$.

Solution:

$$\frac{\partial F}{\partial x} = \frac{\left(x^2+y^2\right)\frac{\partial}{\partial x}(x\cdot y) - xy\frac{\partial}{\partial x}\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2} = \frac{y\left(x^2+y^2\right) - xy\cdot 2x}{\left(x^2+y^2\right)^2} =$$

$$= \frac{y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} \quad \Rightarrow \quad \frac{\partial F}{\partial x}(-1,2) = \frac{2\left(2^2-(-1)^2\right)}{\left[(-1)^2+2^2\right]^2} = \frac{6}{25}.$$

$$\frac{\partial F}{\partial y} = \frac{x\left(x^2+y^2\right) - xy\cdot 2y}{\left(x^2+y^2\right)^2} = \frac{x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} \quad \Rightarrow \quad \frac{\partial F}{\partial y}(-1,2) = \frac{3}{25}.$$

$$y' = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{6}{25}}{\frac{3}{25}} = -2.$$

Exercices 6.7

Find the derivations of the implicit functions:

(1.) $x^3 + x^2y + y^2 = 0$;

(2.) $y^3 = \dfrac{x-y}{x+y}$;

(3.) $xy = arctg\dfrac{x}{y}$

Solution:

(1.) From $F(x,y) = x^3 + x^2y + y^2$ it follows that $\dfrac{\partial F}{\partial x} = 3x^2 + 2xy$; $\dfrac{\partial F}{\partial y} = x^2 + 2y$, so that:

$$y'(x) = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2 + 2xy}{x^2 + 2y}.$$

(2.) From $F(x,y) = y^3 - \dfrac{x-y}{x+y}$ it follows that

$\dfrac{\partial F}{\partial x} = \dfrac{-(x+y)+(x-y)}{(x+y)^2} = -\dfrac{2y}{(x+y)^2}$ and

$\dfrac{\partial F}{\partial y} = 3y^2 - \dfrac{-(x+y)-(x-y)}{(x+y)^2} = 3y^2 + \dfrac{2x}{(x+y)^2} = \dfrac{3y^2(x+y)^2 + 2x}{(x+y)^2}$, so that

$$y'(x) = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{-2y}{(x+y)^2}}{\frac{3y^2(x+y)^2 + 2x}{(x+y)^2}} = \frac{2y}{3y^2(x+y)^2 + 2x}.$$

(3.) From $F(x,y) = xy - arctg\dfrac{x}{y}$ it follows that

$\dfrac{\partial F}{\partial x} = y - \dfrac{1}{1+\frac{x^2}{y^2}} \cdot \dfrac{1}{y} = y - \dfrac{y}{y^2+x^2} = \dfrac{y^3 + x^2y - y}{y^2 + x^2}$ and

$\dfrac{\partial F}{\partial y} = x - \dfrac{1}{1+\frac{x^2}{y^2}} \cdot \dfrac{-x}{y^2} = x + \dfrac{x}{y^2+x^2} = \dfrac{xy^2 + x^3 + x}{y^2 + x^2}$, so that

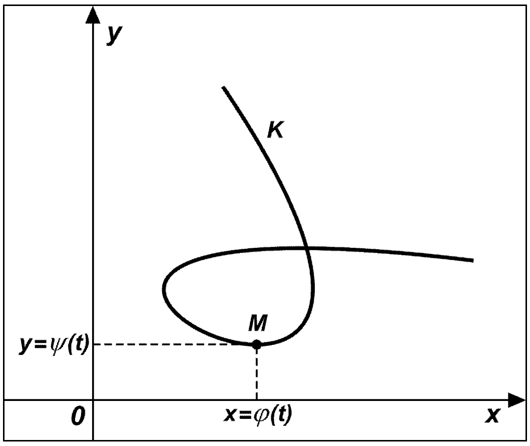
$$y'(x) = f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y^3 + x^2y - y}{xy^2 + x^3 + x} = \frac{y}{x} \cdot \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

6.4 Derivation of the parametrically given function

The trail of a material point T moving across the plane is the curve, for example: line, parabola, ellipse, hyperbola, cosine wave, etc. Each point T can be observed as a vessel sailing along a path (curve). When describing such a movement, it is necessary to know the point coordinates as the function of time, at each moment *t*. If we mark $x=\varphi(t)$ and $y=\psi(t)$ as the coordinates of the point T where φ and ψ are the real functions determined at the interval I during which the movement occurs, it is clear that φ and ψ are derivable functions, because the speed is given by $v(t)=\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}$. Hence, when time t describes the interval $I\subseteq R$, then the point $T(\varphi(t),\psi(t))$ passes at least once through each point of the set, that is $K=\{(\varphi(t),\psi(t)):t\in I\}$.

Parametric equations of the curve (*t* is called the **parameter**) are expressed as:

$$\begin{cases} x=\varphi(t) \\ y=\psi(t), \quad t\in I \end{cases}$$



If φ is strictly monotonous by I (we know that there is an inverse function $t=\varphi^{-1}(x)$), then, by replacing the variable *t* by $\varphi^{-1}(x)$ we get $y=\psi(t)=\psi[\varphi^{-1}(x)]=f(x)$.

According to the composition derivation, it follows that:

$$y'(x_0)=f'(x_0)=\psi'[\varphi^{-1}(x_0)]\cdot[\varphi^{-1}(x_0)]'=\psi'(t_0)\cdot\frac{1}{\varphi'(t_0)}, \text{ that is } f'(x_0)=\frac{\psi'(t_0)}{\varphi'(t_0)}.$$

We can use a simpler way:

$$y'(x)=f'(x)=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)}.$$

It is easily proven that:

$$f''(x)=\frac{\varphi'(t)\cdot\psi''(t)-\varphi''(t)\cdot\psi'(t)}{\left[\varphi'(t)\right]^3}\quad\Rightarrow$$

$$y''(x)=f''(x)=\frac{x'(t)\cdot y''(t)-x''(t)\cdot y'(t)}{\left[x'(t)\right]^3}.$$

Example 1

Find the derivative *f* at the point x_0 = 3, which is a parametrically given by formulas:

$$\begin{cases} x=2t-1 \\ y=t^3 \end{cases}$$

Solution:

First we determine t_0 with the corresponding value $x_0=3$. From $x=2t-1\Rightarrow t=\frac{x+1}{2}$, so that $x_0=3$; $t_0=\frac{3+1}{2}=2$.

Now, let us find $\varphi'(t)$ and $\psi'(t)$:

$\varphi'(t)=x'(t)=2$ and $\psi'(t)=y'(t)=3t^2$.

By using the formula, it follows that $f'(x_0)=\frac{\psi'(t_0)}{\varphi'(t_0)}$, that is: $f'(3)=\left.\frac{3t_0{}^2}{2}\right|_{t_0=2}=6$. Therefore,

$$y'=f'(x)=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\frac{-8}{-3}=\frac{8}{3}.$$

Example 2

Find $f'(x)$ and $f''(x)$ for the function $y=f(x)$, which is a parametrically given by formulas:

$$\begin{cases} x=e^t\cos t, \\ y=e^t\sin t. \end{cases}$$

Solution:

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t} = \frac{\sin t + \cos t}{\cos t - \sin t}.$$

$$y''(x) = \frac{x'(t) \cdot y''(t) - x''(t) \cdot y'(t)}{\left[x'(t)\right]^3} =$$

$$= \frac{\left(e^t \cos t - e^t \sin t\right)\left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)}{\left(e^t \cos t - e^t \sin t\right)^3} -$$

$$- \frac{\left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\left(e^t \sin t + e^t \cos t\right)}{\left(e^t \cos t - e^t \sin t\right)^3} =$$

$$= \frac{e^t\left(\cos t - \sin t\right) 2e^t \cos t + 2e^t \sin t \cdot e^t\left(\sin t + \cos t\right)}{\left(e^t\right)^3\left(\cos t - \sin t\right)^3} =$$

$$= \frac{2e^{2t}\left[\cos^2 t - \sin t \cos t + \sin^2 t + \sin t \cos t\right]}{e^{2t} \cdot e^t\left(\cos t - \sin t\right)^3} = \frac{2}{e^t\left(\cos t - \sin t\right)^3}.$$

Exercices 6.8

Find $y'(x)$ if $\begin{cases} x(t) = a\left(\ln tg \frac{t}{2} + \cos t - \sin t\right), \\ y(t) = a(\sin t + \cos t). \end{cases}$

Solution:

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{a(\cos t - \sin t)}{a\left(\frac{1}{tg \frac{t}{2}} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{1}{2} - \sin t - \cos t\right)} = \frac{\cos t - \sin t}{\left(\frac{1}{\sin t} - \sin t - \cos t\right)} =$$

$$= \frac{\sin t(\cos t - \sin t)}{1 - \sin^2 t - \sin t \cos t} = \frac{\sin t(\cos t - \sin t)}{\cos^2 t - \sin t \cos t} == \frac{\sin t(\cos t - \sin t)}{\cos t(\cos t - \sin t)} = tgt.$$

Exercices 6.9

Find the coefficient of the tangent direction in the graph of a function that is parametrically given:

$$\begin{cases} x(t) = t\ln t, \\ y(t) = \dfrac{\ln t}{t} \end{cases}$$ at the point $t_0 = 1$.

Solution:

As $y'(t) = \dfrac{\frac{1}{t}\cdot t - \ln t}{t^2} = \dfrac{1-\ln t}{t^2}$, and $x'(t) = \ln t + 1$,

It follows that $y'(x) = \dfrac{y'(t)}{x'(t)} = \dfrac{1-\ln t}{t^2(\ln t + 1)}$, so that

$$k_t = y'(x)\Big|_{t_0=1} = \frac{1-\ln 1}{1^2(\ln 1 + 1)} = 1.$$

Exercices 6.10

Find $y'(x)$ for the parametrically given function:

$$\begin{cases} x(t) = arccos\dfrac{1}{\sqrt{1+t^2}}, \\ y(t) = arcsin\dfrac{t}{\sqrt{1+t^2}}. \end{cases}$$

Solution:

Since:

$$y'(t) = \frac{1}{\sqrt{1-\frac{t^2}{1+t^2}}}\cdot\frac{\sqrt{1+t^2} - t\cdot\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{\sqrt{1+t^2}\left(1+t^2-t^2\right)}{\left(1+t^2\right)^{\frac{3}{2}}} = \frac{1}{1+t^2},$$

$$x'(t) = -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\cdot\frac{-\frac{t}{\sqrt{1+t^2}}}{1+t^2} = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2-1}}\cdot\frac{t}{\left(1+t^2\right)\sqrt{1+t^2}} = \frac{t}{|t|\cdot\left(1+t^2\right)}.$$

We obtain $$y'(x) = \frac{y'(t)}{x'(t)} = \frac{\frac{1}{1+t^2}}{\frac{t}{|t|}\frac{1}{1+t^2}} = \frac{|t|}{t} = \begin{cases} 1 & \text{za } t > 0 \\ -1 & \text{za } t < 0 \end{cases}.$$

Exercices 6.11

Find another derivative for $y''(x)$:

(1.) $\begin{cases} x(t) = a(\sin t - t\cos t), \\ y(t) = a(\cos t + t\sin t), \end{cases}$ (2.) $\begin{cases} x(t) = a\cos^3 t, \\ y(t) = a\sin^3 t. \end{cases}$

Solution:

(1.) $x'(t) = a[\cos t - \cos t - t(-\sin t)] = at\sin t$,

$y'(t) = a[-\sin t + \sin t + t\cos t] = at\cos t$.

So that $y'(x) = \frac{y'(t)}{x'(t)} = \frac{at\cos t}{at\sin t} = ctgt$. It follows that:

$$y''(x) = \frac{\frac{d}{dt}[y'(x)]}{x'(t)} = \frac{-\frac{1}{\sin^2 t}}{at\sin t} = -\frac{1}{at\sin^3 t}.$$

(2.) $x'(t) = 3a\cos^2 t(-\sin t)$,

$y'(t) = 3a\sin^2 t(\cos t)$.

So that $y'(x) = \frac{y'(t)}{x'(t)} = -\frac{\sin t}{\cos t} = -tgt$. It follows that

$$y''(x) = \frac{\frac{d}{dt}[y'(x)]}{x'(t)} = \frac{-\frac{1}{\cos^2 t}}{-3a\cos^2 t\sin t} = \frac{1}{3a\cos^4 t\sin t}.$$

Exercices 6.12

Prove that the function $y = f(x)$, given parametrically

$\begin{cases} x(t) = 2t + 3t^2 \\ y(t) = t^2 + 2t^3 \end{cases}$ satisfies the equation $2(y')^3 + (y')^2 - y = 0, \left(y' = \frac{dy}{dx}\right)$.

Solution:

$$y' = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t + 6t^2}{2 + 6t} = t.$$

If y' is inserted into the equation $2(y')^3 + (y')^2 - y = 0$, we obtain

$$2t^3 + t^2 - (t^2 + 2t^3) \equiv 0$$

6.5 The tangent and the normal in the graph of a function

Let a curve be defined by the formula $y = f(x)$, where f is a derivable function. The curve secant $y = f(x)$ passing through the points $(x_0, f(x_0))$ i $(x, f(x))$, where $x_0 \neq x$, is the line with coefficient $tg\varphi = \dfrac{f(x) - f(x_0)}{x - x_0}$.

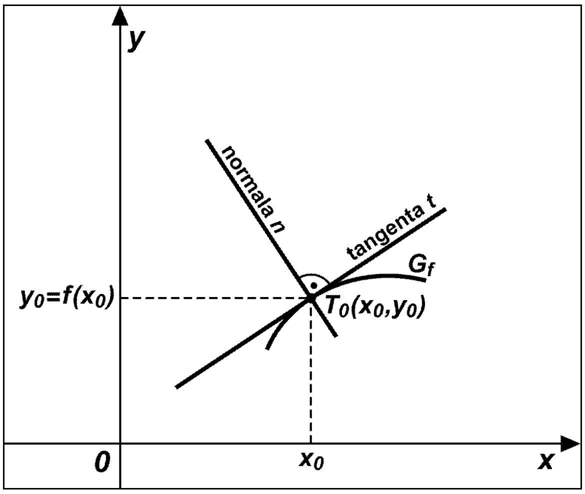
When $x \to x_0$, then the secant line tends to the curve tangent $y = f(x)$ at the point $(x_0, f(x_0))$, whose coefficient of direction is equal to $tg\varphi_0$. Obviously, it is valid that: $x \to x_0 \Rightarrow tg\varphi \to tg\varphi_0 = f'(x_0)$. Therefore:

*the **equation of the tangent*** in the graph of function $y = f(x)$ at the point $T_0(x_0, y_0)$, where $y_0 = f(x_0)$, a $f'(x_0) = k \in R$, is:

$$y - f(x_0) = f'(x_0)(x - x_0);$$

while the ***equation of the normal*** (perpendicular on the tangent) at that point is:

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0).$$



The angle φ at which the graphs of the functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$ intersect at the point $T_0(x_0, y_0)$ is the angle between their tangents at that point, and is calculated according to the formula:

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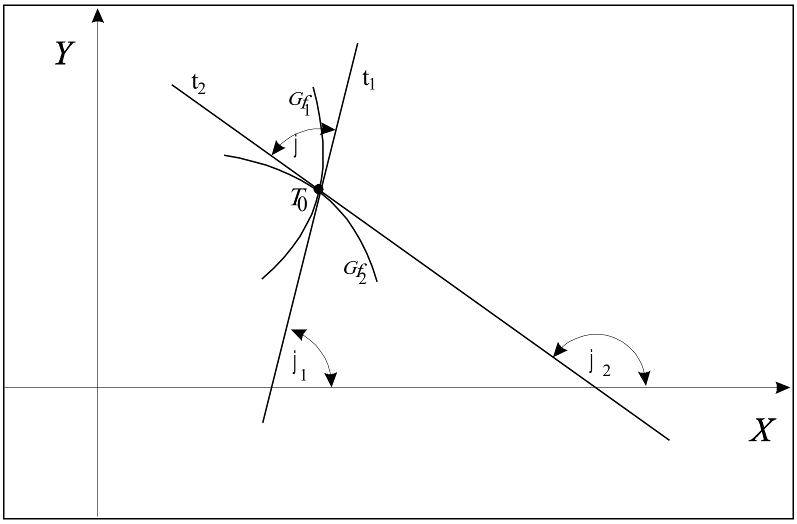
$$tg\varphi = tg(\varphi_2 - \varphi_1) = \frac{tg\varphi_2 - tg\varphi_1}{1 + tg\varphi_2 tg\varphi_2} = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)}.$$

This is the angle that we need to rotate the tangent t_1 of the function f_1 in the positive direction (counter-clockwise) around their mutual point, so that it could be aligned with the tangent t_2 of the function f_2 (see the figure).



Example 1

Find the equation of the tangent and the normal in the graph of function $f(x) = x^3 - 3x + 2$ at a point whose abscissa is $x_0 = 2$.

Solution:

If we insert $x_0 = 2$ into the formula by which the function is given, we get the ordinate of the point *T*0, that is $y_0 = f(x_0) = f(2) = 4$. Now we look for the equation of the tangent and the normal at the point $T_0(2,4)$. We can find the derivative at the point $x_0 = 2$:

$$f'(x) = 3x^2 - 3 \quad \Rightarrow \quad f'(2) = 9.$$

Equation of the tangent: $y - 4 = 9(x - 2)$ or

$$9x - y - 14 = 0.$$

Equation of the normal: $y - 4 = -\frac{1}{9}(x - 2)$ or

$$x + 9y - 38 = 0.$$

Example 2

Find the equation of the tangent in the graph of function $f(x)=x^2-3x+1$ which passes through the point $T_0(2,-2)$. Find the coordinates of the contact.

Solution:

If the contact $D(x_1,y_1)$ of the tangent in the graph of function *f*, its coordinates are $\left(x_1, x_1^2-3x_1+1\right)$. Furthermore, $f'(x)=2x-3$ and $k_t=2x_1-3$. On the other hand,

$$k_t=\frac{y_1-y_0}{x_1-x_0}=\frac{x_1^2-3x_1+1+2}{x_1-2}=2x_1-3\text{, that is}$$

$$x_1^2-3x_1+3=(2x_1-3)(x_1-2)\Rightarrow x_1^2-4x_1+3=0\Rightarrow\begin{cases}(x_1)_1=3,\\(x_1)_2=1.\end{cases}$$

For $(x_1)_1=3\Rightarrow y_1=f(3)=1\Rightarrow D_1(3,1)\Rightarrow k_{t_1}=3$.

For $(x_1)_2=1\Rightarrow y_1=f(1)=-1\Rightarrow D_2(1,-1)\Rightarrow k_{t_2}=-1$.

Hence, there are two tangents:

$$t_1: y-1=3(x-3)\Rightarrow\quad y-3x+8=0\quad\text{and}$$

$$t_2: y+1=-1(x-1)\Rightarrow\quad y=-x.$$

Example 3

Find the angle at which the functions $y=4-\frac{x^2}{2}$ and $y=4-x$ intersect.

Solution:

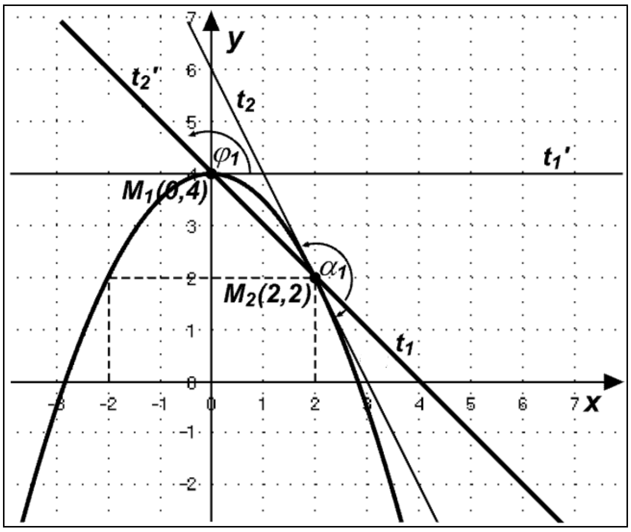
The angle φ at which the graphs of functions f_1 and f_2 intersect at the intersection point $M_1(x_1,y_1)$ is calculated through the well-known formula

$$tg\varphi=\frac{f_2'(x_1)-f_1'(x_1)}{1+f_1'(x_1)f_2'(x_1)}.$$

The intersection points of the given functions are obtained by solving the system of equations:

$$\left\{\begin{array}{c} y = 4 - \dfrac{x^2}{2} \\ y = 4 - x \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} x_1 = 0, & y_1 = 4, \\ x_2 = 2, & y_2 = 2, \end{array}\right.$$

that is, $M_1(0,4); M_2(2,2)$. (see the figure)



We obtain the angle at the point $M_1(0,4)$ by inserting the point into $f_1'(x) = -x$ and $f_2'(x) = -1$; so that $f_1'(0) = 0$ and $f_2'(0) = -1$. Therefore, $tg\varphi_1 = \dfrac{f_2'(0) - f_1'(0)}{1 + f_1'(0) \cdot f_2'(0)} = -\dfrac{1}{1}$ $= -1 \Rightarrow \varphi_1 = 135°$.

Analogously, we obtain the angle at the point $M_2(2,2)$. Let t_1 be the equation of the line, and t_2 equation of the tangent at the point M_2 through the function $y = 4 - \dfrac{x^2}{2}$, that is $f_1'(x) = -1$ and $f_2'(x) = -x$: so that $f_1'(2) = -1$ and $f_2'(2) = -2$. Therefore,

$$tg\alpha_1 = \frac{f_2'(2) - f_1'(2)}{1 + f_1'(2) \cdot f_2'(2)} = \frac{-2+1}{1+(-1)(-2)} = -\frac{1}{3} \Rightarrow .\ \alpha_1 = 161°33'54''$$

Exercices 6.13

Find the equations of the tangent and the normal in the functions with given points:

(1.) $f(x) = 2x^2 - x + 5$ at the point $T_0\left(-\dfrac{1}{2}, 6\right)$;

(2.) $f(x) = x^3 + 2x^2 - 4x - 3$ at the point $T_0(-2, 5)$;

(3.) $f(x)=\sqrt[3]{x-1}$ at the point with the abscissa $x_0=1$.

Solution:

(1.) $f(x)=2x^2-x+5$; $T_0\left(-\frac{1}{2},6\right)$. Find the derivative at the point $x_0=-\frac{1}{2}$:

$$f'(x)=4x-1 \Rightarrow f'\left(-\frac{1}{2}\right)=-3.$$

The equation of the tangent through the point $T_0(x_0, f(x_0))$ has a typical form:

$y-f(x_0)=f'(x_0)(x-x_0)$, so that, for the given function, the equation of the tangent is:

$$y-6=-3\left(x+\frac{1}{2}\right) \quad \text{or} \quad 2y+6x-9=0.$$

The equation of the normal through the point $T_0(x_0, f(x_0))$ has the form:

$y-f(x_0)=-\frac{1}{f'(x_0)}(x-x_0)$, so that

$$y-6=\frac{1}{3}\left(x+\frac{1}{2}\right) \quad \text{or} \quad 6y-2x-37=0.$$

(2.) $f(x)=x^3+2x^2-4x-3$; $T_0(-2,5)$.

$$f'(x)=3x^2+4x-4 \Rightarrow f'(-2)=0.$$

The equation of the tangent is $y-5=0$, while the equation of the normal is $x+2=0$.

<u>Note</u>: if $f'(x_0)=0$ then the equation of the tangent $y=f(x_0)$ (this is a line parallel to the x-axis passing through the point $T_0(x_0, f(x_0))$, and the equation of the normal is $x=x_0$.

(3.) $f(x)=\sqrt[3]{x-1}$ at the point with the abscissa $x_0=1$.

