8.3 SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

In this section we shortly consider basic concepts for second-order differential equations. The second-order linear differential equations with constant coefficients and two solving methods will be considered in detail, i.e. the method of variation of constants and the method of undetermined coefficients.

8.3.1 *Basic concepts for second-order differential equations. Second-order linear differential equations.*

A second-order differential equation can be written in the general (implicit) form

$$F(x, y, y', y'') = 0$$

or in the explicit form

$$y'' = f(x, y, y')$$

where *y=y(x)* is an unknown function.

The general solution of a second-order differential equation involves two arbitrary constants C_1 and C_2. It can be written in the explicit form $y = \varphi(x, C_1, C_2)$ or the implicit form $\Phi(x, y, C_1, C_2) = 0$.

Example 8.12

Consider the simplest second-order differential equation

$$y'' = 6x$$

The unknown function *y(x)* is found by integrating both sides of the equation two times with respect to x:

$$y' = \int 6x dx = 3x^2 + C_1$$

$$y = \int (3x^2 + C_1) dx = x^3 + C_1 x + C_2$$

where C_1, C_2 are arbitrary integration constants.

The general solution of the given equation is:

$$y = x^3 + C_1 x + C_2$$

In order to find only one particular solution of a second-order differential equation, two additional conditions are necessary. These additional conditions can be given as

1) *Initial value conditions*, when the functions' y(x) and y'(x) values are prescribed at defined x_0 value of x:

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$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

2) *Boundary conditions*, when the function's y(x) values are prescribed at different x_1 and x_2 values of x.

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2$$

Example 8.13

Let us consider the differential equation from Example 8.12 as an initial value problem

$$y'' = 6x \text{ and } y(0) = 1\,, y'(0) = 2$$

The general solution of the differential equation from Example 8.11 is

$$y = x^3 + C_1x + C_2.$$

In order to find the corresponding C_1 and C_2 values, we do the following:

1) Substitute $x = 0$ and $y = 1$ (i.e., the initial condition for *y*) into the general solution:

$$1 = 0^3 + C_1 0 + C_2$$

2) Find a *y'(x)* derivative of the general solution y (x) and substitute $x = 0$ and $y' = 2$ into obtained expression:

$$y' = 3x^2 + C_1$$

$$2 = 3 \cdot 0^2 + C_1$$

As a result, we find constants corresponding to the initial conditions:

$$C_1 = 2 \quad \text{and} \quad C_2 = 1$$

The solution of the initial value problem is found by substituting obtained C_1 and C_2 values into the general solution, and the particular solution that satisfies the given initial condition is:

$$y = x^3 + 2x + 1$$

In the following chapter we will consider second-order linear differential equations with constant coefficients.

Definition: Second-order linear differential equations

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A second-order differential equation is called a ***linear differential equation,*** if it can be written in the form

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x)$$

where $a_1(x)$, $a_2(x)$ and $a_3(x)$ are continuous functions and $a_1(x) \neq 0$.

Definition: Homogeneous and Nonhomogeneous linear differential equations

1) If the function $f(x)$ on the left-hand side of a linear equation is not equal to zero $(f(x) \neq 0)$, then the differential equation is called a ***nonhomogeneous linear equation***:

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x)$$

2) If the function $f(x)$ on the left-hand side of a linear equation is equal to zero $(f(x) = 0)$, then the differential equation is called a ***homogeneous linear equation:***

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = 0$$

Definition: Second-order linear differential equations with constant coefficients

A second-order linear differential equation is called *a **linear differential equation with constant coefficients*** if coefficients before y'', y' and y are constants

$$a_1y'' + a_2y' + a_3y = 0$$

where a_1 , a_2 and a_3 are constants and $a_1 \neq 0$.

8.3.2 *Second-order linear Homogeneous differential equations with constant coefficients*

Consider a second-order **linear homogeneous differential equation with constant coefficients**:

$$a_1y'' + a_2y' + a_3y = 0$$

where a_1 , a_2 and a_3 are some constant coefficients and $a_1 \neq 0$.

Solution method:

For each of the linear homogeneous differential equation with constant coefficients can be written the, so-called, *characteristic (*also called *auxiliary)* equation:

$$a_1k^2 + a_2k + a_3 = 0$$

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation. There exist three cases, as follows:

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1. The discriminant of the characteristic quadratic equation $\underline{D > 0}$.

