8.2 FIRST-ORDER DIFFERENTIAL EQUATIONS

The main concept for first-order differential equations will be given in this chapter. The separable variable equations and first-order linear differential equations with their solving methods will be considered in detail. Two solving methods of first-order linear differential equations will be presented, i.e. the method of variation of constants and the Bernoulli method (solving by using substitution).

8.2.1 *Main concept of first-order differential equations*

An ordinary differential equation of the first order can be given in the following standard forms:

a) in implicit form

$$F(x, y, y') = 0,$$

b) in explicit form

$$y' = f(x, y)$$

c) in the differential form, since $y' = dy/dx$

$$P(x, y)dx + Q(x, y)dy = 0$$

where $f(x, y)$, $P(x, y)$, $Q(x, y)$ are functions of x and y, in general.

We also point out that $f(x, y) = -P(x, y)/Q(x, y)$.

Example 8.4

Let us consider a differential equation that is given in the implicit form:

$$(x^2 + 1)y' - 2xy - 3x = 0$$

This equation can be written in explicit form:

$$y' = \frac{2xy + 3x}{x^2 + 1}$$

We can also obtain this equation in a differential form, if we substitute $\mathrm{dy/dx}$ instead of $\mathrm{y'}$ into the first equation and multiply both its sides by *dx*:

$$(2xy + 3x)dx - (x^2 + 1)dy = 0$$

The general solution of a first-order differential equation involves one arbitrary constant C and it can be written in the explicit form $y' = \varphi(x, C)$ or in the implicit form $\Phi(x, y, C) = 0$.

Definition: General integral

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The general solution of a differential equation in the implicit form $\Phi(x, y, C) = 0$ is called a *general integral*.

We get a particular solution of a differential equation if we substitute a defined number instead of a constant *C* into the general solution.

Example 8.5

Let us consider one of the easiest first-order differential equations:

$$y' = 2x.$$

To find the unknown function *y(x)* we use integration with respect to x, taking into account that $y = \int y'dx$. As a result, we have

$$y = \int 2xdx = x^2 + C$$

where C is an integration constant.

The function $y = x^2 + C$ is the **general solution** of the given differential equation, since it satisfies the given equation, i.e. $y' = (x^2 + C)' = 2x$, and contains one essential arbitrary constant C.

Assigning particular values to the arbitrary constant C in the general solution, we get so-called particular solutions. For instance,

$y = x^2 - 2$ is the particular solution which corresponds to $C = -2$,

$y = x^2 - 1$ is the particular solution which corresponds to $C = -1$,

$y = x^2$ is the particular solution which corresponds to $C = 0$,

$y = x^2 + 1$ is the particular solution which corresponds to $C = 1$,

$y = x^2 + 2$ is the particular solution which corresponds to $C = 2$.

Visualization of the obtained solutions, presents a set of parabolas, where each parabola corresponds to a particular value of the constant C (see Fig.1).

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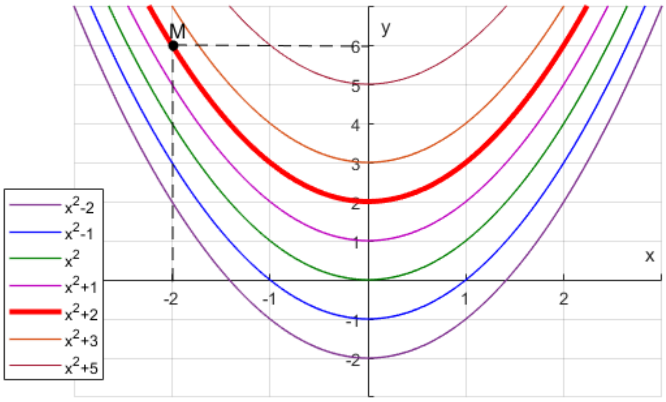

Figure 8.1

Due to the arbitrary constant *C*, the general solution is not just one function, but a set of functions. To each specific numerical value of the constant C in the general solution there corresponds one particular solution.

It is often necessary to find only one particular solution from a set of solutions. In this case, a differential equation is given together with an additional condition $y(x_0) = y_0$, which means that a value of the unknown function is equal to y_0 at some particular value x_0 of the argument. This additional condition is called an ***initial condition***.

Definition: Initial value problem

The ***initial value problem*** (Cauchy problem) is a problem which involves an ordinary differential problem $\mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{y}') = 0$ together with an initial condition $\mathrm{y}(\mathrm{x}_0) = \mathrm{y}_0$ which specifies the value of an unknown function at a given point x_0. That is a problem of finding a particular solution of the differential equation that satisfies the initial condition $\mathrm{y}(\mathrm{x}_0) = \mathrm{y}_0$.