Since $y_0 = f(x_0) = 0$, we should find the equation of the tangent and the normal passing through the point $T_0(1, 0)$. The derivative $f(x) = \sqrt[3]{x-1}$ at the point $x_0 = 1$ is

$$f'(x) = \frac{1}{3 \cdot \sqrt[3]{(x-1)^2}} \Rightarrow f'(x_0) = f'(1) \text{ does not exist, that is}$$

$$f'(1) = \lim_{x \to 1} f'(x) = \lim_{x \to 1} \frac{1}{\left[3 \cdot \sqrt[3]{(x-1)^2}\right]} = \infty .$$

The equation of the tangent is $x - 1 = 0$ and the normal $y = 0$.

Note: if $f'(x_0) \to \infty$ when $x \to 1$, then the equation of the normal is $y - f(x_0) = 0$ and the tangent is $x - x_0 = 0$.

Exercices 6.14

Define the intersection of the tangents on the curve $y = \dfrac{1+3x^2}{3+x^2}$ at the points for which $y = 1$.

Solution:

The conditions of the task imply that

$$\frac{1+3x^2}{3+x^2} = 1 \Rightarrow x^2 = 1 \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}, \text{ that is } T_1(-1,1) \text{ and } T_2(1,1).$$

Let us find the values of the derivative at the points T_1 and T_2. From $y'(x) = \dfrac{16x}{(3+x^2)^2}$ it follows that

$k_1 = y'(-1) = -1$; $k_2 = y'(1) = 1$, so that

$t_1 : y - 1 = -1(x+1) \Rightarrow y = -x$, and

$t_2 : y - 1 = 1(x-1) \Rightarrow y = x$.

Therefore, the intersection of the tangents on the given curve is $S(0,0)$.

Exercices 6.15

Find the equation of the normal in the graph of function $f(x) = x \ln x$ which is parallel to the line $2x - 3y + 3 = 0$.

Solution:

In order to achieve that the normal in the graph of the given function in x_0 is parallel to the line p, it must be valid that $k_p = -\frac{1}{f'(x_0)}$, where k_p is the coefficient of the line p direction. From the given derivative, it follows that

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1, \text{ and}$$

$$f'(x_0) = \ln x_0 + 1.$$

From $y = \frac{2}{3}x + 1 \Rightarrow k_p = \frac{2}{3}$: so that

$$-\frac{1}{\ln x_0 + 1} = \frac{2}{3} \Rightarrow 2\ln x_0 + 2 = -3 \Rightarrow \ln x_0 = -\frac{5}{2} \Rightarrow x_0 = e^{-\frac{5}{2}}.$$

Since $y_0 = f(x_0) = f\left(e^{-\frac{5}{2}}\right) = e^{-\frac{5}{2}} \cdot \ln e^{-\frac{5}{2}} = -\frac{5}{2}e^{-\frac{5}{2}}$, the graph of the given function has a normal parallel to the line $2x - 3y + 3 = 0$ at the point $T_0\left(e^{-\frac{5}{2}}, -\frac{5}{2e^{\frac{5}{2}}}\right)$, hence the equation of the required normal is:

$$y + \frac{5}{2e^{\frac{5}{2}}} = \frac{2}{3}\left(x - e^{-\frac{5}{2}}\right) \text{ or } y = \frac{2}{3}x - \frac{19}{6e^{\frac{5}{2}}}.$$

Exercices 6.16

In the graph of the function $f(x) = x^2 - 2x + 5$ find the point at which the tangent is vertical to the line $y = x$.

Solution:

In order to achieve that the tangent in the graph of the given function in x_0 is vertical to the line p, it must be valid that $f'(x_0) = -\frac{1}{k_p}$, where k_p is the coefficient of the line p direction.

From the derivative of the given function $f'(x) = 2x - 2$ it follows that $f'(x_0) = 2x_0 - 2$.

From $y = x$ it follows that $k_p = 1$, therefore $2x_0 - 2 = -1 \Rightarrow x_0 = \frac{1}{2}$.

Since $y_0 = f(x_0) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 2\cdot\frac{1}{2} + 5 = \frac{17}{4}$, the graph of the given function has the tangent vertical to the line $y = x$ at the point $T_0\left(\frac{1}{2}, \frac{17}{4}\right)$.

Exercices 6.17

Find the angle at which the graph of the function $f(x) = arctg\left(1 + \frac{1}{x}\right)$ intersects the *x*-axis.

Solution:

The angle at which the graph of function *f* intersects the *x*-axis is the angle the tangent at that point makes with the positive direction of the *x*-axis. The desired angle φ is obtained as a solution to the trigonometric equation $f'(x_0) = k_t = tg\varphi$, where x_0 is the zero of the function *f*.

x_0 is obtained by solving the equation

$$arctg\left(1 + \frac{1}{x}\right) = 0 \Rightarrow 1 + \frac{1}{x} = 0 \Rightarrow x_0 = -1.$$

We know that

$$f'(x) = \frac{1}{1 + \left(1 + \frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2 + (x+1)^2}, \text{ so that}$$

$$f'(x_0) = f'(-1) = -1.$$

It follows that $tg\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$.

Exercices 6.18

Find the equations of the tangent and the normal at the point $T(1,1)$ on the graph of the function that is implicitly given by the equation $x^5 + y^5 - xy - 1 = 0$.

Solution:

If we denote $F(x, y) = x^5 + y^5 - xy - 1$, then $F(1,1) = 0$ and

$$\frac{\partial F}{\partial y}\Big|_{(1,1)} = 5y^4 - x\Big|_{(1,1)} = 4 \neq 0, \text{ so that}$$

$$f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{5x^4 - y}{5y^4 - x} \Rightarrow f'(1) = -1.$$

Therefore, the equation of the tangent is: $y - 1 = -1(x-1)$ or $y + x - 2 = 0$.

The equation of the normal: $y - 1 = -\frac{1}{-1}(x-1)$ or $y - x = 0$.

Exercices 6.19

At which point of the parabola $y = x^2 - 2x - 3$ the tangent makes identical angles on the both sides of the coordinate axis?

Solution:

Assume there is a general function $y = f(x)$, then, as required by the task, $tg\,\alpha = tg\,\beta$.

Since the value of the angle that the tangent makes with the axis x is equal to $y'(x)$, and the value of the angle that the same tangent makes with the axis y is equal to $x'(y)$, then $y'(x) = x'(y)$.

From $y = x^2 - 2x - 3 \quad \Rightarrow \quad y'(x) = 2x - 2$.

In order to find $x'(y)$, we mark $F(x, y) = x^2 - 2x - y - 3$. Since

$\frac{\partial F}{\partial x} = 2x - 2$ and $\frac{\partial F}{\partial y} = -1$, it follows that

$$x'(y) = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} = -\frac{-1}{2x-2} = \frac{1}{2x-2}.$$

By insertion, we get

$2x - 2 = \frac{1}{2x-2} \Rightarrow (2x-2)^2 = 1$, so that

$2x - 2 = \pm 1 \Rightarrow x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{2}$.

Since:

$$y(x_1) = y\left(\frac{1}{2}\right) = -\frac{15}{4} \quad \text{and} \quad y(x_2) = y\left(\frac{3}{2}\right) = -\frac{15}{4},$$

these are the required points:

$T_1\left(\frac{1}{2},-\frac{15}{4}\right)$ and $T_2\left(\frac{3}{2},-\frac{15}{4}\right)$.

Exercices 6.20

Find the equation of the tangent and the normal on the parametrically given curve:

$$\begin{cases} x(t) = & ln(cos\, t)+1, \\ y(t) = & tgt + ctgt, \end{cases} \quad \text{u} \quad t_0 = \frac{\pi}{4}.$$

Solution:

For the parameter value $t_0 = \frac{\pi}{4}$ it follows that

$$x_0 = x(t_0) = x\left(\frac{\pi}{4}\right) = \ln\cos\frac{\pi}{4} + 1 = \ln\frac{\sqrt{2}}{2} + 1, \text{ and}$$

$$y_0 = y(t_0) = y\left(\frac{\pi}{4}\right) = tg\frac{\pi}{4} + ctg\frac{\pi}{4} = 1 + 1 = 2.$$

We need to define the equation of the tangent and the normal at the point $T_0\left(ln\frac{\sqrt{2}}{2}+1, 2\right)$.

Since:

$$y'(x) = \frac{y'(t_0)}{x'(t_0)} = \left.\frac{\frac{1}{cos^2 t} - \frac{1}{sin^2 t}}{-\frac{sin\, t}{cos\, t}}\right|_{t_0=\frac{\pi}{4}} = 0, \text{ that is, } k_t = 0,$$

the tangent is parallel to the *x*-axis, while the normal is parallel to the *y*-axis *Y* at the point T_0, so that

$$t: \quad y - 2 = 0 \Rightarrow y = 2, \text{ and}$$

$$n: \quad x - ln\frac{\sqrt{2}}{2} - 1 = 0 \Rightarrow x = ln\frac{\sqrt{2}}{2} + 1.$$

Exercices 6.21

Find the equation of the tangent and the normal at the point (0, 0) on the parametrically given curve:

$$\begin{cases} x(t) = t\,ln\,t, \\ y(t) = \dfrac{ln\,t}{t}, \end{cases}$$

Solution:

$$\left.\begin{array}{l} x_0 = x(t_0) = 0 \Rightarrow \quad t_0\,ln\,t_0 = 0 \\ y_0 = y(t_0) = 0 \Rightarrow \quad \dfrac{ln\,t_0}{t_0} = 0 \end{array}\right\} \Rightarrow t_0 = 1$$

(parameter $t_1 = 0$ is not from the domain of the given functions and is not taken into consideration).

$$k_t = y'(x)\big|_{t_0=1} = \left.\frac{1 - ln\,t}{t^2(1 + ln\,t)}\right|_{t_0=1} = 1.$$

Now it is easy to obtain that the equation of the required tangent is $y = x$, while the equation of the normal is $y = -x$.

6.6 Application of derivations in determining the limit of functions

A typical requirement in determining the limit value or the limit of a function is set, when $x \to c \in R$ or when $x \to \pm\infty$, when the function is given in the form of a product or a quotient of two functions *f* and *g*. However, the result is often a product that is not defined, and to which already known theorems on the limit values of functions cannot be applied. The extension of the rules was given by G.F.A. de L`Hospital, therefore it is often said that **when determining the limits of functions of indeterminate forms of type**: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0 *so called the L'Hospital's Rule is applied.*

Note: The indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are solved by L'Hospital's rule, and other indeterminate forms are reduced to one of these two forms using appropriate transformations.

Theorem (L'Hospital's Rule):

If f and g are derivable functions around the point $c \in R$ where $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$. If $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)} = L \in R$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ and also

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

In particular, if f' and g' are continuous functions at point *c* and $g'(c) \neq 0$, then $L = \frac{f'(c)}{g'(c)}$.

Note:

i) L'Hospital's rule can be applied when $x \to \pm\infty$, for the indeterminate form $\frac{\infty}{\infty}$ and for limits and left-hand and right-hand derivatives.

ii) L'Hospital's rule can be applied several times in succession if one of the indeterminate forms is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

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Example 1

Find limits:

(1.) $\lim\limits_{x\to 0}\dfrac{e^{7x}+10}{3x}$; (2.) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^2}$.

Solution:

$$(1.)\ \lim_{x\to 0}\frac{e^{7x}+10}{3x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(e^{7x}+10\right)'}{(3x)'}=\lim_{x\to 0}\frac{7e^{7x}}{3}=\frac{7}{3};$$

$$(2.)\ \lim_{x\to\infty}\frac{\ln x}{x^2}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{\left(x^2\right)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=\left[\frac{1}{\infty}\right]=0.$$

Example 2

Find limits:

(1.) $\lim\limits_{x\to 0}\dfrac{x-sin\,x}{x^3}$; (2.) $\lim\limits_{x\to\infty}\dfrac{ln^2\,x}{x}$.

Solution:

$$(1.)\ \lim_{x\to 0}\frac{x-\sin x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(x-\sin x)'}{\left(x^3\right)'}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(1-\cos x)'}{\left(3x^2\right)'}=$$

$$=\lim_{x\to 0}\frac{\sin x}{6x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{(6x)'}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6};$$

$$(2.)\ \lim_{x\to\infty}\frac{\ln^2 x}{x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(\ln^2 x\right)'}{(x)'}=\lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}=\left[\frac{\infty}{\infty}\right]=$$

$$=2\lim_{x\to\infty}\frac{(\ln x)'}{(x)'}=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\frac{0}{1}=0.$$

- For the indeterminate form $0\cdot\infty$ the following transformation is used

$$\lim_{x\to c}\left(f(x)\cdot g(x)\right)=\lim_{x\to c}\frac{f(x)}{\frac{1}{g(x)}}\quad or\quad \lim_{x\to c}\left(f(x)\cdot g(x)\right)=\lim_{x\to c}\frac{g(x)}{\frac{1}{f(x)}}$$

  And is reduced to the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

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Example 3

Find limits:

(1.) $\lim\limits_{x\to 0^+} x\ln x$; (2.). $\lim\limits_{x\to 0}\left(\ln(1-\sin x)\cdot ctgx\right)$

Solution:

(1.) $\lim\limits_{x\to 0^+}(x\ln x)=\left[0\cdot(-\infty)\right] = \lim\limits_{x\to 0^+}\dfrac{\ln x}{\dfrac{1}{x}}=\left[\dfrac{-\infty}{+\infty}\right] = \lim\limits_{x\to 0^+}\dfrac{(\ln x)'}{\left(\dfrac{1}{x}\right)'}\lim\limits_{x\to 0^+}\dfrac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=\lim\limits_{x\to 0^+}(-x)=0$;

(2.) $\lim\limits_{x\to 0}\left(\ln(1-\sin x)\cdot ctgx\right)=\left[0\cdot\infty\right] = \lim\limits_{x\to 0}\dfrac{\ln(1-\sin x)}{\dfrac{1}{ctgx}}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{\left(\ln(1-\sin x)\right)'}{\left(\dfrac{1}{ctgx}\right)'}=$

$$=\lim_{x\to 0}\frac{\dfrac{1}{1-\sin x}\cdot(1-\sin x)'}{(tgx)'}=\lim_{x\to 0}\frac{\dfrac{-\cos x}{1-\sin x}}{\dfrac{1}{\cos^2 x}}=\lim_{x\to 0}\frac{-\cos^3 x}{1-\sin x}=\frac{-1}{1}=-1.$$

- <u>For the indeterminate form</u> $\infty-\infty$ an appropriate transformation is used in the following way
  - i) in order to exclude one member, the form is reduced to the indeterminate form $0\cdot\infty$, and then to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$;
  - ii) in order to reduce the form to the common denominator, it is directly reduced to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Example 4

Find limits:

(1.) $\lim\limits_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)$; (2.) $\lim\limits_{x\to -2}\left[\dfrac{1}{x+2}-\dfrac{1}{\ln(x+3)}\right]$.

Solution:

$$\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)=[\infty-\infty]=\lim_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=[\infty\cdot 0]\;=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}=\left[\frac{0}{0}\right]=$$

(1.)

$$=\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}}=\lim_{x\to\infty}e^{\frac{1}{x}}=e^{\lim\limits_{x\to\infty}\frac{1}{x}}=e^0=1$$;

(2.) $\lim\limits_{x\to -2}\left[\frac{1}{x+2}-\frac{1}{ln(x+3)}\right]=[\infty-\infty]=\left[\begin{matrix}\text{the expression is reduced here}\\ \text{to a common denominator}\end{matrix}\right]=$

$$=\lim_{x\to -2}\frac{ln(x+3)-x-2}{(x+2)ln(x+3)}=\left[\frac{0}{0}\right]=\lim_{x\to -2}\frac{\frac{1}{x+3}-1}{ln(x+3)+\frac{x+2}{x+3}}=$$

$$=-\lim_{x\to -2}\frac{x+2}{(x+3)ln(x+3)+(x+2)}=\left[\frac{0}{0}\right]=-\lim_{x\to -2}\frac{1}{ln(x+3)+\frac{x+3}{x+3}+1}=-\frac{1}{2}.$$

- Indeterminate forms of type 1^∞, 0^0, ∞^0 are reduced to the indeterminate form $0\cdot\infty$ using prior logarithmation or using the identity:

$$[f(x)]^{g(x)}=e^{g(x)ln\,f(x)},\;f(x)>0.$$

Example 5

Find limits:

(1.) $\lim\limits_{x\to 0}\left(e^{2x}+x\right)^{\frac{1}{x}}$; (2.) $\lim\limits_{x\to 0^{+0}}x^x$; (3.) $\lim\limits_{x\to\frac{\pi}{2}^{-0}}(tgx)^{ctgx}$.

Solution:

(1.)

$$\lim_{x\to 0}(e^{2x}+x)^{\frac{1}{x}}=[1^\infty]=e^{\lim\limits_{x\to 0}\frac{1}{x}ln(e^{2x}+x)}=e^L,\text{ where}$$

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$$L=\lim_{x\to 0}\frac{ln(e^{2x}+x)}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{1}{e^{2x}+x}(2e^{2x}+1)=3.\quad \text{Now } \lim_{x\to 0}(e^{2x}+x)^{\frac{1}{x}}=e^L=e^3;$$

(2.) $\lim_{x\to 0^{+0}} x^x=[0^0]=e^{\lim_{x\to 0^{+0}}(x\,ln\,x)}=e^L,$ where

$$L=\lim_{x\to 0^{+0}}(x\,ln\,x)=[0\cdot\infty]=\lim_{x\to 0^{+0}}\frac{ln\,x}{\frac{1}{x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0^{+0}}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0.\quad \text{Now } \lim_{x\to 0^{+0}}x^x=e^L=e^0$$
$$=1;$$

(3.)

$$\lim_{x\to\frac{\pi}{2}^{-0}}(tgx)^{ctgx}=[\infty^0]=e^{\lim_{x\to\frac{\pi}{2}^{-0}}ctgx\cdot ln(tgx)}=e^L,\text{ where}$$

$$L=\lim_{x\to\frac{\pi}{2}^{-0}}[ctgx\,ln(tgx)]=[0\cdot\infty]=\lim_{x\to\frac{\pi}{2}^{-0}}\frac{ln(tgx)}{tgx}=\left[\frac{\infty}{\infty}\right]=$$
$$=\lim_{x\to\frac{\pi}{2}^{-0}}\frac{\frac{1}{tgx}\cdot\frac{1}{cos^2x}}{\frac{1}{cos^2x}}=\lim_{x\to\frac{\pi}{2}^{-0}}c\,tgx=0.\ \text{Now } \lim_{x\to\frac{\pi}{2}^{-0}}(tgx)^{ctgx}=e^L=e^0=1.$$

Exercices 6.22

Find limits:

(1.) $\lim_{x\to 0}\frac{e^{5x}-1}{2x}$;

(2.) $\lim_{x\to 0}\frac{x-sin\,x}{x^3}$;

(3.) $\lim_{x\to\infty}\frac{ln^2x}{x}$;

(4.) $\lim_{x\to\infty}\frac{e^x}{x^3}$;

(5.) $\lim_{x\to\infty}\frac{\frac{\pi}{2}-arctgx}{\frac{1}{3}ln\frac{x+1}{x-1}}$;

(6.) $\lim_{x\to 0^{+0}}\frac{\arcsin\sqrt{x}}{\sqrt{2x-x^2}}$.

The solution:

(1.) $$\lim_{x\to 0}\frac{e^{5x}-1}{2x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(e^{5x}-1)'}{(2x)'}=\lim_{x\to 0}\frac{5e^{5x}}{2}=\frac{5}{2};$$

(2.) $$\lim_{x\to 0}\frac{x-sin\,x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(x-sin\,x)'}{(x^3)'}=\lim_{x\to 0}\frac{1-cos\,x}{3x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(1-cos\,x)'}{(3x^2)'}=$$

$$=\lim_{x\to 0}\frac{\sin x}{6x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{(6x)'}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6};$$

(3.) $$\lim_{x\to\infty}\frac{\ln^2 x}{x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln^2 x)'}{(x)'}=\lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}=\left[\frac{\infty}{\infty}\right]=$$

$$=2\lim_{x\to\infty}\frac{(\ln x)'}{(x)'}=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\frac{0}{1}=0;$$

(4.) $$\lim_{x\to\infty}\frac{e^x}{x^3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(e^x)'}{(x^3)'}=\lim_{x\to\infty}\frac{e^x}{3x^2}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(e^x)'}{(3x^2)'}=\lim_{x\to\infty}\frac{e^x}{6x}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{(e^x)'}{(6x)'}=\lim_{x\to\infty}\frac{e^x}{6}=\infty;$$

(5.) $$\lim_{x\to\infty}\frac{\frac{\pi}{2}-arctgx}{\frac{1}{3}\ln\frac{x+1}{x-1}}=\left[\frac{0}{0}\right]=\lim_{x\to\infty}\frac{\left(\frac{\pi}{2}-arctgx\right)'}{\left(\frac{1}{3}\ln\frac{x+1}{x-1}\right)'}=\lim_{x\to\infty}\frac{-\frac{1}{1+x^2}}{-\frac{2}{3(x^2-1)}}=\lim_{x\to\infty}\frac{3(x^2-1)}{2(x^2+1)}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left[3(x^2-1)\right]'}{\left[2(x^2-1)\right]'}=\lim_{x\to\infty}\frac{6x}{4x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(6x)'}{(4x)'}=\frac{6}{4}=\frac{3}{2};$$

(6.) $$\lim_{x\to 0^{+0}}\frac{\arcsin\sqrt{x}}{\sqrt{2x-x^2}}=\left[\frac{0}{0}\right]=\lim_{x\to 0^{+0}}\frac{\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}}{\frac{1(2-2x)}{2\sqrt{2x-x^2}}}=\lim_{x\to 0^{+0}}\frac{\frac{1}{2\sqrt{x}\cdot\sqrt{1-x}}}{\frac{1-x}{\sqrt{x}\cdot\sqrt{2-x}}}=$$

$$=\lim_{x\to 0^{+0}}\frac{\sqrt{2-x}}{2(1-x)\sqrt{1-x}}=\frac{\sqrt{2}}{2}.$$

Exercices 6.23

Find limits:

(1.) $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]$;

(2.) $\lim_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot ctg(x-a)\right]$.

The solution:

$$(1.)\ \lim_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]=\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}\quad=\left[\frac{0}{0}\right]\quad=\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{2x}=\lim_{t\to 0}\frac{x}{2x(1+x)}=\frac{1}{2};$$

$$(2.)\ \lim_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot ctg(x-a)\right]=[0\cdot\infty]\quad=\lim_{x\to a}\frac{\arcsin\frac{x-a}{a}}{tg(x-a)}=$$

$$=\left[\frac{0}{0}\right]\quad=\lim_{x\to \alpha}\frac{\frac{1}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}\cdot\frac{1}{a}}{\frac{1}{\cos^2(x-a)}}=\frac{1}{a}\cdot\lim_{x\to\alpha}\frac{\cos^2(x-a)}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}=\frac{1}{a}.$$

Exercices 6.24

Find limits:

$$(1.)\ \lim_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right];\qquad (2.)\ \lim_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot ctg(x-a)\right].$$

The solution:

$$(1.)\ \lim_{x\to 0}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]=\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}\quad=\left[\frac{0}{0}\right]\quad=\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{2x}=\lim_{t\to 0}\frac{x}{2x(1+x)}=\frac{1}{2};$$

$$(2.)\ \lim_{x\to a}\left[\arcsin\frac{x-a}{a}\cdot ctg(x-a)\right]=[0\cdot\infty]\quad=\lim_{x\to a}\frac{\arcsin\frac{x-a}{a}}{tg(x-a)}=$$

$$=\left[\frac{0}{0}\right]\quad=\lim_{x\to \alpha}\frac{\frac{1}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}\cdot\frac{1}{a}}{\frac{1}{\cos^2(x-a)}}=\frac{1}{a}\cdot\lim_{x\to\alpha}\frac{\cos^2(x-a)}{\sqrt{1-\left(\frac{x-a}{a}\right)^2}}=\frac{1}{a}.$$

Exercices 6.25

Find limits:

$$(1.)\ \lim_{x\to\frac{\pi}{2}}\left(\frac{x}{ctgx}-\frac{\pi}{2\cos x}\right);\qquad (2.)\ \lim_{x\to 0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right);$$

(3.) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-ctg^2 x\right)$.

