8 DIFFERENTIAL EQUATIONS

8.1 THE CONCEPT OF DIFFERENTIAL EQUATIONS

Definition: Differential equation

A *differential equation* is an equation that involves both an unknown function and its derivatives or differentials.

There are ordinary and partial differential equations.

Definition: Ordinary differential equation

A differential equation for a one variable function is called an ordinary differential equation.

The general form of an ordinary differential equation can be written as

$$F\left(x, y, y', y'', y''', \dots, y^{(n)}\right) = 0$$

or

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

where $y(x)$ is an unknown function and $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}, \dots,$ $y^{(n)} = \frac{d^ny}{dx^n}$ are derivatives of the function *y(x).*

Example 8.1

The following two equations,

$$y' + xy = x^3$$

$$y'' - 5y' + 6y = 13\sin(3x)$$

are ordinary differential equations for an unknown one-variable function y=y(x).

These equations can be also written as:

$$\frac{dy}{dx} + xy = x^3$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13\sin(3x)$$

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Definition: Partial differential equation

A differential equation for a function of several variables is called a ***partial differential equation*** (PDE). PDE contains partial derivatives.

Example 8.2

The equation of a string vibration

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2}$$

is a partial differential equation for the function of two variables *U=U(x,t).*

In this topic **only ordinary differential equations** of the first and second order will be considered.

Definition: Order of a differential equation

An order of a differential equation is the order of the highest derivative it contains.

Example 8.3

The first order ODE:	$y' + xy = x^3$
The second order ODE*:*	$y'' - 5y' + 6y = 13sin\,(3x)$
The third order ODE:	$y''' - x\,ln(x) = 0$

Definition: Solution of a differential equation

The ***solution*** of a differential equation is any function that satisfies given equation identically.

It means that the given equation becomes identical after substituting its solution into the differential equation.

Definition: General and particular solutions of a differential equation

A solution of an ordinary differential equation of order n, which involves exactly n (maximum number) of essential arbitrary constants is called a ***general solution***.

A solution of a differential equation obtained by substituting the defined numerical values instead of arbitrary constants in the general solution of a differential equation is called a ***particular solution***.

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Definition: Singular solution of a differential equation

A solution of an ordinary differential equation that does not contain arbitrary constants and cannot be obtained from the general solution is called a singular solution of a differential equation.

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