Thus, the solutions to a differential equation can be viewed as a family of solution curves in the x y-plane. Besides, from a geometric point of view, the initial condition $y(x_0) = y_0$ is the same as a point (x_0, y_0) that the solution curve must pass through.

An initial value problem is solved in the following way

1. Find a general solution to the given differential equation that involves an arbitrary constant C.

2. Substitute x_0 and y_0 from the initial condition into the general solution instead of x and y.

3. Solve the obtained equation with respect to C.

4. Substitute the result back into the general solution.

Example 8.6

Let us take the differential equation from Example 8.5 and consider it as an initial value problem

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$$y' = 2x \text{ and } y(-2) = 6.$$

Thus, the task is to find a particular solution of the given differential equation that satisfies the initial condition $y(-2) = 6$, which means that at x=-2, the value of the solution function is y=6.

The general solution of the differential equation from Example 8.5 is

$$y = x^2 + C$$

In order to find the corresponding C value, we substitute $x = -2$ and $y = 6$ (i.e. the initial condition) into the general solution:

$$6 = (-2)^2 + C$$

$$6 = 4 + C$$

$$C = 2$$

The solution of the initial value problem is found by substituting the obtained C value into the general solution. It means that the particular solution which satisfies the given initial condition is:

$$y = x^2 + 2$$

The graphical interpretation of the result (see Fig.1) means that from the set of curves, only the curve $y = x^2 + 2$ which goes through the point M(-2,6), must be selected.

There are different types of the first order differential equations which are solved by different methods. Separable variable equations and linear equations and their respective solutions are discussed below.

8.2.2 *Separable variable equations*

Consider a differential equation given in the form $y' = f(x, y)$.

Definition:

A first-order differential equation $y' = f(x, y)$ is called a separable variable equation if the function $f(x, y)$ can be factored into the product of two functions of x and y (i.e., it can be divided into multipliers so that each multiplier depends on only one variable):

$$y' = f_1(x) \cdot f_2(y)$$

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where $f_1(x)$ and $f_2(y)$ are continuous functions.

Solution method:

1) Replace y' with $\frac{dy}{dx}$

$$\frac{dy}{dx} = f_1(x) \cdot f_2(y)$$

2) Separate the variables, i.e. move all the y terms (including *dy*) to one side of the equation and all the x terms (including *dx*) to the other side.

For this purpose, multiply both sides of the equation by *dx* and divide by $f_2(y)$:

$$\frac{dy}{f_2(y)} = f_1(x)dx$$

Here we suppose that $f_2(y) \neq 0$.

3) Integrate directly both sides of the equation with respect to their variables.

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx$$

4) Solve the obtained equation if possible.

If *y(x)* cannot be expressed in an explicit form, the expression on the right-hand side of the equation is shifted to the left side. In this case, the general integral of the differential equation is obtained.

5) Dividing by $f_2(y)$ we assume that $f_2(y) \neq 0$. It may cause loss of the solution $f_2(y) = 0$. If there are y values for which $f_2(y) = 0$ and these values satisfy the given differential equation, then these values will also be solutions of the differential equation. Therefore, we should proof if $f_2(y) = 0$ is a solution of the differential equation, and if it is a solution, then check if it is a singular solution.

Let us consider a differential equation given in the form

$$P(x,y)dx + Q(x,y)dy = 0$$

Definition:

Differential equation $P(x,y)dx + Q(x,y)dy = 0$ is called a separable variable equation if each function $P(x,y)$ and $Q(x,y)dy$ can be factored into a product of two functions so that each multiplier depends on only one variable:

$$P_1(x) \cdot P_2(y)dx + Q_1(x) \cdot Q_2(y)dy = 0$$

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Solution method:

1) Separate the variables:

$$Q_1(x) \cdot Q_2(y)dy = -P_1(x) \cdot P_2(y)dx$$

$$\frac{Q_2(y)dy}{P_2(y)} = -\frac{P_1(x)dx}{Q_1(x)}$$

2) Integrate both sides of the obtained expression:

$$\int \frac{Q_2(y)dy}{P_2(y)} = -\int \frac{P_1(x)dx}{Q_1(x)}$$

3) Obtain a solution in an explicit or implicit form.

4) Check out if $P_2(y) = 0$ and $Q_1(y) = 0$ are the singular solutions of the differential equation.