The solution:

(1.) $$\lim_{x\to\frac{\pi}{2}}\left(\frac{x\sin x}{\cos x}-\frac{\pi}{2\cos x}\right)=[\infty-\infty]=\lim_{x\to\frac{\pi}{2}}\frac{2x\sin x-\pi}{2\cos x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\sin x+2x\cos x}{2(-\sin x)}=\frac{2}{-2}=-1.$$

(2.) $$\lim_{x\to 0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right)=[\infty-\infty]=\lim_{x\to 0}\frac{x-\sin x}{x^2\sin x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{1-\cos x}{2x\sin x+x^2\cos x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\sin x}{2\sin x+2x\cos x+2x\cos x+x^2(-\sin x)}=$$

$$=\lim_{x\to 0}\frac{\sin x}{2\sin x+4x\cos x-x^2\sin x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{\cos x}{2\cos x+4\cos x+4x(-\sin x)-2x\sin x-x^2\cos x}=\frac{1}{6}.$$

(3.) $$\lim_{x\to 0}\left(\frac{1}{x^2}-ctg^2 x\right)=[\infty-\infty]=\lim_{x\to 0}\frac{1-x^2ctg^2x}{x^2}=$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2\cos^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{(\sin x-x\cos x)(\sin x+x\cos x)}{x^2\sin x\cdot\sin x};$$

Since

$$\lim_{x\to 0}\frac{\sin x-x\cos x}{x^2\sin x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{2x\sin x+x^2\cos x}=$$

$$=\lim_{x\to 0}\frac{\sin x}{2\sin x+x\cos x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x}{2\cos x+\cos x+x(-\sin x)}=\frac{1}{3},$$

and

$$\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}=2,$$

it is

$$\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x^2\sin x}\right)\cdot\left(\frac{\sin x+x\cos x}{\sin x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x^2\sin x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x+x\cos x}{\sin x}\right)=\frac{2}{3}.$$

Exercices 6.26

Find limits:

(1.) $\lim\limits_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}}$;

(2.) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$;

(3.) ; $\lim\limits_{x\to 1}\left(tg\frac{\pi\cdot x}{4}\right)^{tg\frac{\pi\cdot x}{2}}$;

(4.) $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{tgx}$;

(5.) $\lim\limits_{x\to 0^{+0}} x^{\frac{3}{4+\ln x}}$;

(6.) $\lim\limits_{x\to 0^{+0}}(ctgx)^{\frac{1}{\ln x}}$;

(7.) $\lim\limits_{x\to 0} x^{\frac{1}{\ln\left(e^x-1\right)}}$;

(8.) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

The solution:

(1.) $\lim\limits_{x\to 0}\left(\frac{2}{\pi}arccos()^{\frac{1}{x}}[1^\infty]^{\lim\limits_{x\to 0}\frac{1}{x}ln\left(\frac{2}{\pi}arccos\,x\right)^{L}}\text{where}\right)$

$$\text{L} = \lim_{x\to 0}\frac{\ln\left(\frac{2}{\pi}\arccos x\right)}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\left[\left(\frac{\pi}{2\arccos x}\right)\cdot\left(-\frac{2}{\pi\sqrt{1-x^2}}\right)\right] = -\frac{2}{\pi};$$

$$\lim_{x\to 0}\left(\frac{2}{\pi}\arccos\right)^{\frac{1}{x}} = e^{-\frac{2}{\pi}}$$

(2.) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = [1^\infty] = e^{\lim\limits_{x\to 0}\frac{1}{x^2}ln\frac{\sin x}{x}} = e^L$, where

$$L = \lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{2x} =$$

$$= \lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3} = \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 0}\frac{\cos x + x(-\sin x)-\cos x}{6x^2} = -\lim_{x\to 0}\frac{\sin x}{6x} = -\frac{1}{6};$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$$

(3.) $\lim_{x\to 1}\left(tg\frac{\pi x}{4}\right)^{tg\frac{\pi x}{2}} = [1^\infty] = e^{\lim_{x\to 1}\left[tg\frac{\pi x}{2}\cdot ln\left(tg\frac{\pi x}{4}\right)\right]} = e^L$, where

$$L = \lim_{x\to 1}\left[tg\frac{\pi x}{2}\cdot\ln\left(tg\frac{\pi x}{4}\right)\right] = [\infty\cdot 0] = \lim_{x\to 1}\frac{\ln\left(tg\frac{\pi x}{4}\right)}{ctg\left(\frac{\pi x}{2}\right)} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 1}\frac{\frac{1}{tg\frac{\pi x}{4}}\cdot\frac{1}{cos^2\frac{\pi x}{4}}\cdot\frac{\pi}{4}}{-\frac{1}{sin^2\frac{\pi x}{2}}\cdot\frac{\pi}{2}} = \left(-\frac{1}{2}\right)\cdot\lim_{x\to 1}\frac{2\,sin^2\frac{\pi x}{2}}{2\,sin\frac{\pi x}{4}\cdot cos\frac{\pi x}{4}} =$$

$$= -\lim_{x\to 1}\frac{sin^2\frac{\pi x}{2}}{sin\frac{\pi x}{2}} = -\lim_{x\to 1}\left(sin\frac{\pi x}{2}\right) = -1;$$

$$\lim_{x\to 1}\left(tg\frac{\pi\cdot x}{4}\right)^{tg\frac{\pi\cdot x}{2}} = e^{-1}$$

(4.) $\lim_{x\to\frac{\pi}{2}}(sin\,x)^{tgx} = [1^\infty] = e^{\lim_{x\to\frac{\pi}{2}}[tgx\,ln(sin\,x)]} = e^L$, where

$$L = \lim_{x\to\frac{\pi}{2}}\left[tgx\cdot\ln(\sin x)\right] = [\infty\cdot 0] = \lim_{x\to\frac{\pi}{2}}\frac{\ln(\sin x)}{ctgx} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{sin\,x}\cdot cos\,x}{-\frac{1}{sin^2 x}} = -\lim_{x\to\frac{\pi}{2}}\left[cos\,x\cdot sin\,x\right] = 0;$$

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{tgx} = e^0 = 1$$

(5.) $\lim_{x\to 0^{+0}} x^{\frac{3}{4+\ln x}} = [0^0] = e^{\lim_{x\to 0^{+0}}\frac{3}{4+\ln x}\cdot \ln x} = e^L$, where

$$L = \lim_{x\to 0^{+0}} \frac{3\ln x}{4+\ln x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to 0^{+0}} \frac{\frac{3}{x}}{\frac{1}{x}} = 3;$$

$$\lim_{x\to 0^{+0}} x^{\frac{3}{4+\ln x}} = e^3$$

(6.) $\lim_{x\to 0^{+0}} (ctgx)^{\frac{1}{\ln x}} = [\infty^0] = e^{\lim_{x\to 0^{+0}}\left[\frac{1}{\ln x}\ln(ctgx)\right]} = e^L$, where

$$L = \lim_{x\to 0^{+0}} \frac{\ln(ctgx)}{\ln x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to 0^{+0}} \frac{\frac{1}{ctgx}\cdot\left(-\frac{1}{\sin^2 x}\right)}{\frac{1}{x}} =$$

$$= -\lim_{x\to 0^{+0}} \frac{x}{\cos x \sin x} = -\lim_{x\to 0^{+0}} \frac{2x}{\sin 2x} = \left[\frac{0}{0}\right] = -\lim_{x\to 0^{+0}} \frac{2}{\cos 2x\cdot 2} = -1;$$

$$\lim_{x\to 0^{+0}} (ctgx)^{\frac{1}{\ln x}} = e^{-1}$$

(7.) $\lim_{x\to 0} x^{\frac{1}{\ln(e^x-1)}} = [0^0] = \lim_{x\to 0} e^{\frac{1}{\ln(e^x-1)}\cdot \ln x} = e^{\lim_{x\to 0}\left[\frac{1}{\ln(e^x-1)}\cdot \ln x\right]} = e^L$, where

$$\lim_{x\to 0} \frac{\ln x}{\ln(e^x-1)} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{e^x}{e^x-1}} = \lim_{x\to 0} \frac{e^x-1}{xe^x} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{e^x}{e^x+xe^x} = 1;$$

$$\lim_{x\to 0} x^{\frac{1}{\ln(e^x-1)}} = e^1 = e$$

(8.) $$\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = [1^\infty] = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+\sin x)} = e^{\lim_{x\to 0}\left[\frac{\ln(1+\sin x)}{x}\right]} = e^{\lim_{x\to 0}\left[\frac{\frac{1}{1+\sin x}\cdot\cos x}{1}\right]} = e^1 = e.$$

6.7 Examining the flux and drawing a graph of a continuous real function

Examining the flux of a continuous real function f that is set analytically, i.e. by a formula, consists of the following steps:

1. Area of definition (natural domain), parity and periodicity

In order to determine the natural domain D_f of a given function f it is necessary to know the basic elementary functions and procedures for solving equations or inequations.

Definition (even, odd, and periodic function):

The function f is

a) **even** if

$$f(-x)=f(x) \text{ for all } x\in D_f;$$

b) **odd** if

$$f(-x)=-f(x) \text{ for all } x\in D_f;$$

c) **periodic** if for some $P\neq 0$ for all $x\in D_f$ is:

$$f(x+P)=f(x).$$

Definition (fundamental period of a function):

Suppose that the function f is periodic and q denotes the smallest positive number so that $f(x+q)=f(x)$ for all $x\in D_f$.

Such a number q is called the ***fundamental or basic period*** of the periodic function f.

If the domain D_f is determined, we examine whether the function f is even, odd, or periodic. This is useful to know because it can significantly help in further examination of the flux of a function.

Namely, if the function f is an even function, then its graph is symmetric with respect to the axis y therefore, it is sufficient to examine the flow of the function f only on the set $D_f \cap [0,+\infty\rangle$;

if the function f is an odd function, then its graph is centrally symmetric with respect to the origin, so again it is sufficient to examine the flow of the function f only on the set $D_f \cap [0,+\infty\rangle$;

if the function f is the periodic function with the fundamental period q, then it is sufficient to examine the flow of the function f only on the set $D_f \cap \left[-\frac{q}{2},\frac{q}{2}\right]$.

Important notes:

1) The function f can be **even or odd** only if the set D_f is symmetric with respect to zero.

2) The function f **cannot be periodic** unless it contains one of the trigonometric functions.

2. Intersections or touch points with coordinate axes

The procedure: for the equation $f(x)=0$.

If $x=x_0$ is the solution of that equation, then $N(x_0,0)$ is a **zero-point** of the function f.

If the equation $f(x)=0$ has no solution, then the function f has no zero-point.

The equation $f(x)=0$ can have multiple solutions, i.e. the function f can have multiple zeros (there may even be an infinite number of them).

If $0 \in D_f$, and the value of $f(0)$ is determined, then the point $(0, f(0))$ is the **touch point** or intersection of the function f with an axis y.
If $0 \notin D_f$, then touch point does not exist.

3. Asymptote

Definition (asymptote):
If $a \in \mathcal{R}$ the line $x=a$ is the right vertical asymptote of the function f if

$$\lim_{x\to a^{+0}} f(x)=\infty \text{ or } \lim_{x\to a^{+0}} f(x)=-\infty.$$

The line $x=a$ is the left vertical asymptote of the function f if

$$\lim_{x\to a^{-0}} f(x)=\infty \text{ or } \lim_{x\to a^{-0}} f(x)=-\infty.$$

The line $y=b$ is the right horizontal asymptote of the function f if

$$\lim_{x\to\infty} f(x) = b \in \mathcal{R}.$$

The line $y=c$ is the left horizontal asymptote of the function f if

$$\lim_{x\to\infty} f(x) = c \in \mathcal{R}.$$

If $\lim_{x\to\infty} \frac{f(x)}{x} = k_1 \in \mathcal{R}\backslash\{0\}$ and $\lim_{x\to\infty}[f(x)-k_1x] = l_1 \in \mathcal{R}$. Then

$$y=k_1x+l_1$$

is the **right oblique asymptote** of the function f.
If $\lim_{x\to-\infty} \frac{f(x)}{x} = k_2 \in \mathcal{R}\backslash\{0\}$ and $\mathrm{li}-\mathrm{m}_{x\to\infty} \frac{f(x)}{x} = k_2 \in \mathcal{R}\backslash\{0\}$ then

$$y=k_2x+l_2$$

is the **left oblique asymptote** of the function f.

Important notes:

1) The function f can have a **vertical** asymptote $x = a$ only if a is a point at the edge of the domain D_f where the function is not defined.

2) The function f can have more (even infinitely many) vertical asymptotes.

3) The function f cannot have a ***horizontal*** and an ***oblique*** asymptote on the same side of the graph.

Therefore, the function f cannot have

the right horizontal and right oblique

or

the left horizontal and left oblique asymptote.

However, the function f can have

the left horizontal and right oblique

or

the right horizontal and left oblique asymptote.

4) The function f does not have to have an asymptote.

The procedure: Taking into account the previous notes, all meaningful limits should be determined, and the equations of the corresponding asymptotes should be written. When drawing a graph of the function f, asymptotes are usually drawn with dashed lines.

4. Intervals of monotonicity and points of local extrema

Definition (open interval): The interval J is an *open interval* (in the set $\Re$) if J is one of the intervals in a form of $\langle -\infty, a\rangle$, $\langle a,b\rangle$, $\langle a,\infty\rangle$ for any real numbers a and b (of course, for the interval $\langle a,b\rangle$ where $a<b$).

Definition (monotonic function on open interval, intervals of monotonicity):

Suppose that the function f is defined on an open interval J.

a) The function f is *increasing* on J for each choice of different points $x_1, x_2 \in J$ if

$$f(x_1) \le f(x_2) \text{ where } x_1 < x_2.$$

In that case it is said that f is monotone on the interval J, and the interval J is called the **interval of monotonicity** of the function f.

b) The function f is decreasing on J for each choice of different points $x_1, x_2 \in J$ if

$$f(x_1) \ge f(x_2) \text{ where } x_1 < x_2.$$

In that case it is said that f is monotone on the interval J, and the interval J is called the **interval of monotonicity** of the function f.

Theorem (sufficient condition for monotonicity):

Suppose that the function f is the derivative function on an open interval J.

If $f'(x)>0$ for all $x\in J$, then f is **increasing** on J (i.e. it is monotonous on J).

If $f'(x)<0$ for all $x\in J$, then f is **decreasing** on J (i.e. it is monotonous on J).

Definition (point of local extremum):

Let the function f is defined on an open interval J and if $c\in J$.

a) If $\varepsilon>0$ then

$$f(x)\le f(c) \text{ for all } x\in\langle c-\varepsilon,c\rangle\cup\langle c,c+\varepsilon\rangle,$$

then the function f has a local extremum (namely a local maximum) at the point c, and $M(c,f(c))$ is the point of the local maximum of the function f.

b) If $\varepsilon>0$ then

$$f(x)\ge f(c) \quad \text{for all } x\in\langle c-\varepsilon,c\rangle\cup\langle c,c+\varepsilon\rangle,$$

then the function f has a *local extremum* (namely a local minimum) at the point c, and $m(c,f(c))$ is the point of the local minimum of the function f.

Definition (stationary point and critical point):

If the function f is defined on an open interval J and if $c\in J$.

The point $(c,f(c))$ is a *stationary point* of the function f if $f'(c)=0$.

The point $(c,f(c))$ is a *critical point* of the function f if $f'(c)=0$ or $f'(c)$ does not exist (in a set $\mathfrak{R}$).

Theorem (sufficient condition for the existence of a point of local extremum):

If the function f is defined on an open interval J and if f is derivable on J except eventually at the point $c\in J$.

If $\varepsilon>0$ then

$$f'(x)>0 \text{ for all } x\in\langle c-\varepsilon,c\rangle \text{ and } f'(x)<0 \text{ for all } x\in\langle c,c+\varepsilon\rangle,$$

then the function f has a local maximum at the point c, i.e. $M(c,f(c))$ is the point of the local maximum of the function f.

If $\varepsilon>0$ then

$$f'(x)<0 \text{ for all } x\in\langle c-\varepsilon,c\rangle \text{ and } f'(x)>0 \text{ for all } x\in\langle c,c+\varepsilon\rangle,$$

then the function f has a local minimum at the point c, i.e. $m(c,f(c))$ is the point of the local minimum of the function f.

Definition (function that changes the sign at a point):

If the function f is defined on an open interval J except eventually at the point $c\in J$.

It is said that the function f is a function that changes the sign at a point c if $\varepsilon>0$ then

$$f(x)<0 \text{ for all } x\in\langle c-\varepsilon,c\rangle \text{ and } f(x)>0 \text{ for all } x\in\langle c,c+\varepsilon\rangle$$

or

$$f(x)>0 \text{ for all } x\in\langle c-\varepsilon,c\rangle \text{ and } f(x)<0 \text{ for all } x\in\langle c,c+\varepsilon\rangle.$$

The procedure:

Firstly, the function f' should be determined, and then the set

$$S_1=\{x\in D_f : f'(x)=0 \text{ or } f'(x) \text{ does not exist}\}.$$

Therefore, the natural domain $D_{f'}$ of the function f' should be defined, and all possible solutions of the equation $f'(x)=0$.

The elements of the set S_1 together with the edges of the domain D_f of the function f determine the edges of the interval of monotonicity of the function f.

On each interval of monotonicity, on which the function f' is a continuous function (which has no zeros in that interval), the same procedure is applied:

1) One point of that interval is chosen and the value of the function f' is calculated at that point.
2) If this value is positive (**negative**), then the function f is increasing (**decreasing**) on that interval.

If the function f' is the function that changes sign at the point $c\in S_1$, then $(c,f(c))$ is the point of the local extremum of the function f.

The type of the extremum is determined, a point of the local maximum or local minimum, using the appropriate definition (definition of the point of local extremum).

5. Curvature intervals and inflection points

Definition (curved function on open interval, curvature interval):

Consider the function f which is defined on an open interval J.

a) The function f is ***strictly convex*** on J and if for all $x_1,x_2\in J$ then

$$f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2} \text{ when } x_1<x_2.$$

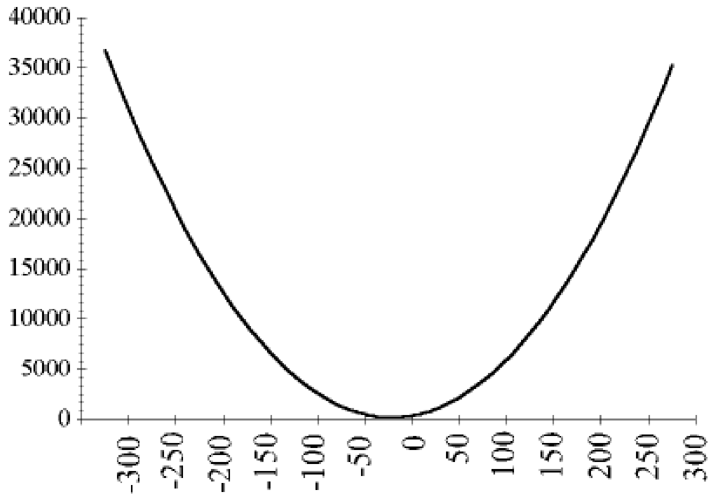
In that case it is said that f is curved on the interval J, and the interval J is called the interval of curvature of the function f.

b) The function f is ***strictly concave*** on J if for all $x_1,x_2\in J$ then

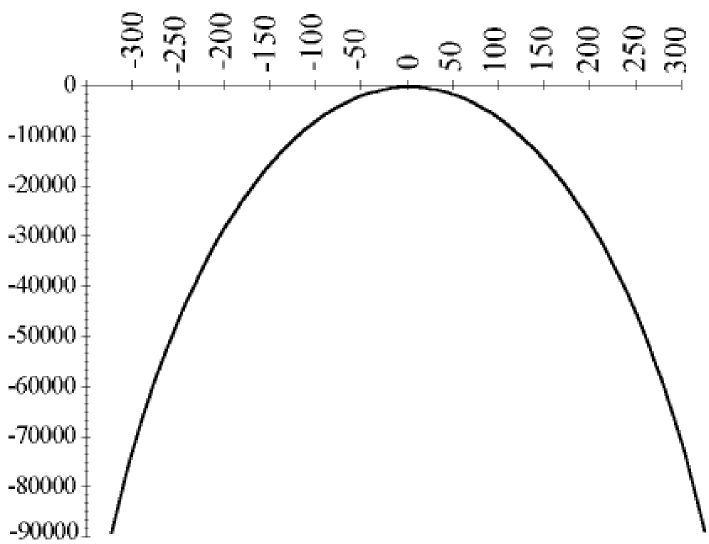
$$f\left(\frac{x_1+x_2}{2}\right)>\frac{f(x_1)+f(x_2)}{2} \text{ when } x_1<x_2.$$

In that case it is said that f is curved on the interval J, and the interval J is called the interval of curvature of the function f.

> The function f is strictly convex (strictly concave) on an open interval J if and only if at each point of that interval the tangent to the graph of the function f is belove (above) the graph of the function f.



Graph of a strictly convex function on an interval



Graph of a strictly concave function on an interval

Theorem (sufficient condition for a curve):

Suppose that the function f is double derivative on the open interval J.

If $f''(x) > 0$ for all $x \in J$, then f is strictly convex on J (i.e. it is curved on the interval J).

If $f''(x) < 0$ for all $x \in J$, then f is strictly concave on J (i.e. it is curved on the interval J).

Definition (point of inflection):

Suppose that the function f is defined on an open interval J and if $c \in J$.

If $\varepsilon > 0$ then f is strictly convex on the interval $\langle c-\varepsilon, c \rangle$ and strictly concave on the interval $\langle c, c+\varepsilon \rangle$, or vice versa, if f has an inflection at the point c, and $I(c, f(c))$ is the inflection point of the function f.

Theorem (sufficient condition for the existence of an inflection point):
If the function f is defined on an open interval J and if f is double derivative on J except eventually at the point $c \in J$. If the function f'' is a function that changes the sign at the point c, then the function f has an inflection at the point c, i.e. $I(c, f(c))$ is the inflection point of the function f.

The procedure:

Firstly, the function f'', is defined, and then the set

$$S_2 = \{x \in D_f : f''(x) = 0 \text{ or } f''(x) \text{ does not exist}\}.$$

Therefore, the natural domain $D_{f''}$ of the function f'' should be defined, and all possible solutions of the equation $f''(x) = 0$.

The elements of the set S_2 together with the edges of the domain D_f of the function f determine the edges of the curvature of the function f.

On each curvature interval, on which the function f'' is a continuous function (which has no zero points on that interval), the same procedure is applied

1) one point of that interval is chosen and the value of the function f'' is calculated at that point.
2) If this value is positive (negative) then the function f is strictly convex (strictly concave) on that interval.

If the function f'' is the function that changes sign at the point $c \in S_2$, then $I(c, f(c))$ is the inflection point of the function f.

6. Graph function

All obtained information about the function f through steps 1-5 should be merged into a coherent image. When drawing a graph, it is possible to detect all inconsistencies, i.e. errors in the previous calculation and correct them.