   In this case, the roots of the characteristic equations ***are real and distinct*** $k_1 \neq k_2$, and the general solution of the homogeneous differential equation in this case has the form:

   $$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

   where C_1 and C_2 are arbitrary real numbers.

2. The discriminant of the characteristic equation $\underline{D = 0}$.

   In this case, the roots **are real and** equal $k_1 = k_2 = k$ (***repeated***), and the general solution of the differential equation has the form:

   $$y = C_1 e^{kx} + C_2 x e^{kx} \quad \text{or} \quad y = (C_1 + C_2 x)e^{kx}$$

3. The discriminant of the characteristic equation $\underline{D < 0}$.

   In this case, the roots **are complex** and conjugate, $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$ $(i = \sqrt{-1})$ and the general solution is written as

   $$y = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$$

Example 8.14

Let us consider the following linear differential equation with constant coefficients:

$$y'' + 3y' - 10y = 0$$

The corresponding characteristic (auxiliary) equation is

$$k^2 + 3k - 10 = 0$$

The discriminant of this equation $D = 49 > 0$; therefore, the roots are real and distinct:

$$k_1 = 2 \quad \text{and} \quad k_2 = -5$$

Then the general solution of the differential equation is

$$y = C_1 e^{2x} + C_2 e^{-5x}.$$

Example 8.15

Consider the equation:

$$y'' - 4y' + 4y = 0$$

Its characteristic (auxiliary) equation is

$$k^2 - 4k + 4 = 0$$

The discriminant of the quadratic equation $D = 0$, and the roots are real and repeated:

$$k_1 = k_2 = 2$$

The general solution of the differential equation is

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$$y = C_1e^{2x} + C_2xe^{2x}$$

Example 8.16

Consider the equation:

$$y'' + 2y' + 10y = 0$$

Its characteristic (auxiliary) equation is:

$$k^2 + 2k + 10 = 0$$

The discriminant of the quadratic equation $D = -36 < 0$, and the roots complex and conjugate:

$$k_1 = -1 + 3i \quad \text{and} \quad k_2 = -1 - 3i$$

The general solution of the differential equation is

$$y = C_1e^{-1\cdot x}\cos 3x + C_2e^{-1\cdot x}\sin 3x$$

8.3.3 *Second-order linear Nonhomogeneous differential equations with constant coefficients*

A **nonhomogeneous** linear differential equation with constant coefficient has the form

$$a_1y'' + a_2y' + a_3y = f(x)$$

where a_1 , a_2 and a_3 are arbitrary constants and $a_1 \neq 0$.

For each nonhomogeneous linear differential equation its related homogeneous differential equation can be written as

$$a_1y'' + a_2y' + a_3y = 0$$

1.1.1.1 Theorem.

A general solution of a nonhomogeneous equation is the sum of the general solution $y_c(x)$ of the related homogeneous equation and a *particular* solution $Y(x)$ of the nonhomogeneous equation:

$$y = y_c(x) + Y(x)$$

There exist two general approaches to find a particular solution $Y(x)$ of a nonhomogeneous differential equation. These are the Method of Undetermined Coefficients, and the Method of Variation of Constants.

1.1.1.2 Method of Variation of Constants

The Lagrangian constant variation method can be used for any type of function *f(x)* on the right-hand side of the nonhomogeneous linear differential equation.

Steps of solving:

1) First, solve an associated homogeneous equation

$$a_1 y'' + a_2 y' + a_3 y = 0$$

and find the general solution $y_c(x)$ of this equation. The general solution of the homogeneous equation contains two constants C_1 and C_2 and can be written in the form

$$y_c = C_1 \cdot y_1 + C_2 \cdot y_2$$

where C_1 , C_2 are constants and functions y_1, y_2 depend on the roots of the characteristic equation.

2) Replace the constants C_1 and C_2 with arbitrary (still unknown) functions $C_1(x)$ and $C_2(x)$ and find the ***general solution*** of the ***given nonhomogenous equation*** in the form

$$y = C_1(x) \cdot y_1 + C_2(x) \cdot y_2$$

3) Taking into account that $y = C_1(x)y_1 + C_2(x)y_2$ satisfies the given nonhomogeneous equation with the right-hand side f(x), it can be shown that the unknown functions $C_1(x)$ and $C_2(x)$ can be determined from the system of two equations:

$$\begin{cases} C'_1(x) \cdot y_1 + C'_2(x) \cdot y_2 = 0 \\ C'_1(x) \cdot y'_1 + C'_2(x) \cdot y'_2 = \dfrac{f(x)}{a_1} \end{cases}$$

4) Find $C_1{}'(x)$ and $C_2{}'(x)$ from the system.