Example 8.7

Solve the differential equation $y' - (x+2) \cdot e^{-y} = 0.$

This equation can be rewritten as

$$y' = (x+2) \cdot e^{-y}$$

This equation is a separable variable equation because the function on the right-hand side of the equation is a product of two functions. One function depends on *x* only and other one depends on *y* only.

We solve the equation by the following steps:

1) Replace y' with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = (x+2) \cdot e^{-y}$$

2) Separate variables, by multiplying both sides of the equation by *dx* and dividing by e^{-y}.

$$\frac{dy}{e^{-y}} = (x+2)dx$$

Since $e^{-y} \neq 0$, we do not miss any solutions dividing by e^{-y} .

3) Now we have an expression that contains only *y* terms on the left-hand side and only *x* terms on the right-hand side. It means that we can integrate both sides of the expression:

$$\int \frac{dy}{e^{-y}} = \int (x+2)dx$$

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Let us find separately both integrals:

$$\int \frac{dy}{e^{-y}} = \int e^{y} dy = e^{y} + C_1$$

$$\int (x+2)dx = \int (x+2)d(x+2) = \frac{(x+2)^2}{2} + C_2$$

4) As a result, we have

$$e^{y} + C_1 = \frac{(x+2)^2}{2} + C_2$$

$$e^{y} = \frac{(x+2)^2}{2} + C_2 - C_1$$

Since C_1 and C_2 are constants, $C_2 - C_1 = C$ is also a constant. Therefore, after integration of both sides of the expression, only one constant C of integration is usually placed on only one side of the expression.

$$e^{y} = \frac{(x+2)^2}{2} + C$$

The solution in the explicit form is

$$y = \ln\left(\frac{(x+2)^2}{2} + C\right)$$

The last expression is the general solution of the given differential equation.

Example 8.8

Solve the differential equation $(y + xy)dx + (x - xy)dy = 0$.

The given equation can be rewritten in the form

$$y \cdot (1 + x)dx + x \cdot (1 - y)dy = 0$$

This equation is a separable variable equation because the functions before *dx* and before *dy* have the form of a product of only one variable functions.

We solve the equation by the following steps:

1) Switch the places of the terms

$$x \cdot (1 - y)dy = -y \cdot (1 + x)dx$$

2) Separate variables by dividing both sides of the equation by y and x ($x \neq 0$ and $y \neq 0$).

$$\frac{x \cdot (1-y)dy}{x \cdot y} = -\frac{y \cdot (1+x)}{x \cdot y}dx$$

$$\frac{(1-y)dy}{y} = -\frac{(1+x)}{x}dx$$

3) Integrate each side of the obtained equation:

$$\int \frac{(1-y)dy}{y} = -\int \frac{(1+x)}{x}dx$$

$$\int \left(\frac{1}{y} - 1\right) dy = -\int \left(\frac{1}{x} + 1\right) dx$$

$$\ln|y| - y = -\ln|x| - x + C$$

$$\ln|y| + \ln|x| - y + x = C$$

As a result, we have obtained the general solution of the equation in the form of a general integral:

$$\ln|y \cdot x| + x - y = C$$

where $x \neq 0$ and $y \neq 0$, to satisfy the condition on an argument of logarithmic function.

4) We check for possibly missed solutions because of dividing by x and y:

Both $x = 0$ and $y = 0$ satisfies the given differential equations, but they cannot be obtained from the general solution, so that $x = 0$ and $y = 0$ are singular solutions of the differential equation.

In general, ***solution method for the separable variable equations is:***

1) Separate variables, i.e., rewrite the equation thus that the terms depending on x and terms depending on y appear on opposite sides, so that there is only one variable on each side of the equation.

2) Integrate one side of obtained expression with respect to y and the other side with respect to x.

3) Simplify.

4) Check for possibly missed solutions, i.e. check for existence of singular solutions of the differential equation.

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8.2.3 *First-order linear differential equations*

Definition: Linear differential equation

A first-order differential equation is called a ***linear differential equation*** if it can be written in the form

$$y' + p(x)y = f(x)$$

where $p(x)$ and $f(x)$ are continuous functions.

Example 8.9

1) If the function $f(x)$ on the right-hand side of the equation is equal to zero, then the differential equation is called a ***homogeneous linear equation***:

$$y' + p(x)y = 0, \quad (f(x) = 0)$$

2) If the function $f(x)$ in on right-hand side of the equation is not equal to zero, then the differential equation is called a ***nonhomogeneous linear equation***:

$$y' + p(x)y = f(x), \quad (f(x) \neq 0)$$

Solution method:

There are two methods to solve a linear differential equation. These are the method of Variation of a Constant and Bernoulli method. Both methods will be considered here.