Example 1

Examine the flux of the function $y = f(x)$ and draw its graph, if:

$$f(x) = \frac{16}{x^2(x-4)}.$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined).
The same is true of its derivatives.

1. Area of definition (natural domain), parity and periodicity

$$D_f = \{x \in \mathcal{R} : \ x^2(x-4) \neq 0\} = \mathcal{R}\backslash\{0,4\} = \langle -\infty, 0\rangle \cup \langle 0,4\rangle \cup \langle 4, \infty\rangle$$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.
The function is not periodic because there are no trigonometric functions in its formula.

2. Intersections or touch points with coordinate axes

$f(x) \neq 0$ for all $x \in D_f$ so the function has no zeros.
$f(0)$ does not exist because $0 \notin D_f$. Therefore, the graph of the function f neither intersects nor touches the axis y.

3. Asymptotes

Possible vertical asymptotes are lines $x=0$ and $x=4$. Namely, 0 and 4 are points on the edge of the domain D_f where the function is not defined.

$$\lim_{x\to 0^{+0}} \frac{16}{x^2(x-4)} = -\infty \Rightarrow x=0$$ is the right vertical asymptote.

$$\lim_{x\to 0^{-0}} \frac{16}{x^2(x-4)} = -\infty \Rightarrow x=0$$ is the left vertical asymptote.

$$\lim_{x\to 4^{+0}} \frac{16}{x^2(x-4)} = \infty \Rightarrow x=4$$ is the right vertical asymptote.

$$\lim_{x\to 4^{-0}} \frac{16}{x^2(x-4)} = -\infty \Rightarrow x=4$$ is the left vertical asymptote.

The function could have horizontal asymptotes because the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are reasonable (in the domain D_f is possible that $x\to\infty$ and $x\to-\infty$).

$$\lim_{x\to\infty} \frac{16}{x^2(x-4)} = 0 \in \mathcal{R}$$

$\Rightarrow y=0$ is the right horizontal asymptote.

$$\lim_{x\to-\infty}\frac{16}{x^2(x-4)}=0\in\mathcal{R}$$

$\Rightarrow y=0$ is the left horizontal asymptote.

The function has right and left horizontal asymptotes so there are no oblique asymptotes.

4. Monotonicity intervals and points of local extrema

$$f'(x)=\frac{-16\left[2x(x-4)+x^2\right]}{x^4(x-4)^2}=\frac{16x(8-3x)}{x^4(x-4)^2}=\frac{16(8-3x)}{x^3(x-4)^2};$$

$D_{f'}=D_f;$

$f'(x)=0$

$$\frac{16(8-3x)}{x^3(x-4)^2}=0\Leftrightarrow 8-3x=0\Leftrightarrow x=\frac{8}{3};$$

$$f\left(\frac{8}{3}\right)=\frac{16}{\left(\frac{8}{3}\right)^2\left(\frac{8}{3}-4\right)}=\frac{16}{\frac{64}{9}\frac{-4}{3}}=-\frac{27}{16}.$$

Therefore, the only critical point of the given function is the stationary point $\left(\frac{8}{3},-\frac{27}{16}\right)$.

$S_1=\left\{\frac{8}{3}\right\}$

The edges of the domain D_f of the function f are

$$-\infty,0,4,\infty$$

so the intervals of monotonicity are:

$$\langle-\infty,0\rangle,\left\langle0,\frac{8}{3}\right\rangle,\left\langle\frac{8}{3},4\right\rangle,\langle4,\infty\rangle.$$

$f'(-1)<0\Rightarrow f$ is decreasing on $\langle-\infty,0\rangle$;

$f'(1)>0\Rightarrow f$ is increasing on $\left\langle0,\frac{8}{3}\right\rangle$;

$f'(3)<0\Rightarrow f$ is decreasing on $\left\langle\frac{8}{3},4\right\rangle$;

$f'(5)<0\Rightarrow f$ is decreasing on $\langle4,\infty\rangle$.

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The point of the local extremum of the function f can only be the critical point of the function f.
Therefore

$$f'(x)>0 \text{ for all } x\in\left\langle 0,\frac{8}{3}\right\rangle \text{ (as } f'(1)>0\text{)}$$

and

$$f'(x)<0 \text{ for all } x\in\left\langle \frac{8}{3},4\right\rangle \text{ (as } f'(3)<0\text{)},$$

then $M\left(\frac{8}{3},-\frac{27}{16}\right)$ is the point of the local maximum of the function f.

5. Curvature intervals and inflection points

$$f''(x)=\frac{16}{x^6(x-4)^4}\left\{-3x^3(x-4)^2-\left[3x^2(x-4)^2+2x^3(x-4)\right](8-3x)\right\}=$$

$$=\frac{16x^2(x-4)}{x^6(x-4)^4}\left\{-3x(x-4)-\left[3(x-4)+2x\right](8-3x)\right\}=$$

$$=\frac{16}{x^4(x-4)^3}\left[-3x(x-4)-(5x-12)(8-3x)\right]=$$

$$=\frac{16}{x^4(x-4)^3}\left(-3x^2+12x-76x+15x^2+96\right)=\frac{16}{x^4(x-4)^3}\left(12x^2-64x+96\right)=$$

$$=\frac{64}{x^4(x-4)^3}\left(3x^2-16x+24\right);$$

$D_{f''}=D_f$;

$f''(x)\neq 0$ for all $x\in D_{f''}$ because the equation $3x^2-16x+24=0$ has no real solutions.

Therefore, $S_2=\varnothing$ the set function has no inflection points.

The edges of the domain D_f of the function f are:

$$-\infty,0,4,\infty$$

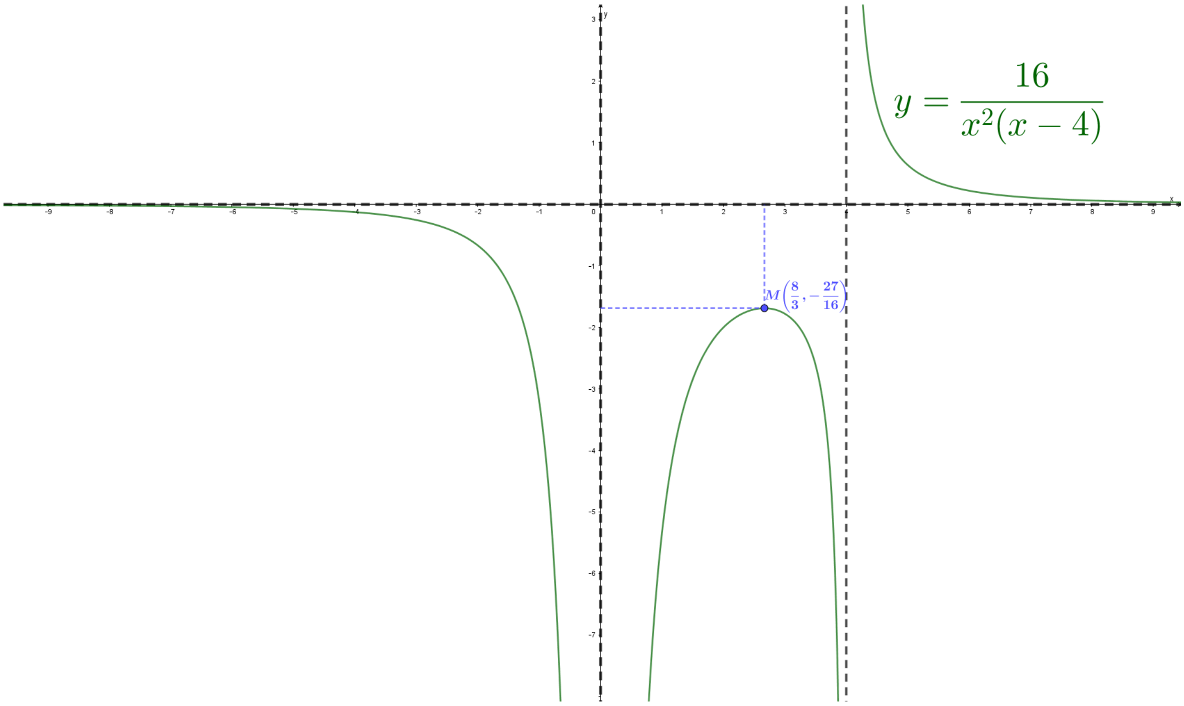
so the curvature intervals are:

$$\langle -\infty,0\rangle,\langle 0,4\rangle,\langle 4,\infty\rangle.$$

$f''(-1)<0\Rightarrow f$ is strictly concave on $\langle -\infty,0\rangle$;
$f''(1)<0\Rightarrow f$ is strictly concave on $\langle 0,4\rangle$;

$f''(5) > 0 \Rightarrow f$ is strictly convex on $\langle 4, \infty \rangle$.

6. Graph function



Example 2

$$f(x) = x \ln^2 x.$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined). The same is true of its derivatives.

1. Area of definition (natural domain), parity and periodicity

$$D_f = \{x \in \mathcal{R} : \ ln(x) \in \mathcal{R}\} = \mathcal{R}^+ = \langle 0, \infty \rangle$$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.
The function is not periodic because there are no trigonometric functions in its formula.

2. Intersections or touch points with coordinate axes

$f(x)=0 \Leftrightarrow x\ln^2 x=0 \Leftrightarrow \ln^2 x=0 \Leftrightarrow \ln x=0 \Leftrightarrow x=1.$

Therefore, $N(1,0)$ is the zero of the function.

$f(0)$ does not exist because $0 \notin D_f$. Therefore, the graph of the given function neither intersects nor touches the axis y.

3. Asymptotes

A possible right vertical asymptote is a line $x=0$. Namely, zero is a point at the edge of the domain D_f where the function is not defined, and only the right limit at that point is sensible because the set function is not defined to the left of zero.

$$\lim_{x\to 0^{+0}} x\ln^2 x=[0\cdot\infty]=\lim_{x\to 0^{+0}}\frac{\ln^2 x}{\frac{1}{x}}=\left[\frac{\infty}{\infty}\right]\overset{\text{LP}}{=}\lim_{x\to 0^{+0}}\frac{2\ln x\cdot\frac{1}{\cancel{x}}}{-\frac{1}{x^{\cancel{2}}}}=\lim_{x\to 0^{+0}}\frac{2\ln x}{-\frac{1}{x}}=$$

$$=\left[\frac{-\infty}{-\infty}\right]=\lim_{x\to 0^{+0}}\frac{\frac{2}{\cancel{x}}}{\frac{1}{x^{\cancel{2}}}}=2\lim_{x\to 0^{+}} x=0.$$

Note: Equality $\overset{\text{LP}}{=}$ is obtained by applying the L'Hospital's Rule.

So, when $x\to 0^{+0}$ then $y\to 0^{+0}$ therefore $x=0$ is not the right vertical asymptote of the given function.

The function could only have a right horizontal asymptote because only the limit $\lim_{x\to\infty} f(x)$ is sensible (in the domain D_f is possible that $x\to\infty$, but not that $x\to-\infty$).

$$\lim_{x\to\infty} x ln^2 x=\infty\notin\mathcal{R}$$

It can be concluded that the given function has no horizontal asymptotes.

The function could have only the right oblique asymptote because only the limits

$\lim_{x\to\infty}\frac{f(x)}{x}=k_1$ and $\lim_{x\to\infty}\left[f(x)-k_1x\right]=l_1$

are reasonable (in the domain D_f is possible that $x\to\infty$, but not that $x\to-\infty$).

$$k_1 = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x ln^2 x}{x} = \infty \notin R$$

so the function has no oblique asymptotes.

4. Monotonicity intervals and points of local extremes

$$f'(x)=\ln^2 x+\cancel{x}\cdot 2\ln x\cdot\frac{1}{\cancel{x}}=\ln x(\ln x+2);$$

$$D_{f'}=D_f;$$

$$f'(x)=0$$
$$\ln x(\ln x+2)=0\Leftrightarrow \ln x=0 \text{ ili } \ln x=-2\Leftrightarrow x=1 \text{ ili } x=e^{-2};$$

$$f(1)=1\ln^2 1=0,\quad f\left(e^{-2}\right)=e^{-2}\ln^2\left(e^{-2}\right)=e^{-2}\cdot(-2)^2=4e^{-2}.$$

Therefore, the critical points of the set function are stationary points $(1,0)$ and $\left(e^{-2},4e^{-2}\right)$.

$$S_1=\left\{e^{-2},1\right\}$$

The edges of the domain D_f of the function f are:

$$0,\infty$$

so the intervals of monotonicity are:

$$\left\langle 0,e^{-2}\right\rangle,\left\langle e^{-2},1\right\rangle,\langle 1,\infty\rangle.$$

$f'\left(e^{-3}\right)>0\Rightarrow f$ is increasing on $\left\langle 0,e^{-2}\right\rangle$;
$f'\left(e^{-1}\right)<0\Rightarrow f$ is decreasing on $\left\langle e^{-2},1\right\rangle$;
$f'(e)>0\Rightarrow f$ is increasing on $\langle 1,\infty\rangle$.

The point of the local extremum of the function f can only be the critical point of the function f.
Therefore

$f'(x)>0$ for all $x\in\left\langle 0,e^{-2}\right\rangle$ (as $f'\left(e^{-3}\right)>0$)

and

$f'(x)<0$ for all $x\in\left\langle e^{-2},1\right\rangle$ (as $f'\left(e^{-1}\right)<0$),

then $M\left(e^{-2},4e^{-2}\right)$ is the point of the local maximum of the function f.

Therefore

$f'(x)<0$ for all $x\in\left\langle e^{-2},1\right\rangle$ (as $f'\left(e^{-1}\right)<0$),

and

$$f'(x) > 0 \text{ for all } x \in \langle 1, \infty \rangle \text{ (as } f'(e) > 0\text{),}$$

then $m(1,0)$ is the point of the local minimum of the function f.

5. Curvature intervals and inflection points

$f''(x) = 2\ln x \cdot \frac{1}{x} + 2 \cdot \frac{1}{x} = \frac{2}{x}(\ln x + 1);$

$D_{f''} = D_f;$

$f''(x) = 0$

$\frac{2}{x}(\ln x + 1) = 0 \Leftrightarrow \ln x + 1 = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = e^{-1};$

$f(e^{-1}) = e^{-1};$

$S_2 = \{e^{-1}\}$

The edges of the domain D_f of the function f are:

$$0, \infty$$

so the curvature intervals are:

$$\langle 0, e^{-1} \rangle, \langle e^{-1}, \infty \rangle.$$

$f''(e^{-2}) < 0 \Rightarrow f$ is strictly concave on $\langle 0, e^{-1} \rangle$;

$f''(1) > 0 \Rightarrow f$ is strictly convex on $\langle e^{-1}, \infty \rangle$.

The inflection point of the function f can only be the point (e^{-1}, e^{-1}).

Therefore

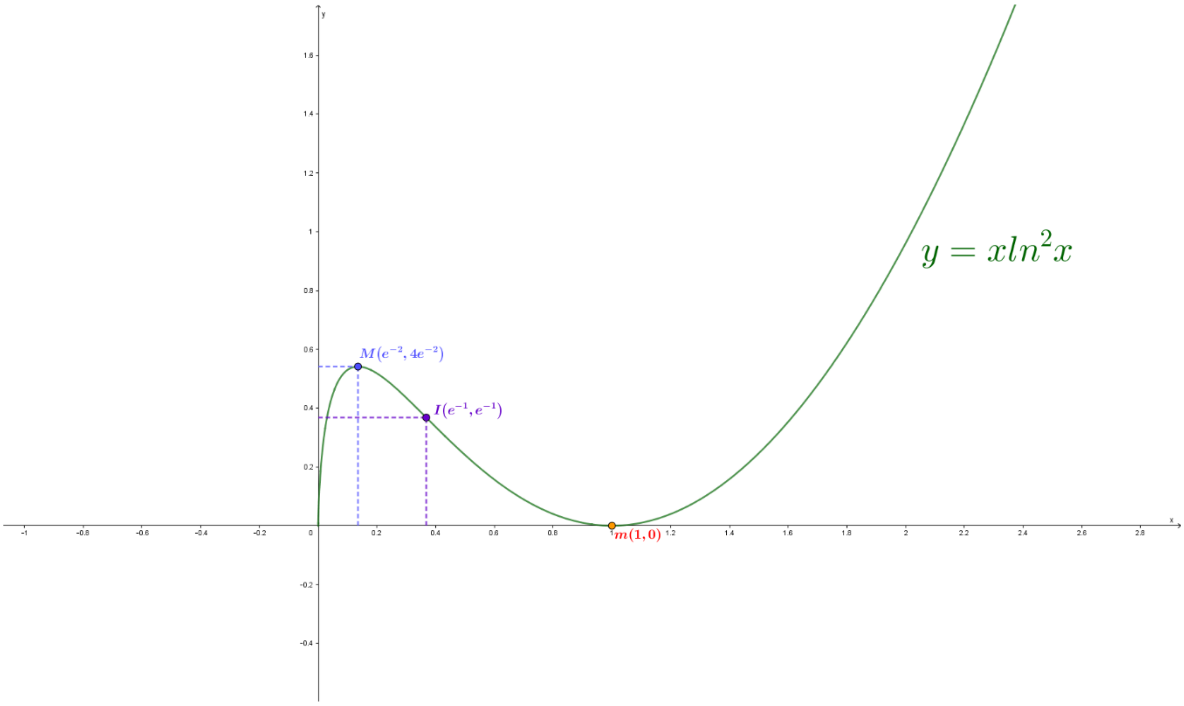
$$f''(x) < 0 \text{ for all } x \in \langle 0, e^{-1} \rangle \text{ (as } f''(e^{-2}) < 0\text{),}$$

and

$$f''(x) > 0 \text{ for all } x \in \langle e^{-1}, \infty \rangle \text{ (as } f''(1) > 0\text{),}$$

then $I(e^{-1}, e^{-1})$ is the point of inflection of the function f.

6. Graph function



Example 3

$$f(x) = xe^{-x^2}.$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined). The same is true of its derivatives.

1. Area of definition (natural domain), parity and periodicity

$D_f = R = \langle -\infty, \infty \rangle$.
For all $x \in D_f$ is:

$$f(-x) = -xe^{-(-x)^2} = -xe^{-x^2} = -f(x)$$

therefore the function f is an odd function. So, the flux of the function only at the set is examined

$$D_f \cap [0, \infty\rangle = R \cap [0, \infty\rangle = [0, \infty\rangle.$$

The function is not periodic because there are no trigonometric functions in its formula.

2. Intersections or touch points with coordinate axes

$f(x)=0 \Leftrightarrow xe^{-x^2}=0 \Leftrightarrow x=0.$

$N(0,0)$ is the zero and intersection with the axis y.

3. Asymptotes

$D_f = R$ so the function has no vertical asymptotes.

The function could only have a right horizontal asymptote because only the limit $\lim_{x\to\infty} f(x)$ is sensible (in the domain D_f is possible that $x\to\infty$).

$$\lim_{x\to\infty} xe^{-x^2}=[\infty\cdot 0]=\lim_{x\to\infty}\frac{x}{e^{x^2}}=\left[\frac{\infty}{\infty}\right]\overset{\text{LP}}{=}\lim_{x\to\infty}\frac{1}{2xe^{x^2}}=0$$

$\Rightarrow y=0$ is the right horizontal asymptote.

The function does not have the right oblique asymptote because it has the right horizontal asymptote.

4. Monotonicity intervals and points of local extremes

$f'(x)=e^{-x^2}-2x^2e^{-x^2}=(1-2x^2)e^{-x^2};$

$D_{f'}=D_f;$

$f'(x)=0$

$(1-2x^2)e^{-x^2}=0 \Leftrightarrow 1-2x^2=0 \overset{x\geq 0}{\Rightarrow} x=\frac{1}{\sqrt{2}}\approx 0.707107;$

$f\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}e^{-1/2}.$

$S_1=\left\{\frac{1}{\sqrt{2}}\right\}$

The edges of the interval $\left[0,\infty\right\rangle$ are:

$$0,\infty$$

so the monotonicity intervals (on the interval $\left[0,\infty\right\rangle$):

$$\left\langle 0,\frac{1}{\sqrt{2}}\right\rangle,\left\langle \frac{1}{\sqrt{2}},\infty\right\rangle.$$

$f'\left(\frac{1}{2}\right)>0\Rightarrow f$ is increasing on $\left\langle 0,\frac{1}{\sqrt{2}}\right\rangle;$

$f'(1)<0 \Rightarrow f$ is decreasing on $\left\langle \frac{1}{\sqrt{2}}, \infty \right\rangle$.

The point of the local extremum of the function f can only be the critical point of the function f.

Therefore

$$f'(x)>0 \text{ for all } x\in\left\langle 0,\frac{1}{\sqrt{2}}\right\rangle \text{ (as } f'\left(\frac{1}{2}\right)>0\text{)}$$

and

$$f'(x)<0 \text{ for all } x\in\left\langle \frac{1}{\sqrt{2}},\infty\right\rangle \text{ (as } f'(1)<0\text{)},$$

then $M\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}e^{-1/2}\right)$ is the point of the local maximum of the function f.

5. Curvature intervals and inflection points

$$f''(x)=-4xe^{-x^2}-2x\left(1-2x^2\right)e^{-x^2}=2x\left(2x^2-3\right)e^{-x^2};$$

$$D_{f''}=D_f;$$

$$f''(x)=0$$

$$2x\left(2x^2-3\right)e^{-x^2} \Leftrightarrow x\left(2x^2-3\right)=0 \overset{x\geq 0}{\Rightarrow} x=0 \text{ ili } x=\sqrt{\frac{3}{2}}\approx 1.22474;$$

$$f(0)=0,\quad f\left(\sqrt{\frac{3}{2}}\right)=\sqrt{\frac{3}{2}}e^{-3/2}.$$

$$S_2=\left\{0,\sqrt{\frac{3}{2}}\right\}$$

The edges of the interval $\left[0,\infty\right\rangle$ are:

$$0,\infty$$

so the curvature intervals (on the interval $\left[0,\infty\right\rangle$):

$$\left\langle 0,\sqrt{\frac{3}{2}}\right\rangle,\left\langle \sqrt{\frac{3}{2}},\infty\right\rangle.$$

$f''(1)<0 \Rightarrow f$ is strictly concave on $\left\langle 0,\sqrt{\frac{3}{2}}\right\rangle$;

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$f''(2)>0 \Rightarrow f$ is strictly convex on $\left\langle \sqrt{\frac{3}{2}},\infty \right\rangle$.