5) By integration find $C_1(x) = \int C_1{}'(x)dx$ and $C_2(x) = \int C_2{}'(x)dx$

6) Substitute the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution.

Example 8.17

Solve the equation:

$$y'' + 9y = \frac{1}{\cos 3x}$$

We solve an associated homogeneous equation

$$y'' + 9y = 0$$

Its characteristic (auxiliary) equation is

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$$k^2 + 9 = 0 \quad => \quad k^2 = -9$$

The roots are complex and conjugate:

$$\mathrm{k}_1 = \sqrt{-9} = 3\mathrm{i} = 0 + 3\mathrm{i} \quad \text{and} \quad \mathrm{k}_2 = -\sqrt{-9} = -3\mathrm{i} = 0 - 3\mathrm{i}$$

The general solution of the associated *homogeneous differential* equation is

$$\mathrm{y_c} = \mathrm{C_1 e^{0\cdot x} cos3x + C_2 e^{0\cdot x} sin3x}$$

or

$$\mathrm{y_c} = \mathrm{C_1 cos3x + C_2 sin3x}$$

where $\mathrm{C_1}$ and $\mathrm{C_2}$ are arbitrary constants.

2) We replace the constants $\mathrm{C_1}$ and $\mathrm{C_2}$ with the arbitrary (still unknown) functions $\mathrm{C_1(x)}$ and $\mathrm{C_2(x)}$ and find the *general solution* of the given *nonhomogeneous* differential equation in the form:

$$\mathrm{y = C_1(x) cos3x + C_2(x) sin3x}$$

3) To determine the unknown functions $\mathrm{C_1(x)}$ and $\mathrm{C_2(x)}$, we write a system of equations for derivatives of the unknown functions

$$\begin{cases} \mathrm{C'_1(x)} \cdot \cos 3x + \mathrm{C'_2(x)} \cdot \sin 3x = 0 \\ \mathrm{C'_1(x)} \cdot (\cos 3x)' + \mathrm{C'_2(x)} \cdot (\sin 3x)' = \dfrac{1}{\cos 3x} \end{cases}$$

The system can be written in the form

$$\begin{cases} \mathrm{C'_1(x)} \cdot \cos 3x + \mathrm{C'_2(x)} \cdot \sin 3x = 0 \\ \mathrm{C'_1(x)} \cdot (-3\sin 3x) + \mathrm{C'_2(x)} \cdot 3\cos 3x = \dfrac{1}{\cos 3x} \end{cases}$$

4) We will solve the system by using Cramer's rule, so that we need to find a determinant of the coefficient matrix:

$$\mathrm{D} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

and the determinants

$$D_1 = \begin{vmatrix} 0 & \sin 3x \\ \dfrac{1}{\cos 3x} & 3\cos 3x \end{vmatrix} = 0 - \frac{\sin 3x}{\cos 3x} = -\tan 3x$$

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$$D_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \dfrac{1}{\cos 3x} \end{vmatrix} = \frac{\cos 3x}{\cos 3x} - 0 = 1$$

Then

$$C'_1(x) = \frac{D_1}{D} = -\frac{\tan 3x}{3} \qquad \text{and} \qquad C'_2(x) = \frac{D_2}{D} = \frac{1}{3}$$

5) We find unknown functions $C_1(x)$ and $C_2(x)$ by integrating:

$$C_1(x) = \int C'_1(x)\,dx \quad \text{and} \quad C_2(x) = \int C'_2(x)\,dx \ .$$

$$C_1(x) = \int \left(-\frac{\tan 3x}{3}\right) dx = -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x}\, dx = \frac{1}{9}\int \frac{d(\cos 3x)}{\cos 3x} = \frac{1}{9}\ln|\cos 3x| + \tilde{C}_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int \frac{1}{3}\, dx = \frac{1}{3}\int 1\, dx = \frac{1}{3}x + \tilde{C}_2$$

Thus,

$$C_1(x) = \frac{1}{9}\ln|\cos 3x| + \tilde{C}_1 \quad \text{and} \quad C_2(x) = \frac{1}{3}x + \tilde{C}_2$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constants.