1. Method of Variation of a Constant

The method consists of the following steps

1) Find a general solution to the corresponding homogeneous equation

$$y' + p(x)y = 0.$$

The general solution of the homogeneous equation contains a constant of integration C.

2) Replace the constant C with a certain (but still unknown) function C(x).

3) Determine the unknown function C(x) by substituting this general solution of the homogeneous equation into the given nonhomogeneous differential equation.

Example 8.10

Solve the differential equation $xy' + y = sinx$ by the *Method of Variation of a Constant.*

This equation is a first-order linear differential equation and can be rewritten in the form

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$$y' + \frac{y}{x} = \frac{sinx}{x}$$

1) We solve the corresponding homogeneous linear equation:

$$y' + \frac{y}{x} = 0$$

This is a separable variable equation. Therefore, we replace y' with $\frac{\mathrm{dy}}{\mathrm{dx}}$, and move the second term from the left-hand side of the equation to the right-hand side:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{y}{x}$$

Then we separate variables, provided that $\mathrm{y} \neq 0$

$$\frac{\mathrm{dy}}{\mathrm{y}} = -\frac{dx}{x}$$

and integrate both sides of the equation:

$$\int \frac{\mathrm{dy}}{\mathrm{y}} = -\int \frac{\mathrm{dx}}{\mathrm{x}}$$

$$\ln|\mathrm{y}| = -\ln|\mathrm{x}| + \mathrm{C}$$

Since C is an arbitrary constant, it may be written in the form $\ln|\mathrm{C}|$:

$$\ln|\mathrm{y}| = -\ln|\mathrm{x}| + \ln|\mathrm{C}|$$

Then, using properties of a logarithm, we have:

$$\ln|\mathrm{y}| = \ln\left|\frac{\mathrm{C}}{\mathrm{x}}\right|$$

As a result, we obtain the general solution of the homogeneous equation in the form

$$\mathrm{y}_0 = \frac{\mathrm{C}}{\mathrm{x}}$$

2) In order to find a general solution of the nonhomogeneous equation, we replace the constant C with an unknown function C(x):

$$\mathrm{y} = \frac{\mathrm{C(x)}}{\mathrm{x}}$$

4) The unknown function C(x) is found by substituting $\mathrm{y} = \frac{\mathrm{C(x)}}{\mathrm{x}}$ into the given nonhomogeneous differential equation together with its derivative y':

$$y' = \left(\frac{C(x)}{x}\right)' = \frac{x \cdot C'(x) - x' \cdot C(x)}{x^2} = \frac{x \cdot C'(x) - C(x)}{x^2} = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$$

After substituting y and y' into the given equation we have

$$\left(\frac{C'(x)}{x} - \frac{C(x)}{x^2}\right) + \frac{C(x)}{x^2} = \frac{sinx}{x}$$

This equation can be simplified as

$$C'(x) = sinx$$

To find the unknown function C(x), we integrate the obtained expression with respect to x:

$$C(x) = \int C'(x)dx = \int sinx\, dx = -cosx + C$$

Substituting the obtained function *C(x)* into the expression for y, we have the general solution of the given nonhomogeneous linear differential equation:

$$y = \frac{-\cos(x) + C}{x}$$

2. Bernoulli method (Solution by using substitution $y = U \cdot V$)

The main idea is that the solution y of a linear differential equation $y' + p(x)y = f(x)$ is sought as a product of two functions $\underline{y = U \cdot V}$, where $U = U(x)$ un $V = V(x)$ are unknown functions. One of these functions can be chosen arbitrarily, but the other function must be chosen the way that the multiplication $U(x) \cdot V(x)$ satisfies the differential equation.

Steps of solving:

1) Substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given linear differential equation:

$$y' + p(x)y = f(x)$$

The equation takes the form

$$U'V + UV' + p(x)UV = f(x)$$

or

$$U'V + U \cdot (V' + p(x)V) = f(x)$$

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2) The function *V* is chosen to make the expression in the brackets equal to zero:

$$V' + p(x)V = 0$$

Then the last equation in the 1st step becomes:

$$U'V + U \cdot 0 = f(x) \quad \text{or} \quad U'V = f(x).$$

As a result, the following system is to be solved

$$\begin{cases} V' + p(x)V = 0 \\ U'V = f(x) \end{cases}$$

3) The equation $V' + p(x)V = 0$ is a separable variable differential equation so that it is solved for V by separating variables:

$$\frac{dV}{dx} = -p(x)V$$

$$\frac{dV}{V} = -p(x)dx$$

$$\int \frac{dV}{V} = -\int p(x)dx$$

$$\ln|V| = -\int p(x)dx$$

$$V = e^{-\int p(x)dx} + C_1$$

Here it is assumed that the constant $C_1 = 0$, because the equation suffices to have only one particular solution.