Therefore

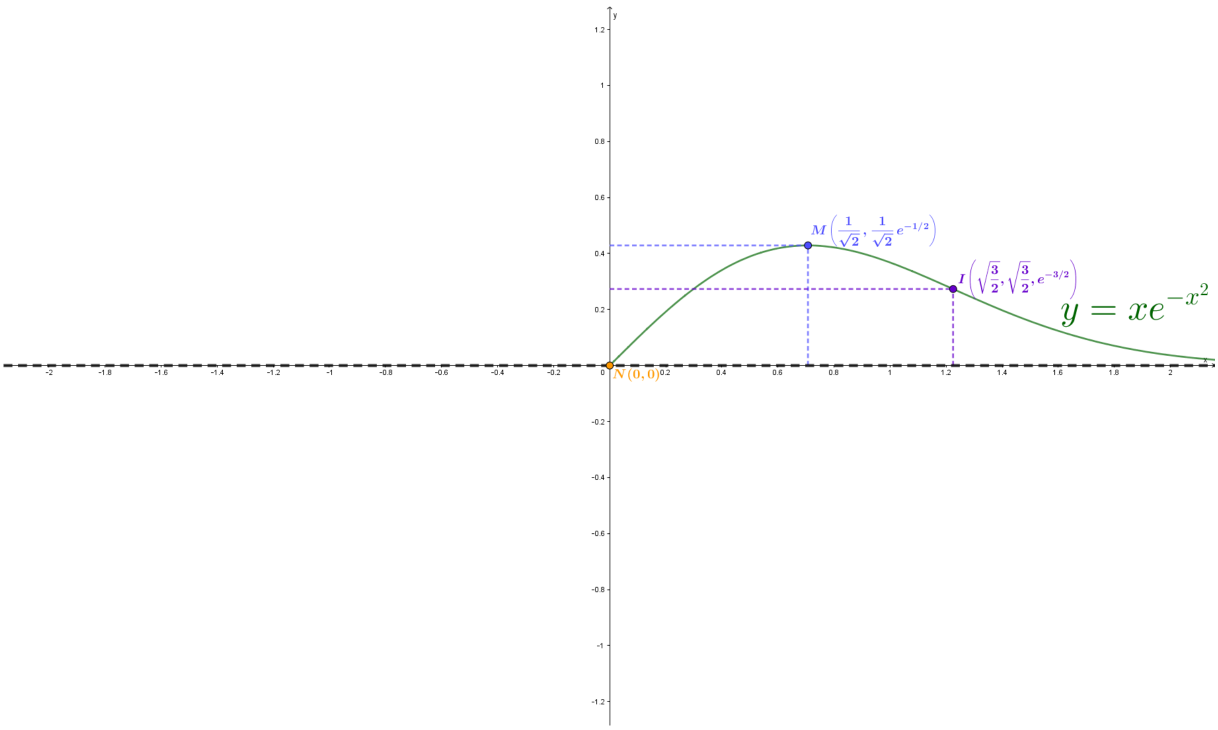
$$f''(x)<0 \text{ for all } x\in\left\langle 0,\sqrt{\frac{3}{2}} \right\rangle \text{ (as } f''(1)<0\text{)},$$

and

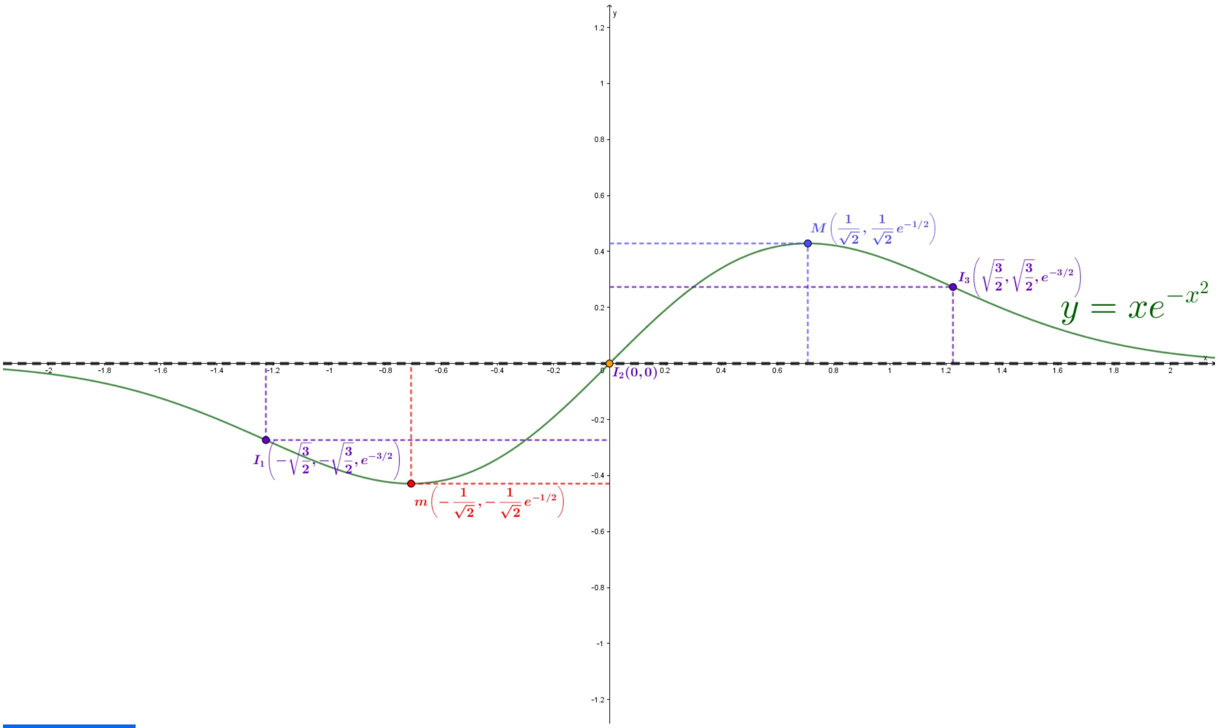
$$f''(x)>0 \text{ for all } x\in\left\langle \sqrt{\frac{3}{2}},\infty \right\rangle \text{ (as } f''(2)>0\text{)},$$

then $I\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}e^{-3/2}\right)$ is the point of the inflection of the function f.

6a. Graph of the function on the interval $\left[0,\infty\right\rangle$



6b. Graph function



Example 4

$$f(x) = x \cdot e^{\frac{1}{x-2}}.$$

Solution:

The function is elementary and therefore continuous (on each point where it is defined). The same is true of its derivatives.

1. Area of definition (natural domain), parity and periodicity $D_f = \mathcal{R}\backslash\{2\} = \langle -\infty, 2\rangle \cup \langle 2, \infty\rangle$

The function is neither even nor odd because the domain is not a symmetric set with respect to zero.
The function is not periodic because there are no trigonometric functions in its formula.

2. Intersections or touch points with coordinate axes

$$f(x) = 0 \Leftrightarrow xe^{\frac{1}{x-2}} = 0 \Leftrightarrow x = 0.$$

$N(0,0)$ is the zero and the intersection with the axis y.

3. Asymptotes

A possible right vertical asymptote is the line $x = 2$. Namely, 2 is the point on the edge of the domain D_f where the function is not defined. Obviously both limits

$$\lim_{x\to 2^{+0}} f(x) \text{ are } \lim_{x\to 2^{-0}} f(x)$$

are sensible.

$\lim\limits_{x\to 2^{+0}} xe^{\frac{1}{x-2}} = \infty$ so $x=2$ is the right vertical asymptote.

$\lim\limits_{x\to 2^{-0}} xe^{\frac{1}{x-2}} = 0.$

The function could have horizontal asymptotes because the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are sensible (in the domain D_f is possible that $x\to\infty$ and $x\to-\infty$).

$$\left.\begin{array}{l}\lim\limits_{x\to\infty} xe^{\frac{1}{x-2}} = \infty \notin R \\ \lim\limits_{x\to-\infty} xe^{\frac{1}{x-2}} = -\infty \notin R\end{array}\right\} \Rightarrow \text{ the function has no horizontal asymptotes.}$$

The function could have oblique asymptotes because all the associated limits are sensible (in the domain D_f is possible that $x\to\infty$ and $x\to-\infty$).

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{\cancel{x}e^{\frac{1}{x-2}}}{\cancel{x}} = 1 = k_1 \in R\setminus\{0\};$$

$$\lim_{x\to\infty}\left[f(x)-k_1x\right] = \lim_{x\to\infty}\left(xe^{\frac{1}{x-2}} - x\right) = [\infty-\infty] = \lim_{x\to\infty} x\left(e^{\frac{1}{x-2}}-1\right) = [\infty\cdot 0] =$$

$$= \lim_{x\to\infty}\frac{e^{\frac{1}{x-2}}-1}{\frac{1}{x}} = \left[\frac{0}{0}\right] \overset{\text{LP}}{=} \lim_{x\to\infty}\frac{-\frac{1}{(x-2)^2}e^{\frac{1}{x-2}}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^2}{(x-2)^2}e^{\frac{1}{x-2}} =$$

$$= \underbrace{\lim_{x\to\infty}\frac{x^2}{x^2-4x+4}}_{=1}\cdot\underbrace{\lim_{x\to\infty} e^{\frac{1}{x-2}}}_{=1} = 1 = l_1 \in R$$

so $y = x+1$ is the right oblique asymptote.

$$\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{\not{x}e^{\frac{1}{x-2}}}{\not{x}}=1=k_2\in R\setminus\{0\};$$

$$\lim_{x\to-\infty}\left[f(x)-k_2x\right]=\lim_{x\to-\infty}\left(xe^{\frac{1}{x-2}}-x\right)=\left[-\infty+\infty\right]=\lim_{x\to-\infty}x\left(e^{\frac{1}{x-2}}-1\right)=\left[-\infty\cdot 0\right]=$$

$$=\lim_{x\to-\infty}\frac{e^{\frac{1}{x-2}}-1}{\frac{1}{x}}=\left[\frac{0}{0}\right]\stackrel{\text{LP}}{=}\lim_{x\to-\infty}\frac{-\frac{1}{(x-2)^2}e^{\frac{1}{x-2}}}{-\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{x^2}{(x-2)^2}e^{\frac{1}{x-2}}=$$

$$=\underbrace{\lim_{x\to-\infty}\frac{x^2}{x^2-4x+4}}_{=1}\cdot\underbrace{\lim_{x\to-\infty}e^{\frac{1}{x-2}}}_{=1}=1=l_2\in R$$

so $y=x+1$ is the left oblique asymptote.

4. Monotonicity intervals and points of local extremes

$$f'(x)=e^{\frac{1}{x-2}}-\frac{x}{(x-2)^2}e^{\frac{1}{x-2}}=e^{\frac{1}{x-2}}\left[1-\frac{x}{(x-2)^2}\right]=\frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}};$$

$$D_{f'}=D_f;$$

$$f'(x)=0$$
$$\frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}}=0\Leftrightarrow x^2-5x+4=0\Leftrightarrow x=1\ \text{ ili }\ x=4.$$

$$f(1)=e^{-1},\quad f(4)=4e^{1/2}.$$

Therefore, the critical points of the given function are stationary points $\left(1,e^{-1}\right)$ and $\left(4,4e^{1/2}\right)$.
$S_1=\{1,4\}$

The edges of the domain D_f of the function f are:

$$-\infty,2,\infty$$

so the intervals of monotonicity are:

$$\langle-\infty,1\rangle,\langle1,2\rangle,\langle2,4\rangle,\langle4,\infty\rangle.$$

$f'(0)>0\Rightarrow f$ is increasing on $\langle-\infty,1\rangle$;

$f'\left(\frac{3}{2}\right)<0\Rightarrow f$ is decreasing on $\langle1,2\rangle$;

$f'(3) < 0 \Rightarrow f$ is decreasing on $\langle 2,4 \rangle$;
$f'(5) > 0 \Rightarrow f$ is increasing on $\langle 4,\infty \rangle$.

The points of the local extremes of the function f can be only critical points of that function.

Therefore

$$f'(x) > 0 \text{ for all } x \in \langle -\infty, 1 \rangle \text{ (as } f'(0) > 0\text{)}$$

and

$$f'(x) < 0 \text{ for all } x \in \langle 1, 2 \rangle \text{ (as } f'\left(\frac{3}{2}\right) < 0\text{)},$$

then $M\left(1, e^{-1}\right)$ is the point of the local maximum of the function f.

Therefore

$$f'(x) < 0 \text{ for all } x \in \langle 2, 4 \rangle \text{ (as } f'(3) < 0\text{)}$$

and

$$f'(x) > 0 \text{ for all } x \in \langle 4, \infty \rangle \text{ (as } f'(5) > 0\text{)},$$

then $m\left(4, 4e^{1/2}\right)$ is the point of the local minimum of the function f.

5. Curvature intervals and inflection points

$$f''(x)=\frac{\left[(2x-5)e^{\frac{1}{x-2}}-\frac{x^2-5x+4}{(x-2)^2}e^{\frac{1}{x-2}}\right](x-2)^2-2(x-2)\left(x^2-5x+4\right)e^{\frac{1}{x-2}}}{(x-2)^4}=$$

$$=\frac{(2x-5)(x-2)-\frac{x^2-5x+4}{x-2}-2\left(x^2-5x+4\right)}{(x-2)^3}e^{\frac{1}{x-2}}=$$

$$=\frac{\cancel{2x^2}-9x+10-\frac{x^2-5x+4}{x-2}\cancel{-2x^2}+10x-8}{(x-2)^3}e^{\frac{1}{x-2}}=\frac{x+2-\frac{x^2-5x+4}{x-2}}{(x-2)^3}e^{\frac{1}{x-2}}=$$

$$=\frac{(x+2)(x-2)-x^2+5x-4}{(x-2)^4}e^{\frac{1}{x-2}}=\frac{\cancel{x^2}-4\cancel{-x^2}+5x-4}{(x-2)^4}e^{\frac{1}{x-2}}=\frac{5x-8}{(x-2)^4}e^{\frac{1}{x-2}};$$

$$D_{f''}=D_f;$$

$$f''(x)=0$$

$$\frac{5x-8}{(x-2)^4}e^{\frac{1}{x-2}}=0\Leftrightarrow 5x-8=0\Leftrightarrow x=\frac{8}{5}.$$

$$f\left(\frac{8}{5}\right)=\frac{8}{5}e^{-5/2}.$$

$$S_2=\left\{\frac{8}{5}\right\}$$

The edges of the domain D_f of the function f are:

$$-\infty,2,\infty$$

so the curvature intervals are:

$$\left\langle-\infty,\frac{8}{5}\right\rangle,\left\langle\frac{8}{5},2\right\rangle,\left\langle 2,\infty\right\rangle.$$

$f''(1)<0\Rightarrow f$ is strictly concave on $\left\langle-\infty,\frac{8}{5}\right\rangle$;

$f''\left(\frac{9}{5}\right)>0\Rightarrow f$ is strictly convex on $\left\langle\frac{8}{5},2\right\rangle$;

$f''(3)>0\Rightarrow f$ is strictly convex on $\left\langle 2,\infty\right\rangle$.

Therefore

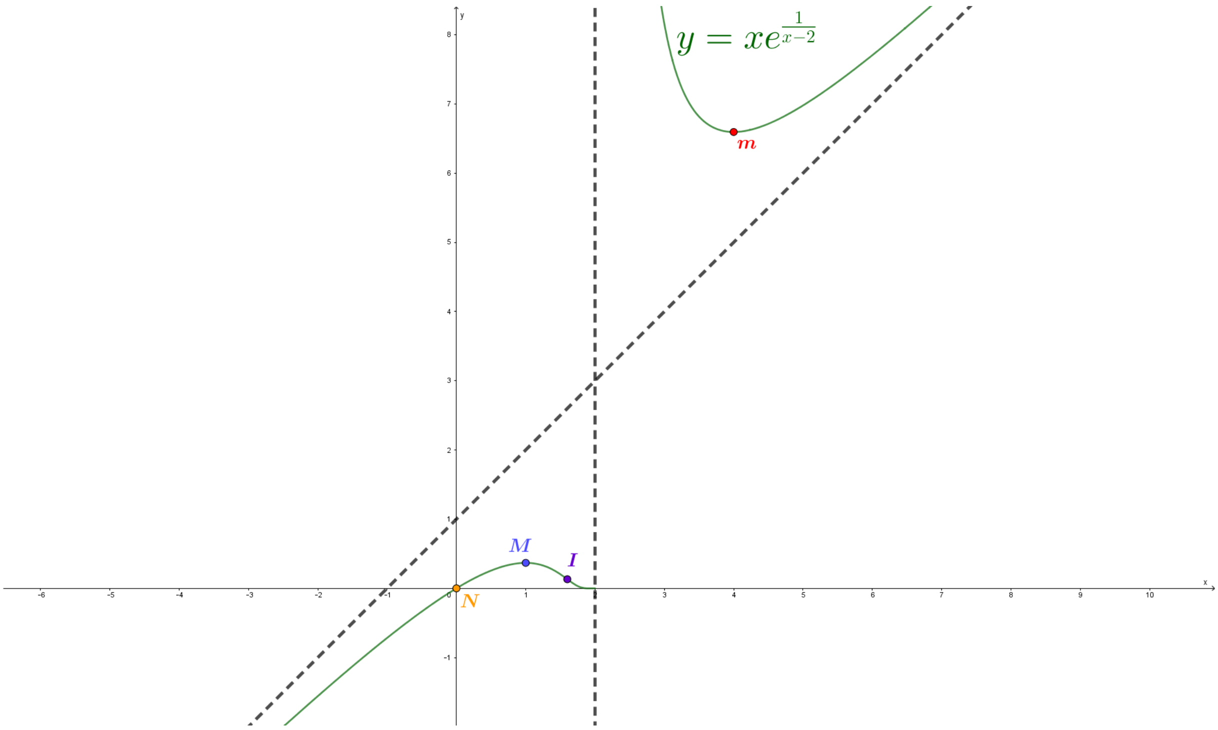
$$f''(x) < 0 \text{ for all } x \in \left\langle -\infty, \frac{8}{5} \right\rangle \text{ (as } f''(1) < 0\text{)},$$

and

$$f''(x) > 0 \text{ for all } x \in \left\langle \frac{8}{5}, 2 \right\rangle \text{ (as } f''\left(\frac{9}{5}\right) > 0\text{)},$$

then $I\left(\frac{8}{5}, \frac{8}{5}e^{-5/2}\right)$ is the point of inflection of the function f.

6. Graph function



Example 5

$$f(x)=\sqrt{|x|}\left(2-\ln x^2\right).$$

Solution:
The function is elementary and therefore continuous (on each point where it is defined). The same is true of its derivatives.

1. Area of definition (natural domain), parity and periodicity

$$D_f=R\setminus\{0\}=\langle-\infty,0\rangle\cup\langle0,\infty\rangle.$$

For all $x\in D_f$:

$$f(-x)=\sqrt{|-x|}\left[2-\ln(-x)^2\right]=\sqrt{|x|}\left(2-\ln x^2\right)=f(x)$$

So the function f is a pair function. Therefore, it is sufficient to examine the flux of the function only for the set

$$D_f\cap[0,\infty\rangle=R\cap[0,\infty\rangle=\langle0,\infty\rangle.$$

for $x>0$ is:

$$f(x)=\sqrt{x}\left(2-2\ln x\right)=2\sqrt{x}\left(1-\ln x\right).$$

The function is not periodic because there are no trigonometric functions in its formula.

2. Intersections or touch points with coordinate axes

$$f(x)=0\Leftrightarrow 2\sqrt{x}(1-\ln x)=0\Leftrightarrow 1-\ln x=0\Leftrightarrow \ln x=1\Leftrightarrow x=e.$$

Therefore, $N(e,0)$ is the only zero of the set function on the interval $\langle0,\infty\rangle$.

$f(0)$ does not exist $0\notin D_f$. Therefore, the graph of the function neither intersects nor touches the axis y.

3. Asymptotes

$$\lim_{x\to0^{+0}}f(x)=\lim_{x\to0^{+0}}2\sqrt{x}(1-\ln x)=[0\cdot\infty]=\lim_{x\to0^{+0}}\frac{1-\ln x}{\frac{1}{2\sqrt{x}}}=\left[\frac{\infty}{\infty}\right]\overset{\text{LP}}{=}$$

$$\overset{\text{LP}}{=}\lim_{x\to0^{+0}}\frac{-\frac{1}{\cancel{x}}}{-\frac{1}{4\cancel{x}\sqrt{x}}}=\lim_{x\to0^{+0}}4\sqrt{x}=0$$

so the function has no vertical asymptotes on the right side of the graph;

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2\sqrt{x}(1-\ln x) = -\infty \notin R$

so the function has no right horizontal asymptote;

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2(1-\ln x)}{\sqrt{x}} = \left[\frac{-\infty}{\infty}\right] \stackrel{\text{LP}}{=} \lim_{x\to\infty} \frac{-\frac{2}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{-4}{\sqrt{x}} = 0 = k_1;$$

$k_1 \notin R\setminus\{0\}$ so the function has no right oblique asymptote.

4. Monotonicity intervals and points of local extremes

$$f'(x) = \frac{1}{\sqrt{x}}(1-\ln x) + 2\sqrt{x}\cdot\frac{-1}{x} = \frac{1-\ln x-2}{\sqrt{x}} = -\frac{1+\ln x}{\sqrt{x}}$$

$D_{f'} = D_f \cap [0,\infty\rangle;$

$f'(x) = 0$

$-\frac{1+\ln x}{\sqrt{x}} = 0 \Leftrightarrow 1+\ln x = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = e^{-1}.$

$f(e^{-1}) = 4\sqrt{e^{-1}} = 4e^{-1/2}.$

$S_1 = \{e^{-1}\}$

The edges of the interval $\langle 0,\infty\rangle$ are:

$$0,\infty$$

so the monotonicity intervals (on the interval $\langle 0,\infty\rangle$):

$$\langle 0,e^{-1}\rangle, \langle e^{-1},\infty\rangle.$$

$f'(e^{-2}) > 0 \Rightarrow f$ is increasing on $\langle 0,e^{-1}\rangle$;

$f'(1) < 0 \Rightarrow f$ is decreasing on $\langle e^{-1},\infty\rangle$.

The point of the local extremum of the function f can only be the critical point of the function f.

Therefore

$f'(x) > 0$ for all $x \in \langle 0,e^{-1}\rangle$ (as $f'(e^{-2}) > 0$)

and

$f'(x) < 0$ for all $x \in \langle e^{-1},\infty\rangle$ (as $f'(1) < 0$), then $M\left(e^{-1},4e^{-1/2}\right)$ is the point of the local maximum of the function f.

5. Curvature intervals and inflection points

$$f''(x)=-\frac{\frac{1}{x}\sqrt{x}-\frac{1}{2\sqrt{x}}(1+\ln x)}{x}=\frac{\frac{1}{2\sqrt{x}}(1+\ln x)-\frac{2}{2\sqrt{x}}}{x}=\frac{\ln x-1}{2x\sqrt{x}};$$

$$D_{f''}=D_f\cap\left[0,\infty\right\rangle;$$

$$f''(x)=0$$
$$\frac{\ln x-1}{2x\sqrt{x}}=0\Leftrightarrow \ln x-1=0\Leftrightarrow x=e.$$

$$f(e)=0.$$

$$S_2=\{e\}$$

The edges of the interval $\langle 0,\infty\rangle$ are:

$$0,\infty$$

so the curvature intervals (on the interval $\langle 0,\infty\rangle$):

$$\langle 0,e\rangle,\langle e,\infty\rangle.$$

$f''(1)<0\Rightarrow f$ is strictly concave on $\langle 0,e\rangle$;
$f''\left(e^2\right)>0\Rightarrow f$ is strictly convex on $\langle e,\infty\rangle$.