6) Substitute the obtained functions $C_1(x)$ and $C_2(x)$ into the form of general solution of the nonhomogeneous differential equation:

$$y = \left(\frac{1}{9}\ln|\cos 3x| + \tilde{C}_1\right)\cos 3x + \left(\frac{1}{3}x + \tilde{C}_2\right)\sin 3x$$

As the result, the general solution of given nonhomogeneous differential equation is

$$y = \tilde{C}_1\cos 3x + \tilde{C}_2\sin 3x + \frac{1}{9}\ln|\cos 3x|\cdot\cos 3x + \frac{1}{3}x\cdot\sin 3x$$

or

$$y = C_1\cos 3x + C_2\sin 3x + \frac{1}{9}\ln|\cos 3x|\cdot\cos 3x + \frac{1}{3}x\cdot\sin 3x$$

where C_1 and C_2 are also arbitrary constants.

Note that the sum of the two first terms in the obtained solution is the general solution for associated homogenous differential equation, and the sum of the last two terms is the particular solution of the nonhomogeneous differential equation.

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8.3.4 *Method of Undetermined Coefficients*

Consider second-order nonhomogeneous differential equations with right-hand functions that has derivatives that vary little (in type of function) from their parent functions. These functions are: polynomial $P_n(x)$ functions, exponential functions $e^{\alpha x}$, trigonometric functions (sine and cosine ($\sin\beta x$, $\cos\beta x$)), as well as the sum, difference and multiplication of these functions. In this case we can predict the form of solution of this differential equation taking into account the form of its right-hand function.

The main idea of the Method of Undetermined Coefficients is to construct the form of a particular solution Y(x) of the given nonhomogenous equation corresponding to the form (based on the form) of a function f(x) on the right side of the equation. Y(x) is written down as a function with undefined coefficients, then is substituted into the equation and the coefficients are found.

As was mentioned before, this method works only for a restricted class of functions on the right-hand side of the equation, such as

$$f(x) = P_n(x)e^{\alpha x}$$

$$f(x) = (P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)) \cdot e^{\alpha x}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degrees n and m, respectively.

In both cases a choice for the particular solution should match the structure of the right-hand side function of the nonhomogeneous equation. It depends on the right side of the equation as well as on the roots of the characteristic equation.

Let us consider in detail how to construct the form of a particular solution Y(x) of a given nonhomogenous equation.

Consider three cases for a function on the right-hand side of the equation:

1) $f(x) = P_n(x)e^{\alpha x}$ $(\beta = 0)$

The particular solution has the same form as f(x), only instead of polynomial $P_n(x)$ we write polynomial with undefined coefficients. Furthermore, if the coefficient α in the *argument of the exponential* function *coincides with a root* of the auxiliary (characteristic) equation, the particular solution will contain the additional factor x^s, where s is the order of the root α in the characteristic equation.

This means that the particular solution *Y* is written down in the form

$$Y = \widetilde{P_n}(x)e^{\alpha x} \cdot x^s$$

where

a) $\widetilde{P_n}(x)$ is a polynomial of order *n* with unknown coefficients, i.e.

if n=0, then $\widetilde{P_0}(x) = A$;

if n=1, then $\widetilde{P_1}(x) = Ax + B$;

if n=2, then $\widetilde{P_2}(x) = Ax^2 + Bx + C$;

and so on.

b) To find the power s of factor x^s ,we compare the coefficient α in the power of the exponential function with the roots $k_1$and k_2 of the auxiliary equation:

if $\alpha \neq k_1$ and $\alpha \neq k_2$ then $\underline{s = 0}$;

if $\alpha = k_1$ and $\alpha \neq k_2$ OR $\alpha \neq k_1$ and $\alpha = k_2$ then $\underline{s = 1}$;

if $\alpha = k_1 = k_2$ then $\underline{s = 2}$;

2) $f(x) = e^{\alpha x}(N\cos(\beta x) + M\sin(\beta x))$, where *N, M* are constants

$f(x) = e^{\alpha x}N\cos(\beta x) \iff f(x) = e^{\alpha x}(N\cos(\beta x) + \mathbf{0} \cdot \sin(\beta x))$ *(M=0)*

$f(x) = e^{\alpha x}M\sin(\beta x) \iff f(x) = e^{\alpha x}(\mathbf{0} \cdot \cos(\beta x) + M \cdot \sin(\beta x))$ *(N=0)*

The particular solution has the same form as f(x) only instead of constants *N and M* we write unknown coefficients. Furthermore, if the number $\alpha + \beta i$ *coincides with a root* of the auxiliary (characteristic) equation, the particular solution will contain the additional multiplier x^s, where s is the order of the root $\alpha + \beta i$ in the characteristic equation.