4) Substitute the obtained V back into the equation $U'V = f(x)$:

$$U' \cdot e^{-\int p(x)dx} = f(x)$$

6) Solve the last equation for U.

7) Finally, substitute the obtained U and V into *y = UV* and get a general solution.

Example 8.11

Solve the linear differential equation $xy' + y = sinx$ by *using substitution y=UV.*

Beforehand, we write the equation in the form

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$$y' + \frac{y}{x} = \frac{sinx}{x}$$

1) We substitute the function $\mathrm{y} = \mathrm{U} \cdot \mathrm{V}$ and its derivative $\mathrm{y}' = \mathrm{U}' \cdot \mathrm{V} + \mathrm{U} \cdot \mathrm{V}'$ into the given differential equation:

$$U'V + UV' + \frac{UV}{x} = \frac{sinx}{x}$$

or

$$U'V + U\left(V' + \frac{V}{x}\right) = \frac{sinx}{x}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' + \frac{V}{x} = 0$$

Then the equation can be written as a system of two equations:

$$\begin{cases} V' + \frac{V}{x} = 0 \\ U'V = \frac{sinx}{x} \end{cases}$$

3) We solve the first equation of the system for V. This is a separable variable equation:

$$\frac{\mathrm{dV}}{\mathrm{dx}} = -\frac{V}{x}$$

$$\frac{\mathrm{dV}}{\mathrm{V}} = -\frac{dx}{x}$$

$$\int \frac{\mathrm{dV}}{\mathrm{V}} = -\int \frac{\mathrm{dx}}{\mathrm{x}}$$

$$\ln|\mathrm{V}| = -\ln|\mathrm{x}| + \mathrm{C}_1$$

where we assume $\mathrm{C}_1 = 0$ so that:

$$\ln|\mathrm{V}| = -\ln|\mathrm{x}| \quad \rightarrow \quad \ln|\mathrm{V}| = \ln|\mathrm{x}^{-1}|$$

$$\mathrm{V} = \frac{1}{\mathrm{x}}$$

4) We substitute the obtained V back into the second equation of the system:

$$\mathrm{U'V} = \frac{\mathrm{sinx}}{\mathrm{x}}$$

$$U' \cdot \frac{1}{x} = \frac{sinx}{x}$$

We simplify and solve the obtained equation for U:

$$U' = sinx$$

$$\mathrm{U} = \int \mathrm{U'dx} = \int \mathrm{sin}x\, dx = -\mathrm{cos}x + C$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

$$\mathrm{y} = \mathrm{UV} = (-\mathrm{cos}x + C) \cdot \frac{1}{x}$$

$$\mathrm{y} = \frac{\mathrm{C} - \mathrm{cos}x}{\mathrm{x}}$$

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8.2.4 *Exercises*

Exercise 8.1.

Solve the initial value problem (Cauchy problem)

$$y' = y \cdot \cot x\,, \qquad y\left(\frac{\pi}{2}\right) = 3.$$

Solution:

This equation is a separable variable equation because the function on the right-hand side of the equation is a product of two functions. One function depends on x only and other one depends on y only.

We solve the equation in the following steps:

1) Replace y' with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y \cdot \cot x$$

2) We separate variables, multiplying both sides of the equation by *dx* and dividing by y:

$$\frac{dy}{y} = \cot x \cdot dx$$

Here we suppose $y \neq 0$.

3) We have the equation that contains only y terms on the left-hand side and only x terms on the right-hand side. It means that we can integrate both sides of the equation:

$$\int \frac{dy}{y} = \int \cot x dx$$

Let us find the integral on the right-hand side of the expression:

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$$

4) As a result, we have

$$\ln|y| = \ln|\sin x| + C$$

where C is a constant.