Therefore

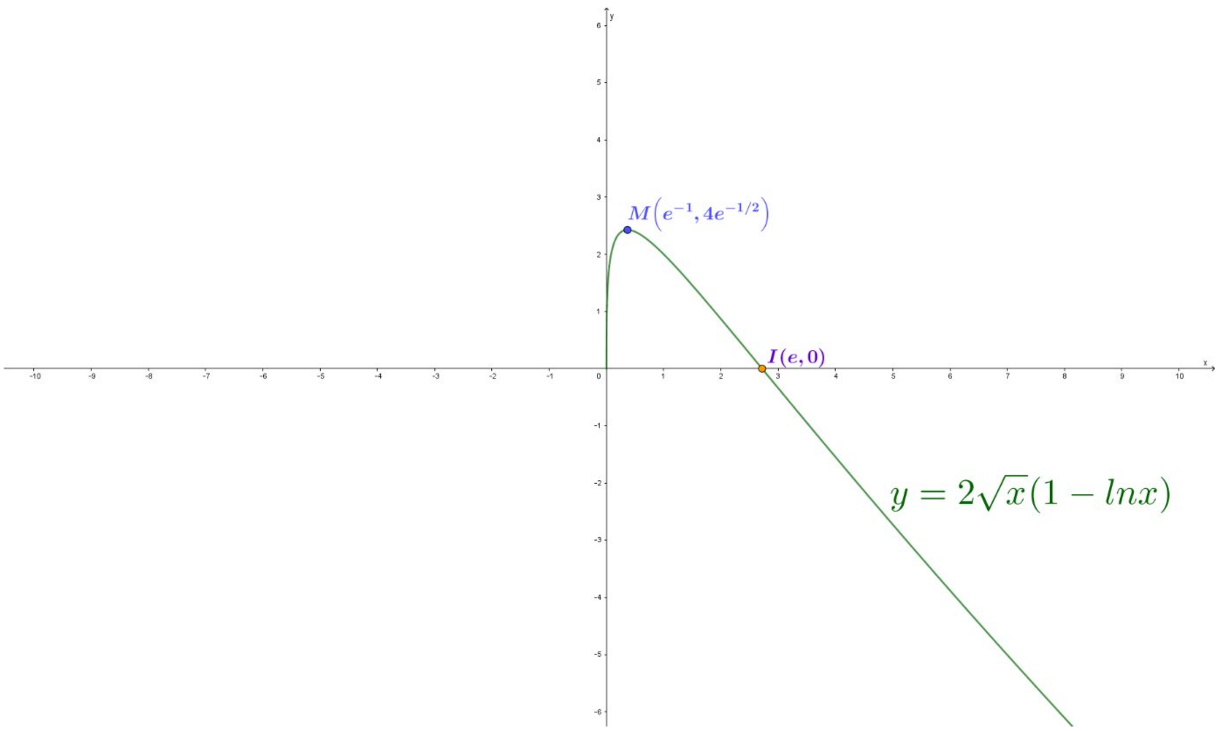
$$f''(x)<0 \text{ for all } x\in\langle 0,e\rangle \text{ (as } f''(1)<0\text{)},$$

and

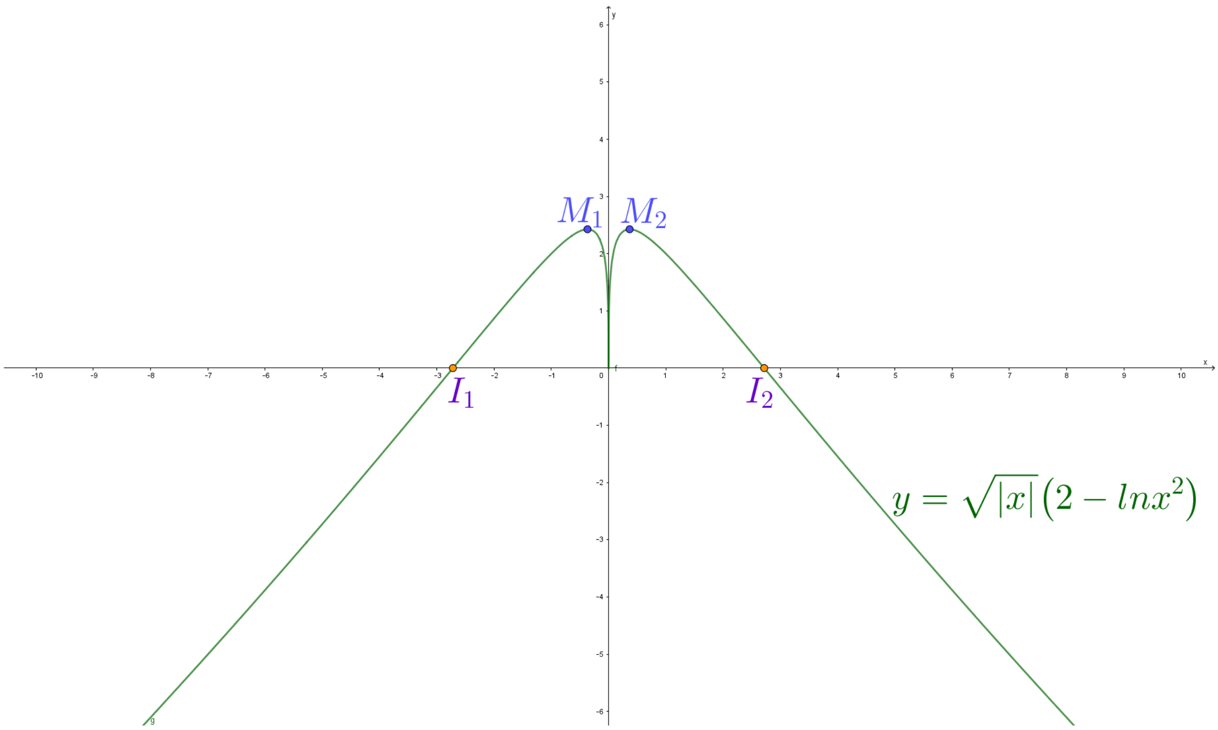
$$f''(x)>0 \text{ for all } x\in\langle e,\infty\rangle \text{ (as } f''\left(e^2\right)>0\text{)},$$

then $I(e,0)$ is the point of the inflection of the function f.

6a. Graph of the function on the interval $\langle 0, \infty \rangle$



6b. Graph function



Example 6

Example of a function where the point of the local extremum is a critical point that is not stationary:

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

$D_f = R = \langle -\infty, \infty \rangle;$

$$f'(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

$$\left.\begin{aligned} f'_d(0) &= \lim_{t\to 0^{+0}} \frac{f(0+t)-f(0)}{t-0} = \lim_{t\to 0^{+0}} \frac{t}{t} = 1 \\ f'_l(0) &= \lim_{t\to 0^{-0}} \frac{f(0+t)-f(0)}{t-0} = \lim_{t\to 0^{+0}} \frac{-t}{t} = -1 \end{aligned}\right\} \Rightarrow f'(0) = \lim_{t\to 0} \frac{f(0+t)-f(0)}{t-0} \text{ does not exist.}$$

$f(0)=0.$

Therefore, $(0,0)$ is a critical point that is not stationary.

$S_1 = \{0\}$

The edges of the domain of the function f are:

$$-\infty, \infty$$

so the intervals of monotonicity are:

$$\langle -\infty, 0\rangle, \langle 0, \infty\rangle.$$

Therefore

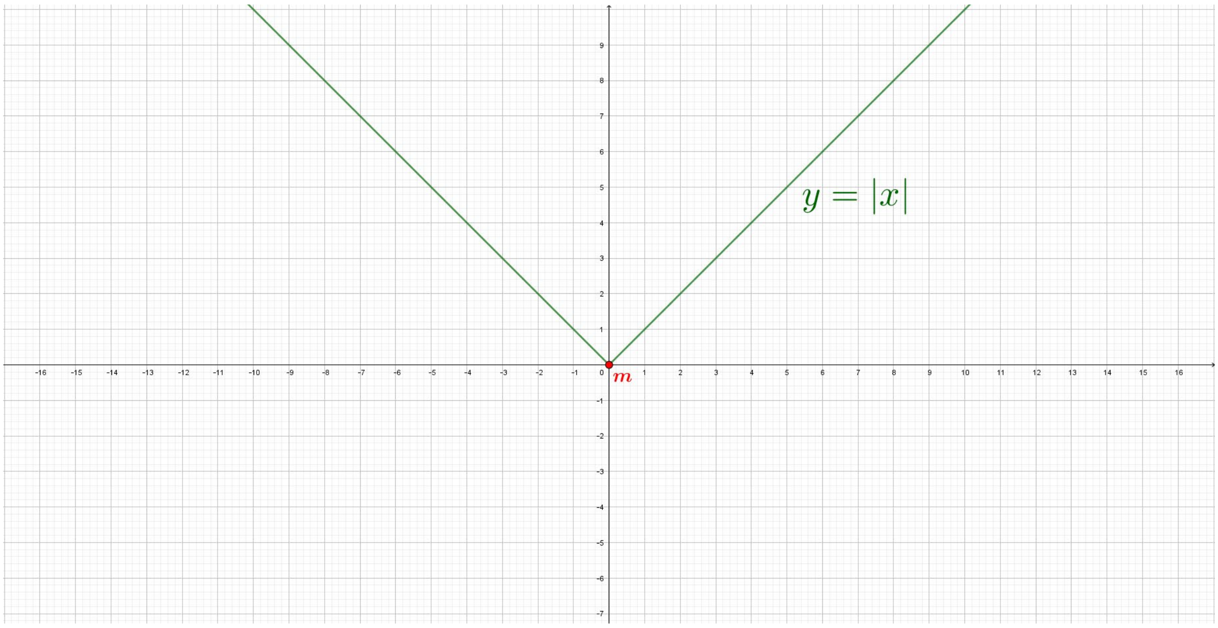
$$f'(x) = -1 < 0 \text{ for all } x < 0$$

and

$$f'(x) = 1 > 0 \text{ for all } x > 0,$$

then $m(0,0)$ is the point of the local minimum of the function f.

Graph function $y=|x|$:


$y = |x|$
m

6.8 Exercises

Task 1. Prove that:

(1.) $\left(\frac{a+bx}{c+dx}\right)' = \frac{bc-ad}{(c+dx)^2}$;

(2.) $\left(\frac{2x+3}{x^2-5x+5}\right)' = \frac{-2x^2-6x+25}{(x^2-5x+5)^2}$;

(3.) $\left(\frac{2}{2x-1}-\frac{1}{x}\right)' = \frac{1-4x}{x^2(2x-1)^2}$;

(4.) $\left[(x-1)\cdot e^x\right]' = xe^x$;

(5.) $\left(\frac{e^x}{x^2}\right)' = \frac{e^x(x-2)}{x^3}$;

(6.) $\left(\frac{1}{t^2+t+1}\right)' = -\frac{2t+1}{(t^2+t+1)^2}$.

Task 2. Find the derivatives:

(1.) $y=(1+3x+5x^2)^4$;

(2.) $y=(3-\sin x)^3$;

(3.) $y=\sqrt[3]{\sin^2 x}+\frac{1}{\cos^2 x}$;

(4.) $y=\sqrt[3]{2e^x+2^x+1}+\ln^5 x$;

(5.) $y=\sin 3x+\cos\frac{x}{5}+tg\sqrt{x}$;

(6.) $y=\sin(x^2-5x+1)+tg\frac{a}{x}$;

(7.) $y=arctg(\ln x)+\ln(arctg\, x)$;

(8.) $y=\ln^2 arctg\left(\frac{x}{3}\right)$;

(9.) $y=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$;

(10.) $y=\ln\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}+2arctg\sqrt{\sin x}$;

(11.) $y=\frac{3}{4}\ln\frac{x^2+1}{x^2-1}+\frac{1}{4}\ln\frac{x-1}{x+1}+\frac{1}{2}arctg\, x$;

(12.) $y=\frac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$;

(13.) $y=\frac{\sin t}{\cos^2 t}+\ln\frac{1+\sin t}{\cos t}$;

(14.) $y=e^x arctg\, e^x-\ln\sqrt{1+e^{2x}}$.

(15.) $y=\frac{\sin^2 x}{1+ctgx}+\frac{\cos^2 x}{1+tgx}$

(16.) $y=arctg\, e^x-\ln\sqrt{\frac{e^{2x}}{e^{2x}+1}}$

Solution:

(1.) $y'=4(1+3x+5x^2)^3(3+10x)$;

(2.) $y'=-3(3-\sin x)^2\cos x$;

(3.) $y' = \dfrac{2\cos x}{3\cdot\sqrt[3]{\sin x}} + \dfrac{2\sin x}{\cos^3 x}$; (4.) $y' = \dfrac{2e^x + 2^x \ln 2}{3\cdot\sqrt[3]{(2e^x + 2^x + 1)^2}} + \dfrac{5\ln^4 x}{x}$;

(5.) $y' = 3\cos 3x - \dfrac{1}{5}\sin\dfrac{x}{5} + \dfrac{1}{2\sqrt{x}}\cdot\dfrac{1}{\cos^2\sqrt{x}}$;(6.) $y' = (2x-5)\cos(x^2-5x+1) - \dfrac{a}{x^2}\cdot\dfrac{1}{\cos^2\dfrac{a}{x}}$;

(7.) $y' = \dfrac{1}{1+\ln^2 x}\cdot\dfrac{1}{x} + \dfrac{1}{arctg\, x}\cdot\dfrac{1}{1+x^2}$; (8.) $y' = 2\ln arctg\left(\dfrac{x}{3}\right)\cdot\dfrac{1}{arctg\dfrac{x}{3}}\cdot\dfrac{3}{9+x^2}$;

(9.) $y' = \dfrac{\sqrt{x^2+1}}{x}$; (10.) $y' = \dfrac{2}{\cos x\sqrt{\sin x}}$;

(11.) $y' = \dfrac{x^2-3x}{x^4-1}$; (12.) $y' = \dfrac{\arcsin x}{(1-x^2)^{\frac{3}{2}}}$;

(13.) $y' = \dfrac{2}{\cos^3 t}$; (14.) $y' = e^x arctg\, e^x$;

(15.) $y' = -\cos 2x$; (16.) $y' = \dfrac{e^x - 1}{e^{2x}+1}$.

Task 3. Prove that the function $y = xe^{-\frac{x^2}{2}}$ satisfies the equation $x\cdot y' = (1-x^2)y$.

Task 4. Prove that the function $y = \dfrac{1+\ln x}{x - x\ln x}$ satisfies the equation $2x^2\cdot y' - x^2y^2 - 1 = 0$.

Task 5. Find the derivatives:

(1.) $f(x) = 10^{xtgx}$; (2.) $g(x) = \sqrt[x]{(1+x)^2}$;

(3.) $y = x^{\sin x}$; (4.) $y = \dfrac{(x-2)^9}{\sqrt{(x-1)^5\cdot(x-3)^{11}}}$;

(5.) $y = x^2\sqrt{\dfrac{2x-1}{x+1}}$.

Solution:

(1.) $f'(x) = 10^{xtgx}\left(tgx + \dfrac{x}{\cos^2 x}\right)\cdot\ln 10$;

(2.) $g'(x)=2\cdot\sqrt[x]{(1+x)^2}\left[\frac{1}{x(x+1)}-\frac{ln(x+1)}{x^2}\right]; \forall x\in R\setminus\{0,-1\};$

(3.) $y'=x^{\sin x}\left(\cos x\cdot \ln x+\frac{\sin x}{x}\right);$

(4.) $y'=\frac{(x-2)^8\cdot(x^2-7x+1)}{\sqrt{(x-1)^7\cdot(x-3)^{13}}};\ \forall x\in R\setminus\{1,3\};$

(5.) $y'=\frac{x(8x^2+7x-4)}{2(x+1)(2x-1)}\cdot\sqrt{\frac{2x-1}{x+1}};\ \forall x\in R\setminus\left\{\frac{1}{2},-1\right\}.$

Task 6. Find the partial derivatives:

(1.) $F(x,y)=ln\left(x+\sqrt{x^2+y^2}\right);$

(2.) $F(x,y)=ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x};$

(3.) $F(x,y)=arcsin\frac{\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}}.$

Solution:

(1.) $\frac{\partial F}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}$, $\frac{\partial F}{\partial y}=\frac{y}{x^2+y^2+x\sqrt{x^2+y^2}};$

(2.) $\frac{\partial F}{\partial x}=\frac{-2}{\sqrt{x^2+y^2}}$, $\frac{\partial F}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}};$

(3.) $\frac{\partial F}{\partial x}=\frac{\sqrt{2}xy}{(x^2+y^2)\sqrt{x^2-y^2}}$, $\frac{\partial F}{\partial y}=\frac{-\sqrt{2}x^2}{(x^2+y^2)\sqrt{x^2-y^2}}.$

Task 7. The function is given as $F(x,y)=\sqrt{\frac{y}{x}(y^2+1)}$. Define $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ at the point $T(2,1)$

.

Solution: $\frac{\partial F}{\partial x}(2,1)=-\frac{1}{4};\ \frac{\partial F}{\partial y}(2,1)=1$

Task 8. At the point $(1,1)$ find the partial derivatives of $F(x,y)=\frac{\sqrt{x^2+y^2}}{1-x-y}.$

Solution: $\frac{\partial F}{\partial x}=\frac{x-xy+y^2}{(1-x-y)^2\sqrt{x^2+y^2}}\Rightarrow\frac{\partial F}{\partial x}(1,1)=\frac{\sqrt{2}}{2};$

$$\frac{\partial F}{\partial y}=\frac{x^2-xy+y}{(1-x-y)^2\sqrt{x^2+y^2}}\Rightarrow\frac{\partial F}{\partial y}(1,1)=\frac{\sqrt{2}}{2}.$$

Task 9. Find the derivative $y=f(x)$ of the implicitly given function:

(1.) $e^y=x+y$; (2.) $\ln y+\frac{x}{y}=C$; (3.) $arctg\frac{y}{x}=\frac{1}{2}\ln(x^2+y^2)$;

(4.) $\sqrt{x^2+y^2}=C\cdot arctg\frac{y}{x}$; (5.) $e^{\frac{y}{x}}=arctg\sqrt{x^2+y^2}$.

Solution:

(1.) $y'=\frac{1}{x+y-1}$; (2.) $y'=\frac{y}{x-y}$; (3.) $y'=\frac{x+y}{x-y}$;

(4.) $y'=\frac{Cy+x\sqrt{x^2+y^2}}{Cx-y\sqrt{x^2+y^2}}$; (5.) $y'=\frac{x^3+y\cdot e^{\frac{y}{x}}(1+x^2+y^2)\sqrt{x^2+y^2}}{x\cdot e^{\frac{y}{x}}(1+x^2+y^2)\sqrt{x^2+y^2}-x^2y}$.

Task 10. Calculate the derivation y' of the given function at the stated points:

(1.) $(x+y)^3=27(x-y)$ at $T(2,1)$; (2.) $ye^y=e^{x+1}$ at $T(0,1)$;

(3.) $y^2=x+ln\frac{y}{x}$ at $T(1,1)$.

Solution:

(1.) $y'\big|_{(2,1)}=\frac{27-3(x+y)^2}{3(x+y)^2+27}\Big|_{(2,1)}=0$; (2.) $y'\big|_{(0,1)}=\frac{e^{x+1}}{ye^y}\Big|_{(0,1)}=\frac{1}{2}$;

(3.) $y'\big|_{(1,1)}=\frac{y(x-1)}{x(2y^2-1)}\Big|_{(1,1)}=0$.

Task 11. Find $y'(x); a,b,c\in R$ if:

(1.) $\begin{cases}x(t)=\frac{a\sin t}{1+b\cos t},\\ y(t)=\frac{c\cdot\cos t}{1+b\cos t}.\end{cases}$ (2.) $\begin{cases}x(t)=\frac{3at}{1+t^3},\\ y(t)=\frac{3at^2}{1+t^3}.\end{cases}$

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(3.) $\begin{cases} x(t) = \dfrac{cos^3 t}{\sqrt{cos\,2t}}, \\ y(t) = \dfrac{sin^3 t}{\sqrt{cos\,2t}}, \end{cases}$

Solution:

(1.) $y'(x) = -\dfrac{c\,sin\,t}{a(b + cos\,t)}$; (2.) $y'(x) = \dfrac{t(2 - t^3)}{1 - 2t^3}$;

(3.) $y'(x) = -tg3t$.

Task 12. Find $y''(x)$ if

(1°) $\begin{cases} x(t) = & arctgt, \\ y(t) = & \dfrac{1}{2}t^2. \end{cases}$ (2°) $\begin{cases} x(t) = & ln\,t, \\ y(t) = & \dfrac{1}{1-t}. \end{cases}$

(3°) $\begin{cases} x(t) = & arcsin\,t, \\ y(t) = & \sqrt{1-t^2}. \end{cases}$ (4°) $\begin{cases} x(t) = & ln\left(t + \sqrt{1+t^2}\right), \\ y(t) = & \sqrt{1+t^2}. \end{cases}$

Solution:

(1.) $y''(x) = (1+t^2)(1+3t^2)$; (2.) $y''(x) = \dfrac{(1+t)\cdot t}{(1-t)^3}$;

(3.) $y''(x) = -\sqrt{1-t^2}$; (4.) $y''(x) = \sqrt{1+t^2}$.

Task 13. Define the equation of the tangent of the function $y = x^2 - 4x + 3$ at the left zero-point.

Solution:

Zero-points are $x_1 = 1$ and $x_2 = 3$; left $T_0(1,0)$; $t:\ y + 2x - 2 = 0$.

Task 14. On the function $f(x) = x^2 + 3x - 4$ place the tangent that is parallel to the line $2x - 3y + 1 = 0$.

Solution:

$$k_t = \frac{2}{3} = (2x_0 + 3) \Rightarrow x_0 = -\frac{7}{6};\; y_0 = -\frac{221}{36};\; t:\quad 36y - 24x + 193 = 0.$$

Task 15. Find the equation of the tangent and the normal for functions:

(1.) $f(x) = ln(cos\, x)$ at the point $x_0 = 2\pi$.

(2.) $f(x) = \dfrac{8}{x^2 + 4}$ at the point $T(2, f(2))$.

Solution:

(1.) $x_0 = 2\pi;\, y_0 = 0;\, f'(x_0) = 0;$
$t:\; y = 0;\;\; n: x - 2\pi = 0;$

(2.) $x_0 = 2;\, y_0 = 1;\, k_t = f'(2) = -\dfrac{1}{2};$
$t: x + 2y - 4 = 0;\;\; n: 2x - y - 3 = 0.$

Task 16. From the point $T(4,1)$ find the tangent on the curve $y = \dfrac{x-1}{x}$ and define the contact points.

Solution:

$$D\left(2, \frac{1}{2}\right);\, t: 4y - 4 = 0.$$

Task 17. At which point of the parabola $y = x^2 + 2x + 1$ the tangent makes identical angles on both sides of the coordinate axis?

Solution:

$$T_1\left(-\frac{1}{2}, \frac{1}{4}\right);\; T_2\left(-\frac{3}{2}, \frac{1}{4}\right).$$

Task 18. Find the equation of the tangent and the normal on the parametrically given curve:

(1.) $\begin{cases} x(t) = \dfrac{1+t}{t^3}, \\ y(t) = \dfrac{3}{2t^2} + \dfrac{1}{2t}; \end{cases}$ at the point $T_0(2,2)$;

(2.) $\begin{cases} x(t) = \dfrac{2t}{t+2}, \\ y(t) = \dfrac{t}{t-1}; \end{cases}$ for $t_0 = 2$;

(3.) $\begin{cases} x(t) = & \sin t, \\ y(t) = & a^t; \end{cases}$ at the point for which $t_0 = 0$.

Solution:

(1.) $t:\ 7x - 10y + 6 = 0,$ $n:\ 10x + 7y - 34 = 0;$ (2.) $t: 4x + y - 6 = 0,$ $n: x - 4y + 7 = 0;$

(3.) $t: y - x\ln a - 1 = 0,$ $n: y + \dfrac{1}{\ln a}x - 1 = 0.$

Task 19. Define the equation of the tangent on the curve $y = \left(\sin^2 x + \frac{1}{2}\right)^{tgx}$ at the point for which the abscissa is $x_0 = \frac{3\pi}{4}$.

Solution:

$$4y - 4x + 3\pi - 4 = 0;\ T_0\left(\frac{3\pi}{4}, 1\right);\ y'\left(\frac{3\pi}{4}\right) = 1.$$

Task 20. Find the angle at which the parabolas intersect:

(1.) $y = 4 - \frac{x^2}{2}$ and $y = \frac{x^2}{2}$; (2.) $y = x^2$ and $y^2 = x$.

Solution:

(1.) $\varphi = 126°52'$; (2.) $\varphi_1 = 36°50'$ and $\varphi_2 = 90°$.