This means that the particular solution *Y* is written down in the form

$$Y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \cdot x^s$$

where A and B are unknown coefficients.

To find the power s of multiplier x^s , we compare the number $\alpha + \beta i$ with the roots $k_1$and k_2 of the auxiliary equation:

if $\alpha + i\beta \neq k_1$ and $\alpha + i\beta \neq k_2$ then $\underline{s = 0}$;

if $\alpha + i\beta = k_1$ or $\alpha + i\beta = k_2$ then $\underline{s = 1}$.

3) $f(x) = e^{\alpha x}(P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x))$

$$f(x) = e^{\alpha x} P_n(x)\cos(\beta x) \quad \Longleftrightarrow \quad f(x) = e^{\alpha x}(P_n(x)\cos(\beta x) + \mathbf{0}\cdot\sin(\beta x))$$

$$f(x) = e^{\alpha x} Q_n(x)\sin(\beta x) \quad \Longleftrightarrow \quad f(x) = e^{\alpha x}(\mathbf{0}\cdot\cos(\beta x) + Q_n(x)\cdot\sin(\beta x))$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of order n and m respectively.

In these cases the particular solution is found in the form

$$Y = e^{\alpha x}\left(\widetilde{P_k}(x)\cos(\beta x) + \widetilde{Q_k}(x)\sin(\beta x)\right)\cdot x^s$$

where $\widetilde{P_k}(x)$ and $\widetilde{Q_k}(x)$ are *polynomials* of order k *with unknown coefficients* and $k = \max(n, m)$.

To find the power s of multiplier x^s , we compare the number $\alpha + \beta i$ with the roots k_1 and k_2 of the auxiliary equation:

if $\alpha + i\beta \neq k_1$ and $\alpha + i\beta \neq k_2$ then $\underline{s = 0}$;

if $\alpha + i\beta = k_1$ or $\alpha + i\beta = k_2$ then $\underline{s = 1}$.

The unknown coefficients are determined by substitution of the expected type of the particular solution into the original nonhomogeneous differential equation.

<u>Scheme of solving:</u>

1) Solve the corresponding homogeneous differential equation $\mathrm{a_1 y'' + a_2 y' + a_3 y = 0}$;

2) By the form of function f(x) on the right-hand side of the equation, write down the form of a particular solution Y with undefined coefficients;

3) Find Y' and Y'';

4) Determine the undefined coefficients A, B, C by substitution of the particular solution Y and its derivatives into the given original nonhomogeneous differential equation.

5) Substitute obtained coefficients into the form of the particular solution Y.

6) Write down the general solution of the given nonhomogeneous differential equation as

$$\mathrm{y = y_c(x)} + Y(\mathrm{x})$$

where $\mathrm{y_c(x)}$ is the general solution of the related homogeneous equation, and $Y(\mathrm{x})$ is a particular solution of the given nonhomogeneous equation.

Example 8.18

Let us solve the equation:

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$$y'' - 2y' = x^2 + 5x - 1$$

1) First, we solve the associated ***homogeneous equation***:

$$y'' - 2y' = 0$$

The auxiliary equation for this equation is:

$$k^2 - 2k = 0 \quad => \quad k \cdot (k-2) = 0$$

The roots of the auxiliary equation are real and distinct:

$$k_1 = \mathbf{0} \quad \text{and} \quad k_2 = 2$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1 e^{0 \cdot x} + C_2 e^{2 \cdot x}$$

or

$$y_c = C_1 + C_2 e^{2x}$$

where C_1 and C_2 are constants.

2) We write down the form for the particular solution Y, taking into account form of function $\underline{f(x) = x^2 + 5x - 1}$ on the right-hand side of the equation.

The function $f(x)$ can be written in the form: $f(x) = (x^2 + 5x - 1) \cdot e^{\mathbf{0} \cdot x}$.

In this case the particular solution Y has the form: $Y = \widetilde{P_n}(x) e^{\alpha x} \cdot x^s$.

a) The function $f(x)$ has a polynomial with degree 2 before the exponential function (n=2), therefore the polynomial $\widetilde{P_n}(x)$ must also be a polynomial with degree 2, but with undefined coefficients: $\widetilde{P_2}(x) = Ax^2 + Bx + C$

b) The coefficient in the argument of the exponential function is ***α=0***. It **coincides with one root** of the auxiliary (characteristic) equation: $\alpha = k_1 = 0$, therefore $s = 1$ and the particular solution will contain the additional factor x^1 .