In order to simplify the obtained solution, we can write $\ln|C_1|$ on the right side of the expression instead of C, where C_1 is also a constant ($C_1 \neq 0$). It is allowed due to both C and $\ln|C_1|$ being arbitrary constants.

$$\ln|y| = \ln|\sin x| + \ln|C_1|$$

According to the properties of logarithmic functions, we have

$$\ln|y| = \ln|C_1 \cdot \sin x|$$

As a result, we obtain the **general solution** of the given differential equation in the explicit form:

$$y = C_1 \cdot \sin x\,, \qquad (C_1 \neq 0\,).$$

4) Check for possibly missed solutions due to dividing by y:

The $y = 0$ satisfies the given differential equations, but it will be not the singular solution if we rewrite the obtained general solution of the differential equation in the form

$$y = C \cdot \sin x\,,$$

where C is an arbitrary constant (it can be equal by zero).

In this case the solution $y = 0$ can be obtained from the general solution at C=0, therefore it is the particular solution of the given equation.

As a result, the general solution of the given differential equation is

$$y = C \cdot \sin x$$

5) To solve the initial value problem, we should find only one particular solution of the differential equation that satisfies the initial condition $y\left(\frac{\pi}{2}\right) = 3$, i.e. the value of the function y(x) must be equal to 3 at $x = \frac{\pi}{2}$. In order to find this particular solution, we insert $y = 3$ and x= $\frac{\pi}{2}$ into the general solution.

$$3 = C \cdot \sin\frac{\pi}{2}$$

$$3 = C \cdot 1$$

$$C = 3\,.$$

We insert the obtained value of the constant C onto the general solution and get a **particular solution** of the given initial value problem.

$$y = 3 \cdot \sin x$$

Exercise 8.2.

Solve the initial value problem (Cauchy problem)

$$2(x^2 y + y)y' + \sqrt{1 + y^2} = 0\,, \qquad y(0) = 2.$$

Solution:

This equation can be rewritten as

$$2y(x^2 + 1)y' = -\sqrt{1 + y^2}$$

This equation is a separable variable equation because the function before y' on the left-hand side of the equation is a product of two functions. One function depends on x only and the other one depends on y only. The function on the right-hand side of the equation depends only on y.

We solve the equation by the following steps:

1) At first, we replace y' with $\frac{dy}{dx}$:

$$2y(x^2+1)\frac{dy}{dx}=-\sqrt{1+y^2}$$

2) We separate the variables, multiplying both sides of the equation by *dx* and dividing by (x^2+1) and by $\sqrt{1+y^2}$:

$$\frac{2ydy}{\sqrt{1+y^2}}=-\frac{dx}{(x^2+1)}$$

Since $\sqrt{1+y^2}\neq 0$ and $(x^2+1)\neq 0$, we do not miss any solution dividing by (x^2+1) and by $\sqrt{1+y^2}$.

3) Now we have an expression that contains only y terms on the left-hand side and only x terms on the right-hand side. This means that we can integrate both sides of the expression:

$$\int\frac{2ydy}{\sqrt{1+y^2}}=-\int\frac{dx}{(x^2+1)}$$

Let us find the integral in the left-hand side of the expression:

$$\int\frac{2ydy}{\sqrt{1+y^2}}=\int\frac{d(y^2)}{\sqrt{1+y^2}}=\int(1+y^2)^{-\frac{1}{2}}d(1+y^2)=2(1+y^2)^{\frac{1}{2}}+C$$

4) As a result, we have

$$2\sqrt{1+y^2}=-\text{arctan}x+C$$

or

$$2\sqrt{1+y^2}+\text{arctan}x=C$$

where C is an arbitrary constant.

The last expression is the general solution of the given differential equation.

5) To solve the initial value problem, we should find only one particular solution of the differential equation that satisfies the initial condition $y(0)=2$. In order to find this particular solution, we insert $y=2$ and x$=0$ into the general solution.

$$2\sqrt{1+2^2}+\text{arctan}0=C$$

$$2\sqrt{5}+0=C \quad \rightarrow \quad C=2\sqrt{5} \quad .$$

We insert the obtained value of the constant C onto the general solution and get the particular solution of the given initial value problem in implicit form

$$2\sqrt{1+y^2}+\text{arctan}x=2\sqrt{5}$$

Exercise 8.3.

Solve the differential equation $(e^{2x}+3)dy+y\cdot e^{2x}dx=0.$

Solution:

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This equation is a separable variable equation because the function before *dy* depends only on x and the function *dx* has the form of a product of only one variable functions.