Task 21. Find the equation of the tangent and the normal on the curves that are given by an implicit equation:

(1.) $3x^5 - y - 2x + 1 = 0$, at the point $T(1,2)$;

(2.) $8x^2 - 9y^2 - 72 = 0$, at the point $T(-9,-8)$;

(3.) $x \cdot e^{-\frac{y}{2}} - y \cdot e^{-\frac{x}{2}} = 0$, at the point with the abscissa $x_0 = 0$.

Solution:

(1.) $\begin{matrix} t: & 13x - y - 11 = 0, \\ n: & x + 13y - 27 = 0; \end{matrix}$; (2.) $\begin{matrix} t: & x - y + 1 = 0, \\ n: & x + y - 1 = 0; \end{matrix}$;

(3.) $T_0(0,-2),\ k_t = e-1$;

$t: y = (e-1)x - 2$; $n: (e-1)y + x + 2(e-1) = 0$.

Task 22. Find limits:

(1.) $\lim_{x\to 1}(1-x)tg\frac{\pi x}{2}$ (2.) $\lim_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$ (3.) $\lim_{x\to\infty}\frac{\pi-2arctgx}{e^{\frac{2}{x}}-1}$

(4.) $\lim_{x\to\infty}[\ln(x+1)-\ln x]$ (5.) $\lim_{x\to 0}x^{\frac{3}{4+\ln x}}$ (6.) $\lim_{x\to 0}\frac{1-\cos x}{x}$

(7.) $\lim_{x\to 0}\frac{\sin \pi x}{\ln x}$ (8.) $\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}$ (9.) $\lim_{x\to 0}\frac{\sin x-x}{x^2}$

Solution:

(1.) $\frac{2}{\pi}$ (2.) $\frac{1}{2}$ (3.) 1 (4.) 0 (5.) e^3

(6.) 0 (7.) $-\pi$ (8.) 1 (9.) 0

Task 23. Consider the function $f(x)=x^3-\frac{3}{2}x^2-18x$. The points c=3,−2 satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Solution:

f has a local maximum at −2 and a local minimum at 3 .

For tasks 2 - 6, determine:

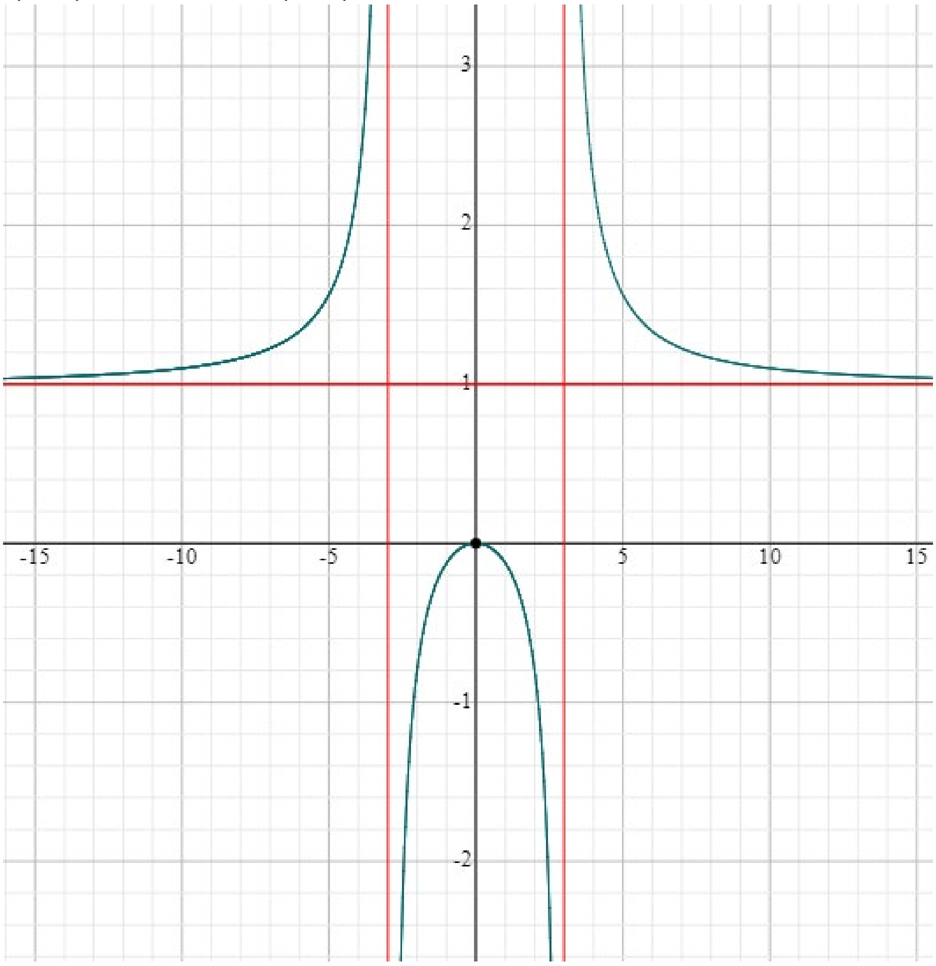
a) intervals where f is increasing or decreasing
b) local minima and maxima of f
c) intervals where f is concave up and concave down
d) the inflection points of f.
e) Sketch the curve.

Task 24.

$$f(x)=-\frac{x^2}{9-x^2}$$

Solution:

a) $D_f = \mathbb{R} \setminus \{3, -3\}$,
b) N (0,0),
c) V.A. $x = \pm 3$, H.A. $y = 1$,
d) $f'(x) = \frac{-18x}{(9-x^2)^2}, f''(x) = -\frac{54x^2+162}{(9-x^2)^3}$, $T_{max}(0, 0)$.



Task 25.

$$f(x) = \frac{4x-12}{(x-2)^2}$$

Solution:

a) $D_f = \mathbb{R} \setminus \{2\}$,

b) S_x (3,0), S_y (0,-3),

c) V.A. $x = 2$, H.A. $y = 0$,

d) $f'(x) = \frac{-4(x-4)}{(x-2)^3}, f''(x) = \frac{8x-40}{(x-2)^4}$, $T_{max}(4,1)$, $I\left(5, \frac{8}{9}\right)$.

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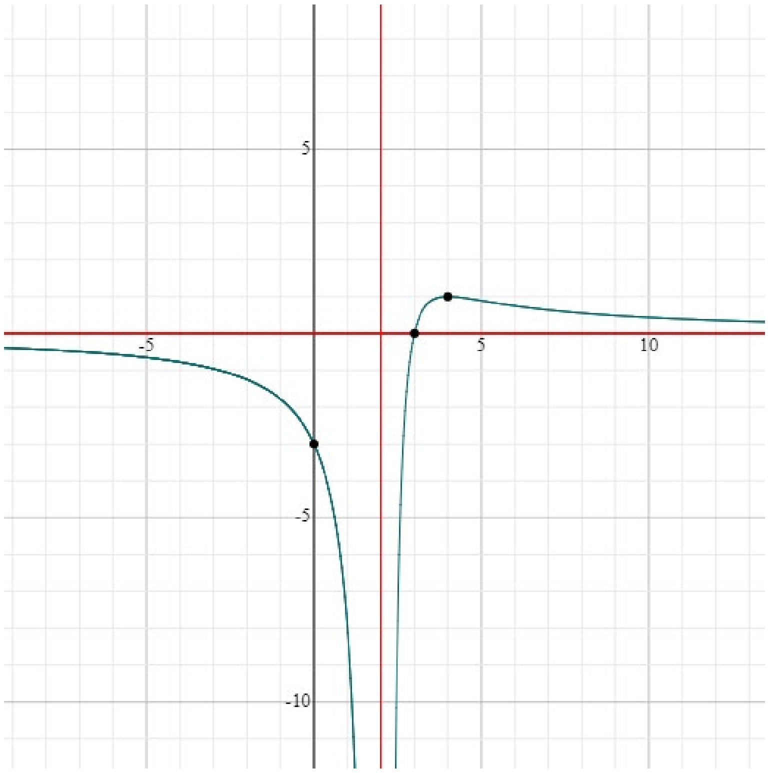

Task 26.

$$f(x)=\frac{x^2-4x+3}{x^2-2x}$$

Solution:

a) $D_f = \mathbb{R} \setminus \{0,2\}$,

b) N_1 (3,0), N_2 (1,0),

c) V.A. $x = 0,\ x = 2$, H.A. $y = 1$,

d) $f'(x) = \frac{2x^2-6x+6}{(x^2-2x)^2}, f''(x) = \frac{-4x^5+26x^4-72x^3+96x^2-48x}{(x^2-2x)^4}$, there are no minimum either maximum.

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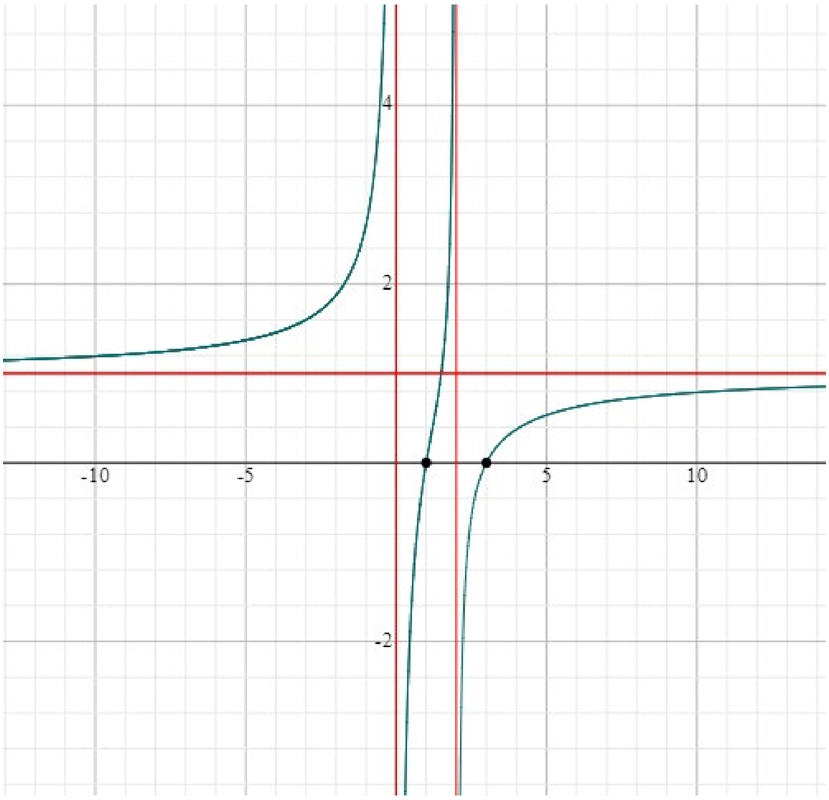


Task 27.

$$f(x) = \frac{\ln(2x)}{x^2}$$

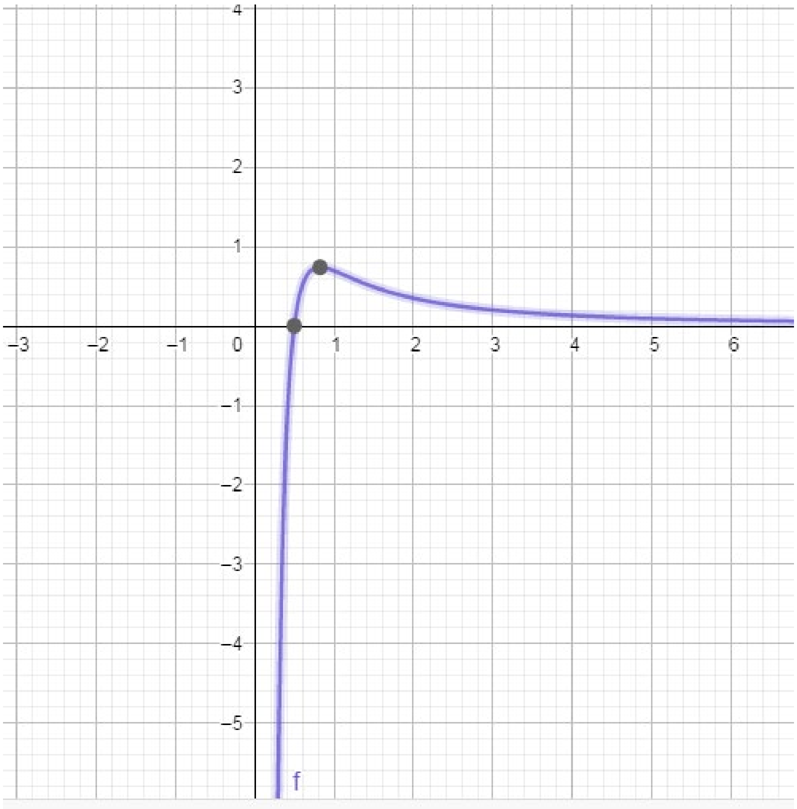
Solution:

a) $D_f = \langle 0, \infty \rangle$,

b) N $(\frac{1}{2}, 0)$,

c) V.A. $x = 0$; H.A. $y = 0$,

d) $f'(x) = \frac{1-2x^2 \ln(2x)}{x^5}$, $f''(x) = \frac{6x^2 ln2x - 2x - 5}{x^6}$, $T_{max}(\frac{\sqrt{e}}{2}, \frac{2}{e})$



Task 28.

$$f(x) = \frac{x}{(1 + x^2)} \cdot e^x$$

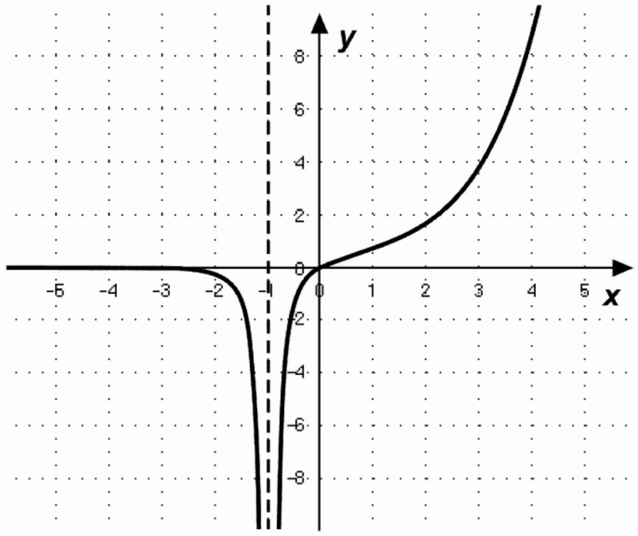
Solution:

a) $D_f = \mathbb{R} \setminus \{-1\}$,

b) N (0, 0),

c) V.A. $x = -1$; L.H.A. $y = 0$,

d) $f'(x) = \frac{e^x(x^2+1)}{(1+x)^3}, f''(x) = \frac{e^x(x^3+3x-2)}{(1+x)^2}$, there are no minimum either maximum.


y
x
8
6
4
2
0
-2
-4
-6
-8
-5
-4
-3
-2
-1
1
2
3
4
5

6.9 Connections and applications

Examples of applications in the maritime domain.

Maritime affairs does not include only sailing or underway actions and navigation but it brings together terms from vessels, employees and companies over shipbuilding to trade, transport and management. Calculus can help us solve many types of real-world problems in maritime affairs.

We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.). Derivatives are met in many problems in maritime domain especially relating velocity and position and generally, rates of change related quantities.

6.9.1 Related Rates

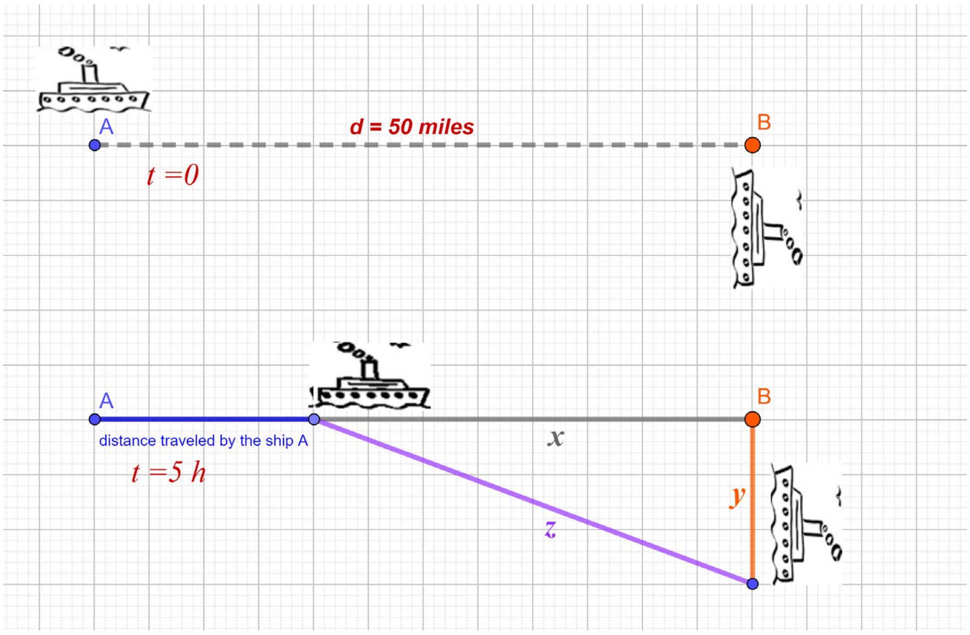
Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 1:

Ship A is 50 miles west of ship B. The ship A is sailing east at 10 knots, and the ship B is sailing south at 15 knots. Find the rate of change of the distance between the ships after 5 hours.

<u>Solution:</u>

<u>Step 1:</u> Draw a picture placing the problem and introducing the variables.



Let y denote the distance sailed by the ship B, x denote the distance between the current position of the ship A and starting position of the ship B and z denote the ***current*** distance between the ships. Notice that x, y and z are functions of time and d does not depend on time - it is the initial distance between ships (*fixed number*).

Step 2: Since x denotes the horizontal distance between the current position of the ship A (at the time t) and the start point of the ship B, then $\frac{dx}{dt}$ represents the speed of the ship A.

It is told the speed of the ship A is 10 knots (mph) and it implies that the distance x decreases 10 miles every hour. Therefore, $\frac{dx}{dt} = -10\ mph$.

Similarly, y denotes the distance between the B ship position at the time t and its start point. $\frac{dy}{dt}$ represents the speed of the ship B. It is told the speed of the ship B is 15 knots (mph) and it implies that the distance y increases 15 miles every hour. Therefore, $\frac{dy}{dt} = 15\ mph$.

Since, it is asked to find the **rate of change** in the distance z between the ships after 3 hours, we need to find $\frac{dz}{dt}$ when $t = 3\ h$.

Step 3: Find the value of x and y after 3 hours of sailing. We will use input values directly in the formulas.

$$x = d - v_A \cdot t = 50 - 10 \cdot 3 = 30\ miles$$

$$y = v_B \cdot t = 15 \cdot 3 = 45\ miles$$

Reminder:
The formula for distance d, speed v and time t:
$v = \frac{d}{t}$ or $d = v \cdot t$ or

Note, z is the hypotenuse of the right triangle with side x and side y from above figure. Thus,

$$z = \sqrt{x^2 + y^2} = \sqrt{30^2 + 45^2} = 54.08 \text{ miles}.$$

Reminder:

$$c^2 = a^2 + b^2$$

Step 4: Find the **rate of change** in the distance z *with the respect to time*. It will be done by determination z′ given that x′ = −35 and y′ = 50 .

We can again use the Pythagorean Theorem here. First, write it down and differentiate the equation using Implicit Differentiation.

$$z^2 = x^2 + y^2 \Rightarrow$$

$$2zz' = 2xx' + 2yy'$$

$$z' = \frac{2xx' + 2yy'}{2z} = \frac{2 \cdot 30 \cdot (-30) + 2 \cdot 45 \cdot 45}{2 \cdot 54.08} = 20.8 \text{ miles}$$

x, y, and z are all changing with time and so the equation id differentiated using Implicit Differentiation.

Therefore, after 3 hours the distance between ships is changing with the rate of 20.8 miles per hour.

Example 2:

A ship sails according the law:

$$s = \left(1272.7 \cdot ln\frac{1+6\cdot e^{0.055t}}{7} - 50t\right) \quad [m]$$

The start velocity of the ship according this voyage should be determined.

<u>Solution:</u>

Let *s* is the distance travelled by a ship and it is changing with time. So it can be denoted ***s(t)***.

Since velocity ***v*** is the instantaneous rate of change of travelled distance with respect to time *t* we need to find the value of the derivative s′(t).

$$v = \frac{ds}{dt} = 1272.7 \cdot \frac{7}{1+6\cdot e^{0.055t}} \cdot \frac{6}{7} \cdot e^{0,055t} \cdot 0.055 - 50$$

$$v = \frac{420}{1+6\cdot e^{0.055t}} - 50$$

To get the start velocity of the ship it is needed to calculate the s′(t) at $t = 0$.

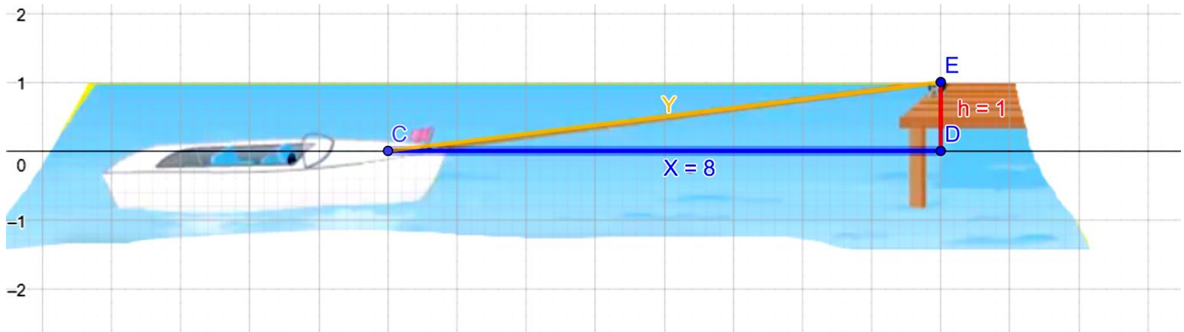
$$v_0 = \frac{420}{1+6} - 50 = 10\, m/s$$

Example 3:

A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 1 m higher than the front of the boat. The rope is being pulled through the ring at the rate of 1 m/sec. How fast is the boat approaching the dock when 8 m of rope are out?

<u>Solution:</u>

Step 1: Draw a picture.



The boat is approaching to the dock. This distance is unknown and let x denote that distance. It is known that the pulley is 1 meter higher than the front of the boat and let h denote this height. It is the constant value.

y denotes the length of the rope that the boat is pulled. From the figure above it can be noted that the angle $\sphericalangle CDE$ is the right angle.

Since the rope is being pulled at the rate of 1 m/sec, we know that $\frac{dy}{dt} = -1\ m/sec$. It is the negative value because the length of the rope is shorter and shorter by pulling the boat (it is shorter for 1 meter per second).

If the boat is apart 8 m from the dock, it is needed to find how fast the boat is approaching to the dock, i.e. the rate of change in the distance *d* between the boat and the dock per second. We need to find $\frac{dx}{dt} = ?\ m/s$ when x is 8m.

Note that both x and y are functions of time, and the height h is the constant.

Step 2: From the right triangle CDE we can use the **Pythagorean Theorem** to write an equation relating x and y (h=1m):

$$y^2 = x^2 + 1^2$$

Reminder
If h is a constant then $\frac{dh}{dt} = 0$.