Thus, the particular solution of the differential equation *Y* has the form:

$$Y = (Ax^2 + Bx + C) \cdot e^{\mathbf{0} \cdot x} \cdot x^1 = (Ax^2 + Bx + C) \cdot x$$

or

$$Y = Ax^3 + Bx^2 + Cx$$

3) We find first- and second-order derivatives for Y:

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$$Y' = (Ax^3 + Bx^2 + Cx)' = 3Ax^2 + 2Bx + C$$

$$Y'' = (3Ax^2 + 2Bx + C)' = 6Ax + 2B$$

4) We substitute them into the given nonhomogeneous differential equation:

$$y'' - 2y' = x^2 + 5x - 1.$$

As a result, we have:

$$6Ax + 2B - 2(3Ax^2 + 2Bx + C) = x^2 + 5x - 1$$

We simplify the left-hand expression:

$$-6Ax^2 + \ 6Ax - 4Bx + 2B - 2C = x^2 + 5x - 1$$

We group coefficients with the same powers of x on the left-hand side of the equation:

$$-6Ax^2 + \ (6A - 4B)x + 2B - 2C = 1 \cdot x^2 + 5x + (-1)$$

The right and left sides of the equation are equal for every $x \in R$. It would be possible only if the coefficients at the same powers of x on the right and left sides of the equation are equal:

The coefficients at x^2: $-6A = 1$;

The coefficients at x: $6A - 4B = 5$;

The coefficients at x^0: $2B - 2C = -1$.

Solve the obtained system of equations: $\begin{cases} -6A = 1 \\ 6A - 4B = 5 \\ 2B - 2C = -1 \end{cases}$

From the first equation of the system we have: $A = -\frac{1}{6}$.

From the second equation: $6A - 4B = 5$ ➔ $6 \cdot \left(-\frac{1}{6}\right) - 4B = 5$ ➔ $-4B = 6$

$$B = -\frac{3}{2}$$

From the third equation: $2B - 2C = -1$ ➔ $2 \cdot \left(-\frac{3}{2}\right) - 2C = -1$ ➔ $-2C = 2$

$$C = -1$$

Substitute the obtained coefficients into the form of the particular solution Y:

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$$Y = -\frac{1}{6}x^3 - \frac{3}{2}x^2 - x$$

As a result, the general solution of the given nonhomogeneous equation is :

$$y = y_c + Y = C_1 + C_2e^{2x} \; -\frac{1}{6}x^3 + \frac{3}{2}x^2 + 2x$$

Example 8.19

Solve the equation:

$$y'' + y' - 2y = xe^{2x}$$

1) The associated homogeneous equation is

$$y'' + y' - 2y = 0$$

Its auxiliary equation is

$$k^2 + k - 2 = 0$$

The roots for this equation are real and distinct:

$$k_1 = 1 \quad \text{and} \quad k_2 = -2$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1e^{1 \cdot x} + C_2e^{-2 \cdot x}$$

where C_1 and C_2 are constants.

2) We construct the form of a particular solution *Y* by taking into account the form of the function on the right-hand side of the equation $(x) = xe^{2x}$.

For such function (x) , the particular solution *Y* has the form: $Y = \widetilde{P_n}(x)e^{\alpha x} \cdot x^s$.

a) The function $f(x)$ has a polynomial with degree 1 before the exponential function (n=1), therefore the polynomial with undefined coefficients $\widetilde{P_n}(x)$ must also be a polynomial with degree 1: $\widetilde{P_1}(x) = Ax + B$.

b) The coefficient in the power of the exponential function is $\alpha = 2$. It ***does not* coincide with any root** of the auxiliary (characteristic) equation: $\alpha \neq k_1$ and $\alpha \neq k_2$, therefore, $s = 0$ and the particular solution does not contain an additional factor.