We solve the equation by the following steps:

1) Switch the places of the terms

$$(e^{2x}+3)dy = -y \cdot e^{2x}dx$$

2) We separate variables by dividing both sides of the equation by y and by $e^{2x}+3$ ($y \neq 0$).

$$\frac{dy}{y} = -\frac{e^{2x}}{e^{2x}+3}dx$$

3) We integrate each side of the obtained equation:

$$\int \frac{dy}{y} = -\int \frac{e^{2x}}{e^{2x}+3}dx$$

$$\int \frac{dy}{y} = -\frac{1}{2}\int \frac{d(e^{2x}+3)}{e^{2x}+3}$$

$$\ln|y| = -\frac{1}{2}\ln|e^{2x}+3| + \ln|C|$$

We simplify the obtained solution by using the properties of a logarithmic function:

$$\ln|y| = \ln(e^{2x}+3)^{-\frac{1}{2}} + \ln|C|$$

$$\ln|y| = \ln\left|C \cdot (e^{2x}+3)^{-\frac{1}{2}}\right|$$

$$y = C \cdot (e^{2x}+3)^{-\frac{1}{2}}$$

As a result, we obtain the general solution of the given differential equation in the form:

$$y = \frac{C}{\sqrt{e^{2x}+3}}$$

4) We check for possibly missed solutions due to dividing by y:

$y = 0$ satisfies the given differential equations, and it can be obtained from the general solution at C=0. Therefore, it is not a singular solution.

Excercise 8.4.

Solve the initial value problem (Cauchy problem)

$$y' + ytanx = \frac{1}{cosx}, \qquad y(\pi) = 0.$$

Solution:

This is a first-order linear differential equation, which is to be solved using substitution y=UV.

1) We substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given differential equation:

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$$\mathrm{U'V} + \mathrm{UV'} + \mathrm{UV} \cdot tanx = \frac{1}{\cos x}$$

$$\mathrm{U'V} + \mathrm{U}(\mathrm{V'} + \mathrm{V} \cdot tanx) = \frac{1}{\cos x}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' + \mathrm{V} \cdot tanx = 0$$

Then the last equation in step 1 can be written as a system of two equations:

$$\begin{cases} V' + \mathrm{V} \cdot tanx = 0 \\ \mathrm{U'V} = \dfrac{1}{\cos x} \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{\mathrm{dV}}{\mathrm{dx}} = -\mathrm{V} \cdot \tan x$$

$$\frac{\mathrm{dV}}{\mathrm{V}} = -\tan x \, dx$$

$$\int \frac{\mathrm{dV}}{\mathrm{V}} = -\int \tan x \, dx$$

The right-hand side is equal to

$$-\int \tan x \, dx = -\int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} d(cosx) = \ln|\cos x| + C_1$$

Therefore,

$$\ln|\mathrm{V}| = \ln|\cos x| + \mathrm{C}_1$$

Assuming $\mathrm{C}_1 = 0$, we have

$$\ln|\mathrm{V}| = \ln|\cos x|$$

$$\mathrm{V} = \cos x$$

4) We substitute the obtained V back into the second equation of the system:

$$U'V = \frac{1}{\cos x}$$

$$U'cosx = \frac{1}{\cos x}$$

We simplify and solve the obtained above equation:

$$U' = \frac{1}{\cos^2 x}$$

$$\mathrm{U} = \int \mathrm{U'dx} = \int \frac{1}{\cos^2 x} \mathrm{dx} = \tan x + \mathrm{C}$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

$$y = UV = (\tan x + C) \cdot \cos x = \left(\frac{\sin x}{\cos x} + C\right) \cdot \cos x = \sin x + C \cdot \cos x$$

As a result, the general solution of the differential equation is

$$y = \sin x + C \cdot \cos x$$

6) We solve the initial value problem, it means that we should find only one particular solution of the differential equation that satisfies the initial condition $y(\pi) = 0$, i.e. the value of the function y(x) must to be equal to 0 at $x = \pi$. In order to find this particular solution, we substitute $y = 0$ and $x = \pi$ into the general solution.

$$0 = \sin\pi + C \cdot \cos\pi$$

$$0 = 0 + C \cdot (-1) \;\Rightarrow\; -C = 0 \quad \Rightarrow\; C = 0.$$

We substitute the obtained value of the constant C onto the general solution and get the particular solution of the given initial value problem

$$y = \sin x$$

Exercise 8.5.

Solve the differential equation $y' - 3x^2 y = x \cdot e^{x^3}$.

Solution:

This is a first-order linear differential equation. We will solve it by the substitution y=UV.