Step 3: Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation:

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + 0$$

$$\frac{2y}{2x}\frac{dy}{dt} = \frac{2x}{2x}\frac{dx}{dt}$$

$$\frac{2 \cdot 8.06m}{2 \cdot 8\,m} \cdot (-1\ m/s) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{8.06}{8} m/s = -1.011 m/s$$

We can use the Pythagorean theorem to determine the lenght **y** when x=8 m, and the height is 1 m . Solving the equation:

$$y^2 = x^2 + h^2$$

$$y = \sqrt{x^2 + h^2}$$

$$y = \sqrt{8^2 + 1^2} = \sqrt{65} \approx 8.06 \text{ m}$$

Example 4:

The atmospheric pressure P varies with altitude above sea level x in accordance with the law:

$$P(x) = P_0 \cdot e^{-0.12104\,x}$$

where P_0 is the atmospheric pressure at sea level. If the atmospheric pressure is 1013 millibars at sea level, how fast the atmospheric pressure is changing with respect to altitude at an altitude of 20 km.

Solution:

The rate of pressure change is derivation of the function $P(x)$ with respect to the altitude x. Thus,

$P'(x) = 1013 \cdot (e^{-0.12104\,x})' = 1013 \cdot (-0.12104) \cdot (e^{-0.12104\,x}) = -122.61 \cdot (e^{-0.12104\,x})$

$P'(20) = -122.61 \cdot (e^{-0.12104 \cdot 20}) = 10.8939$ milibars per km

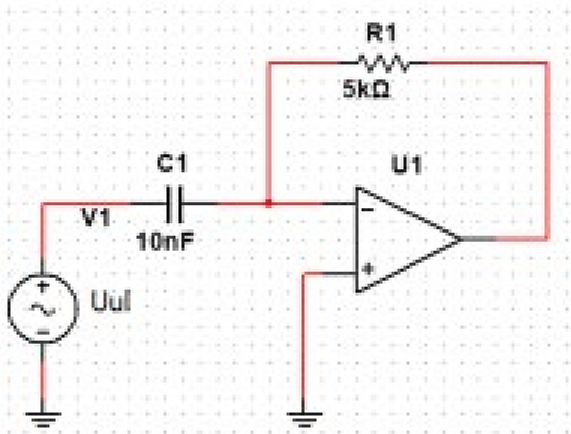
Example 5:

The output signal of an ideal operational amplifier in a derivative connection should be analytically determine. Graphically compare the input and output signal if the following values are known.

$U_{ul}(t) = 10\sin(2\pi \cdot 3000t)$

$R1 = 5k\Omega$

$C1 = 10nF$



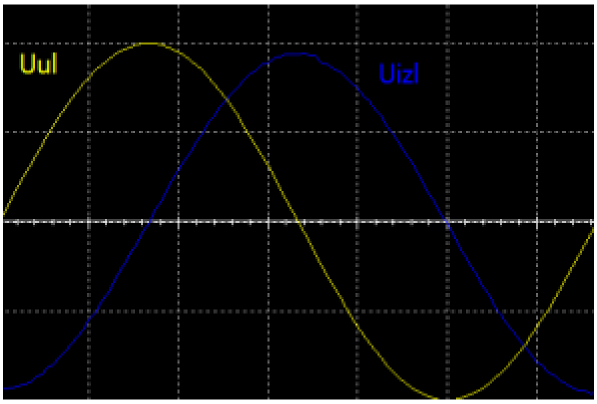
Solution:

The equation for output signal is:

$$U_{izl} = -R1 \cdot C1 \cdot \frac{dU_{ul}}{dt}$$

$$U_{izl} = -5 \cdot 10^3 \cdot 10 \cdot 10^{-9} \cdot 10 \cdot \frac{d[\sin(2\pi \cdot 3000t)]}{dt}$$

$$U_{izl} = -9.42 \cdot \cos\,(2\pi \cdot 3000t)$$



Example 6:

The law of rotational motion of the steam turbine during putting in operation should be determined. It is known that the increasing of angular velocity is proportional to third power of time and in the moment $t = 3\ s$ the velocity of rotation of the turbine's rotor is $n = 810\ min^{-1}$.

<u>Solution:</u>

From described problem, the law of rotation motion is proportioned third potential of time and can be expressed as:

$$\varphi = k \cdot t^3.$$

It can be said that the angular velocity is equal to change in angle over a change in time. So if we want to express it in calculus sense it would be the derivative the angle with respect to time:

$$\omega = \frac{d\varphi}{dt} = 3 \cdot\ k \cdot t^2.$$

Known values can help us to get the proportionality constant *k* from previous equation.

$$k = \frac{\omega}{3 \cdot t^2} = \frac{\pi \cdot n}{3 \cdot 30 \cdot t^2}$$

$$k = \frac{\pi \cdot 810}{3 \cdot 30 \cdot 9} = \pi.$$

The law of rotation motion of a steam turbine is:

$$\varphi = \pi \cdot t^3.$$

The angular velocity and angular acceleration are as follows:

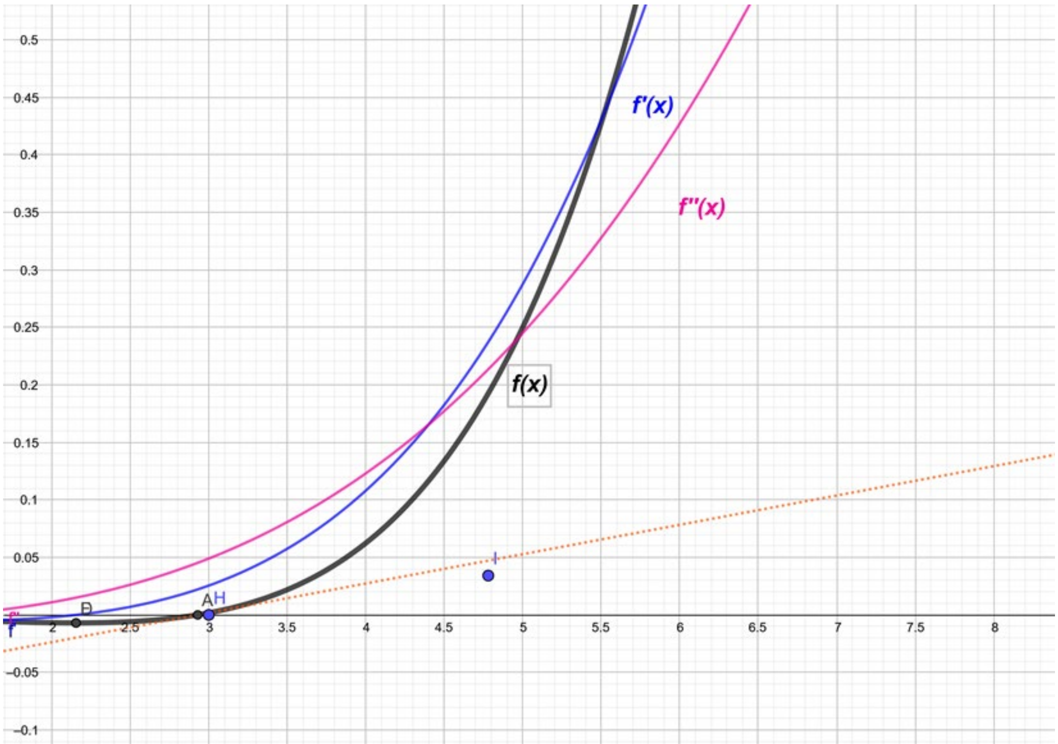
$$\omega = 3 \cdot \pi \cdot t^2$$

$$\varepsilon = 6 \cdot \pi \cdot t.$$

Example 7:

The bending of the steel truss is given by the equation $f(x) = 10^{-4}(x^5 - 25x^2)$, where x denotes the distance from the girder. Calculate the second derivative (change in the direction coefficient of the tangent) for x = 3.

Solution:



$$y' = \frac{dy}{dx} = \frac{d(10^{-4}x^5)}{dx} - \frac{d\left(10^{-4}25x^2\right)}{dx} = 10^{-4}(5x^4 - 50x)$$

$$y'' = 10^{-4}(20x^3 - 50)$$

$$y''_{x=3} = 0.049 \left[\frac{1}{m}\right]$$

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6.9.2 Optimization problem (minimum, maximum)

Many important applied problems in maritime affairs involve finding the maximum or minimum value of some function like as the minimum time to rich the distance by a ship, the maximum profit, the minimum cost for doing a task, the maximum power and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Guide to solve the problem: max (min) of y(x)

a. Determine stationary point: $\frac{dy}{dx} = 0$

Find the value of x .

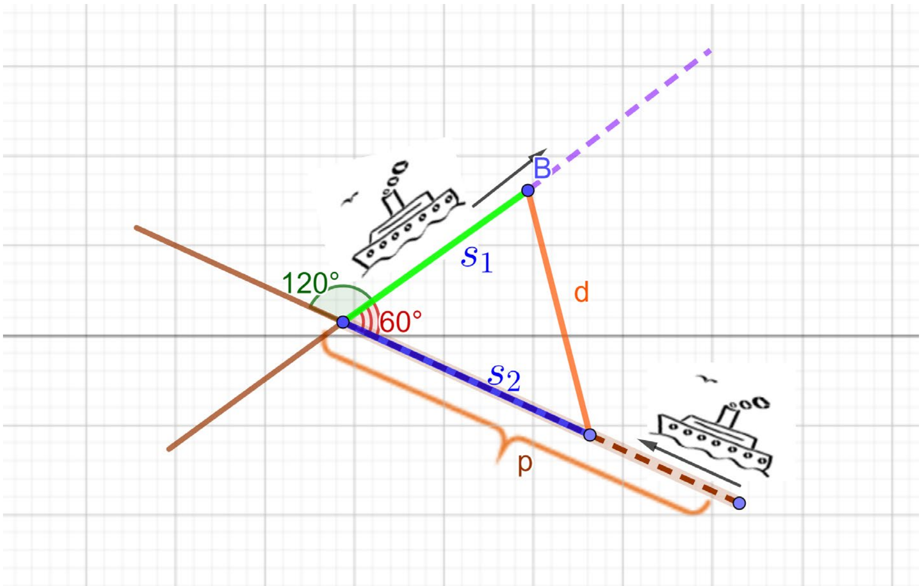
b. Find the value of y , when x = ?

c. Determine the nature of stationary point, $\frac{d^2y}{dx^2} = 0$

Example 8:

Two fishing boats sail in the same plane, in the direction, at the same speed, in knots. The sailing directions close an angle of 120°. At one point one of these boats is at the intersection of their directions, while the other boat is at p knots away from the intersection. Find the time when the distance between the boats will be the shortest and what it will be.

Solution:



$$s = vt \Rightarrow \quad s_1 = vt, \quad s_2 = p - vt$$

$$d^2 = {s_1}^2 + {s_2}^2 - 2\, s_1 s_2 \cos(60°) = v^2t^2 + \left(p^2 - 2pvt + v^2t^2\right) - 2vt(p - vt) \cdot \frac{1}{2}$$

$$d^2 = 3\,v^2t^2 - 3pvt + p^2$$

The distance ***d*** varies with time ***t*** and it is a function of time *d(t)*. To find the shortest distance we should solve the equation $d'(t) = 0$.

First we have to find $d'(t)$.

$$d'(t) = \left(\sqrt{3\,v^2t^2 - 3pvt + p^2}\right)' = \frac{3\,v^2(2t) - 3pv\cdot 1 + 0}{2\sqrt{3\,v^2t^2 - 3pvt + p^2}} = \frac{3(2\,v^2t - pv)}{2\sqrt{3\,v^2t^2 - 3pvt + p^2}}$$

$d'(t) = 0$ if $2\,v^2t - pv = 0 \;\Rightarrow\; t = \frac{p}{2v}$

Answer: The shortest distance between boats will be at the time $t = \frac{p}{2v}$.

The length of the shortest distance will be as follows.

$$d = \sqrt{3\,v^2t^2 - 3pvt + p^2} = \sqrt{3\,v^2\frac{p^2}{4v^2} - 3pv\frac{p}{2v} + p^2} = \sqrt{\frac{3p^2}{4} - \frac{3p^2}{2} + p^2} = \sqrt{\frac{p^2}{4}}$$

$$d = \frac{p}{2}$$

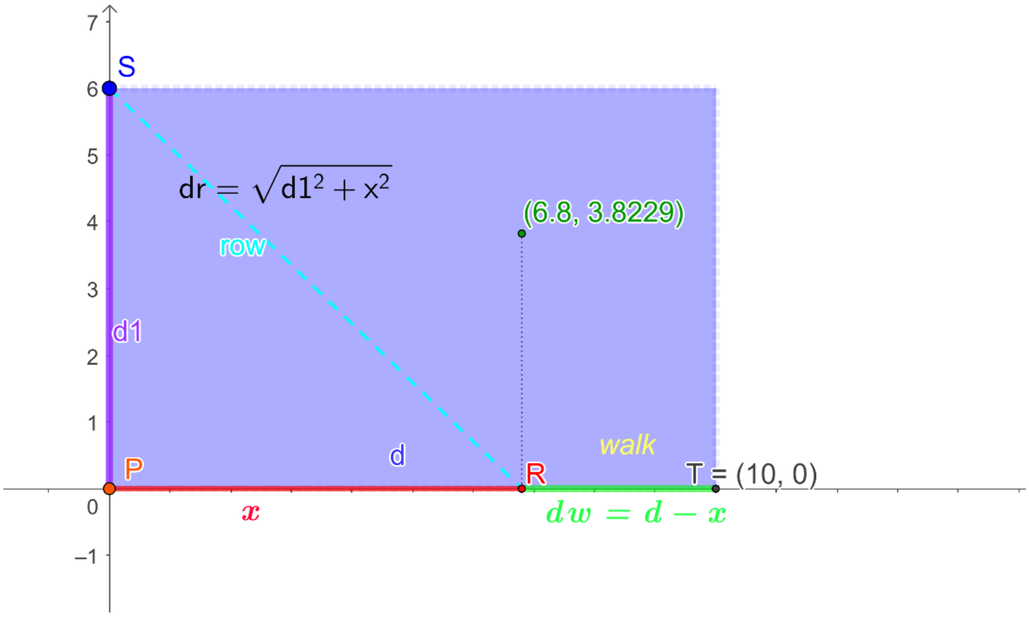
Example 9:

A man is in a boat at 6 miles offshore, at the point S, and wants to get to a town Q on the shore. Point S is d1 = 6 miles away from the closest point P on the shore, point Q is at the distance d =10 miles down the shore from P.

If the man rows with a speed of $v_r = 3$ miles per hour and walks with a speed of $v_w = 4$ miles per hour at what point R should he land his boat in order to get from point S to point Q in the shortest possible time?

Solution:

Step 1: Draw a picture introducing the variables.



Step 2:

Let is note x the distance down the shore where the boat is landed. On the figure x is the length of $\overline{PR}$. Then the length $\overline{RT}$ is $d - x = 10 - x$ $[miles]$.

The question asks us to find **the point** R which **minimizes** rowing **time**.

$$t_r = \frac{d_r}{v_r}$$

$$t_r = \frac{\sqrt{36 + x^2}}{3}$$

The formula for distance d, speed (velocity) and time t
v = d/t

Step 3: We have to find the walking time.

Using Pythagoras' theorem for the right triangle ΔSPR

$$d_r = \sqrt{d_1^2 + x^2}$$

$$= \sqrt{6^2 + x^2} = \sqrt{36 + x^2}$$

$$t_w = \frac{d_w}{v_w}$$

$$t_w = \frac{10 - x}{4}$$

Step 4:

Since we want to minimize total time by setting the distance x, we should look for a function $t(x)$ representing the total time to rich the point Q from the point S when x is the distance down the shore where the boat is landed. Total time has to be converted into a function minimization problem:

$$t(x) = t_r(x) + t_w(x) = \frac{\sqrt{36 + x^2}}{3} + \frac{10 - x}{4}$$

Step 5. To solve this minimization problem (find the minimum of t(x)) we should determine the first derivative with respect to distance x.

$$t'(x) = t_r'(x) + t_w'(x) = \frac{2 \cdot x}{3 \cdot 2 \cdot \sqrt{36 + x^2}} + \frac{(-1)}{4} = \frac{x}{3 \cdot \sqrt{36 + x^2}} - \frac{1}{4}$$

Setting t'(x)=0 ⇒

$$\frac{4x - 3 \cdot \sqrt{36 + x^2}}{4 \cdot 3 \cdot \sqrt{36 + x^2}} = 0$$

$$4x - 3 \cdot \sqrt{36 + x^2} = 0$$

$$4x = 3 \cdot \sqrt{36 + x^2} \, /^2$$

$$16x^2 = 9 \cdot (36 + x^2)$$

$$7x^2 = 324$$

$$x^2 = \frac{324}{7}$$

$$x = \frac{18}{\sqrt{7}} \approx 6.8$$

We get $x = \frac{18}{\sqrt{7}} \approx 6.8$ *as the only critical value and calculate*

$$t(6.8) = \frac{\sqrt{36 + 6.8^2}}{3} + \frac{10 - 6.8}{4} \approx 3.8229 \text{ } hours$$

Step 6:

We have to find a local minimum.

$$t''(x) = \left(\frac{x}{3 \cdot \sqrt{36 + x^2}} - \frac{1}{4}\right)' = \left(\frac{x}{3 \cdot \sqrt{36 + x^2}}\right)' - \left(\frac{1}{4}\right)' =$$

$$= \frac{3 \cdot \sqrt{36 + x^2} - x \cdot 3 \cdot 2x \cdot \frac{1}{2\sqrt{36 + x^2}}}{9 \cdot (36 + x^2)} = \frac{3 \cdot (36 + x^2) - 3x^2}{9 \cdot (36 + x^2) \cdot \sqrt{36 + x^2}}$$

$$t''(6.8) = \frac{3 \cdot (36 + 6.8^2) - 3 \cdot 6.8^2}{9 \cdot (36 + 6.8^2) \cdot \sqrt{36 + 6.8^2}} > 0$$

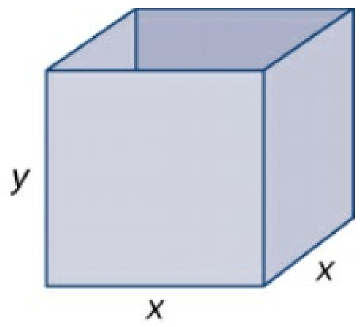
Since, $t''(6.8) > 0$, *there must be a local minimum at x=6.8 , and since this is the only critical value it must be a global minimum as well.*

Example 10:

A rectangular storage container for bulk cargo with an open top, a square base and a volume of $5000 \, m^3$ is to be constructed. What should the dimensions of the container be to minimize the surface area of the container? What is the minimum surface area?

<u>Solution:</u>

Let the variable x represent the length of each side of the square base; let y represent the height of the container and S denotes the surface area of the open-top box.



The surface area of the open-top container is calculated according following formula:

$$S = 4xy + x^2.$$

Volume of this container is:

$$V = x^2 y = 5000\, m^3$$

$$\Rightarrow y = \frac{5000}{x^2}$$

Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x \cdot \frac{5000}{x^2} + x^2$$

$$S(x) = \frac{20\,000}{x} + x^2, x > 0$$

Critical point:

$$S'(x) = -\frac{20\,000}{x^2} + 2x = 0 \Rightarrow x^3 = 10\,000 \Rightarrow x = 10\sqrt[3]{10}$$

$$\Rightarrow y = \frac{1}{2}\sqrt[3]{100}$$

$$S''(x) = 2 \cdot \frac{20\,000}{x^3} + 2$$

$$S''(10\sqrt[3]{10}) > 0$$

Therefore, *S*(*x*) has the minimum at the critical point $x = 10\sqrt[3]{10}$. It implies that is the dimensions of the container should be $x = 10\sqrt[3]{10}, y = \frac{1}{2}\sqrt[3]{100}$.

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$$S(x) = \frac{20\,000}{x} + x^2 = \frac{20\,000}{10\sqrt[3]{10}} + \left(10\sqrt[3]{10}\right)^2 = 300\sqrt[3]{10^2}$$

Example 11:

Owners of a boat rental company have determined that if they charge customers p euros per day to rent a boat, where $(50 \leq p \leq 200)$, then the number of boats n they rent per day can be modelled by the linear function $n(p) = 1000 - 5p$. If they charge €50 per day or less, they will rent all their boats. If they charge €200 per day or more, they will not rent any boats. Assuming the owners plan to charge customers between €50 per day and €200 per day to rent a boat, how much should they charge to maximize their revenue?

Solution:

From described problem, *p* denotes the price charged per boat per day, *n* the number of rented boats per day and *R* revenue per day. We have to find the maximum revenue *R*.

The revenue per day is determined with the number of boats rented per day times the price charged per boat per day. Thus,

$$R = n \cdot p = (1000 - 5p) \cdot p = -5p^2 + 1000p$$

According with the constraint that owners plan to charge between 50 and 200 euro per boat per day, the problem is to find the maximum revenue $R(p)$ (it must be satisfied $p \in [50, 200]$).

R is a continuous function over the closed, bounded interval [50, 200] and it has an absolute maximum in that interval.

$$R'(p) = -10p + 1000 = 0 \Rightarrow p = 100$$

$$R(100) = 50\,000$$

$$p = 50 \Rightarrow R(50) = 37\,500$$

$$p = 200 \Rightarrow R(200) = 0$$

The maximum revenue is reached for $p = 100$.

As conclusion: owners should charge 100 euro per boat per day to maximize their revenue.

Example 12:

The cargo *G* lowers according the law; $x = 80 \cdot t^{2,5}$, where $x[m]$ i $t[s]$. By lowering the cargo, the drum on which the rope holding the load G is wound is rotated. The angular velocity and angular acceleration of the drum must be determined.

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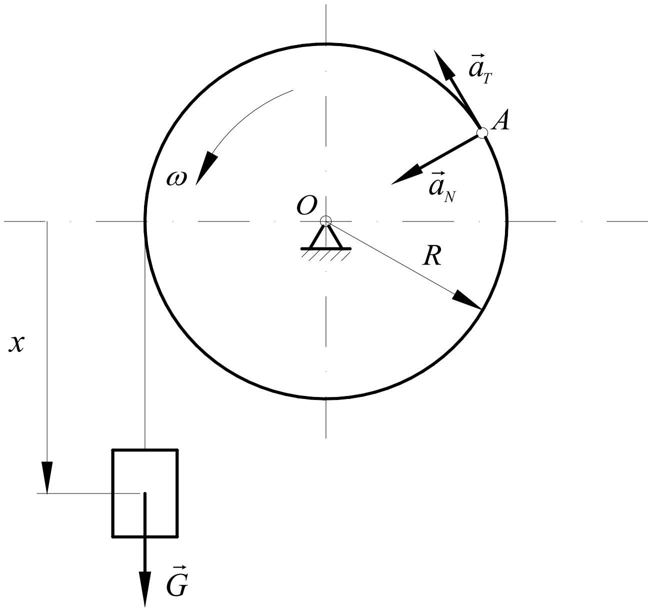

Solution:

The translational motion of the load brings the drum into rotational motion. With this movement, the speed of lowering the load is equal to the circumferential speed of rotational movement.

The speed of lowering the cargo is

$$v = \frac{dx}{dt} = \frac{d(80 \cdot t^{2,5})}{dt} = 200 \cdot t^{1,5} \left[\frac{m}{s}\right]$$

The angular velocity of the drum is obtained using the circumferential speed:

$$v = R \cdot \omega, \omega = \frac{v}{R} = \frac{200 \cdot t^{1,5}}{R}$$

$$\omega = 1000 \cdot t^{1,5} \left[s^{-1}\right]$$

The angular acceleration of the drum is

$$\varepsilon = \frac{d\omega}{dt} = \frac{d(1000 \cdot t^{1,5})}{dt} = 1500 \cdot t^{0,5} \quad \left[s^{-2}\right]$$