1) We substitute the function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the given differential equation:

$$U'V + UV' - 3x^2 \cdot UV = x \cdot e^{x^3}$$

$$U'V + U(V' - V \cdot 3x^2) = x \cdot e^{x^3}$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' - V \cdot 3x^2 = 0$$

The last equation in step 1 can be written as a system of two equations:

$$\begin{cases} V' - V \cdot 3x^2 = 0 \\ U'V = x \cdot e^{x^3} \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{dV}{dx} = V \cdot 3x^2$$

$$\frac{dV}{V} = 3x^2\, dx$$

$$\int \frac{dV}{V} = \int 3x^2\, dx$$

$$\ln|V| = x^3 + C_1$$

On assuming $C_1 = 0$, we have:

$$\ln|V| = x^3 \quad \rightarrow \quad V = e^{x^3}$$

4) We substitute the obtained *V* back into the second equation of the system:

$$U'V = x \cdot e^{x^3} \quad \rightarrow \quad U'e^{x^3} = x \cdot e^{x^3}$$

We simplify and solve the equation for *U*:

$$U' = x$$

$$U = \int U'dx = \int x\,dx = \frac{x^2}{2} + C$$

5) Substituting the obtained U and V into y = UV, we get the general solution for the given linear differential equation:

$$y = UV = \left(\frac{x^2}{2} + C\right) \cdot e^{x^3}$$

Exercise 8.6.

Solve the initial value problem

$$(1 + x^2)y' = 2xy + (1 + x^2)^2, \qquad y(1) = 4.$$

Solution:

First, we rewrite the given equation in the form

$$(1 + x^2)y' - 2xy = (1 + x^2)^2$$

We divide both sides of the equation by $(1 + x^2)$:

$$y' - \frac{2xy}{(1 + x^2)} = (1 + x^2)$$

Now it is clear, that this differential equation is a first-order linear differential equation, which can be solved by substitution *y=UV* (Bernoulli method).

1) We substitute function $y = U \cdot V$ and its derivative $y' = U' \cdot V + U \cdot V'$ into the differential equation:

$$U'V + UV' - \frac{2x \cdot UV}{(1 + x^2)} = (1 + x^2)$$

$$U'V + U\left(V' - \frac{2x \cdot V}{(1 + x^2)}\right) = (1 + x^2)$$

2) According to the method, we equate to zero the expression in the brackets:

$$V' - \frac{2x \cdot V}{(1 + x^2)} = 0$$

The last equation in step 1 can be written as a system of two equations:

$$\begin{cases} V' - \dfrac{2x \cdot V}{(1+x^2)} = 0 \\ \mathrm{U'V} = (1+x^2) \end{cases}$$

3) We solve the first equation of the system by separating variables:

$$\frac{dV}{dx} = \frac{2x \cdot V}{(1+x^2)}$$

$$\int \frac{\mathrm{dV}}{\mathrm{V}} = \int \frac{2x}{(1+x^2)}\, dx$$

To evaluate integral on the right-hand side of the equation, we use $2xdx = d(x^2) = d(1+x^2)$,

$$\int \frac{2x}{(1+x^2)}\, dx = \int \frac{1}{(1+x^2)}\, d(x^2+1) = \ln|x^2+1| + C_1$$

Therefore,

$$\ln|\mathrm{V}| = \ln|x^2+1| + \mathrm{C}_1$$

Assuming $\mathrm{C}_1 = 0$, we have

$$\ln|\mathrm{V}| = \ln|x^2+1| \quad \rightarrow \quad \mathrm{V} = x^2+1$$

4) We substitute the obtained *V* back into the second equation of the system:

$$U'V = x^2+1$$

$$U'(x^2+1) = x^2+1$$

Simplify and solve the equation for *U*:

$$U' = 1$$

$$\mathrm{U} = \int \mathrm{U'dx} = \int 1\, \mathrm{dx} = x + \mathrm{C}$$

5) Substituting the obtained U and V into *y = UV* we get the general solution for the given linear differential equation:

$$\mathrm{y} = \mathrm{UV} = (\mathrm{x}+\mathrm{C}) \cdot (x^2+1)$$

As a result, the general solution of the differential equation is

$$\mathrm{y} = (\mathrm{x}+\mathrm{C}) \cdot (x^2+1)$$

In order to find a particular solution, we substitute $\mathrm{y} = 4$ and x$= 1$ into the general solution.

$$4 = (1+\mathrm{C}) \cdot (1^2+1)$$

$$4 = (1+C) \cdot 2 \quad \rightarrow \quad \mathrm{C}+1 = 2 \quad \rightarrow \quad \mathrm{C} = 1$$

Substitute the obtained value of the constant C onto the general solution to get the particular solution of the given initial value problem:

$$\mathrm{y} = (\mathrm{x}+1) \cdot (x^2+1) = x^3 + x^2 + x + 1$$

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