Thus, a particular solution Y of the differential equation Y has the form

$$Y = (Ax + B) \cdot e^{2 \cdot x} \cdot x^0 = (Ax + B) \cdot e^{2 \cdot x}$$

3) We find first- and second-order derivatives for Y:

$$Y' = \left((Ax + B)e^{2 \cdot x}\right)' = (Ax + B)' \cdot e^{2x} + (Ax + B) \cdot (e^{2 \cdot x})' = Ae^{2x} + (Ax + B) \cdot 2e^{2 \cdot x}$$

It can be also written as $Y' = (2Ax + 2B + A)e^{2x}$.

$$Y" = \left((2Ax + 2B + A)e^{2x}\right)' = (2Ax + 2B + A)' \cdot e^{2x} + (2Ax + 2B + A) \cdot (e^{2x})' =$$

$$= 2\text{A}e^{2x} + (2Ax + 2B + A) \cdot 2e^{2x} = (4Ax + 4B + 4A) \cdot e^{2x}$$

So, $Y" = (4Ax + 4B + 4A) \cdot e^{2x}$.

4) Substitute the obtained Y", Y' and Y into the given nonhomogeneous differential equation:

$$y'' + y' - 2y = xe^{2x}.$$

As a result, we have:

$$(4Ax + 4B + 4A) \cdot e^{2x} + (2Ax + 2B + A)e^{2x} - 2(Ax + B) \cdot e^{2 \cdot x} = xe^{2x}$$

We simplify the expression:

$$(4Ax + 4B + 4A + 2Ax + 2B + A - 2Ax - 2B) \cdot e^{2x} = xe^{2x}$$

$$4Ax + 4B + 5A = x$$

or $$4Ax + 4B + 5A = 1x + 0.$$

The right and left sides of the equation are equal for every $\forall x \in R$. That would only be possible if the coefficients at the same powers of x on the right-hand side and left-hand side of the equation are equal.

The coefficients at x: $4A = 1$;

The coefficients at x^0: $4B + 5A = 0$.

That leads us to solving the system: $\begin{cases} 4A = 1 \\ 4B + 5A = 0 \end{cases}$.

It follows from the first equation of the system: $A = \frac{1}{4}$.

It follows from the second equation: $4B + 5 \cdot \frac{1}{4} = 0$ ➔ $4B = -\frac{5}{4}$ ➔ $B = -\frac{5}{16}$.

We substitute the obtained coefficients into the form of the particular solution Y:

$$Y = (Ax + B) \cdot e^{2x} = \left(\frac{1}{4}x - \frac{5}{16}\right) \cdot e^{2x}$$

As a result, the general solution of the given nonhomogeneous equation is

$$y = y_c + Y = C_1 e^x + C_2 e^{-2x} + \left(\frac{1}{4}x - \frac{5}{16}\right) \cdot e^{2x}$$

Example 8.20

Solve the equation

$$y'' + y = 3\cos x + 2\sin x.$$

1) The associated homogeneous equation is

$$y'' + y = 0$$

Its auxiliary equation is

$$k^2 + 1 = 0 \;\rightarrow\; k^2 = -1 \;\rightarrow\; k^2 = \pm\sqrt{-1} \;= \pm i$$

The roots of the auxiliary equation are complex and conjugated:

$$k_1 = i = 0 + 1 \cdot i \quad \text{and} \quad k_2 = -i = 0 - 1 \cdot i$$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1 e^{0 \cdot x} \cos(1 \cdot x) + C_2 e^{0 \cdot x} \sin(1 \cdot x)$$

$$y_c = C_1 \cos x + C_2 \sin x$$

2) The function on the right-hand side of the equation is $\underline{f(x) = 3cosx + 2sinx}$.

The function $f(x)$ can be written as $f(x) = e^{0x}\left(3cos(1 \cdot x) + 2sin(1 \cdot x)\right)$.

For such function $f(x)$, the particular solution Y has the form:

$$Y = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right) \cdot x^s.$$

The power of the exponential function in function $f(x)$ is $\alpha = 0$ and the coefficient before *x* in the argument of cosine and sine is $\boldsymbol{\beta = 1}$.

The number $\boldsymbol{\alpha + i\beta = 0 + 1 \cdot i = i}$ coincides with one root of the auxiliary (characteristic) equation: $\alpha + i\beta = k_1$, therefore, $s = 1$ and the particular solution contains the factor x^1.

Thus, the particular solution Y of the differential equation Y has the form:

$$Y = e^{0\cdot x}(\mathrm{A}\cos x + \mathrm{B}\sin x) \cdot x^1 = (\mathrm{A}\cos x + \mathrm{B}\sin x) \cdot \mathrm{x}$$

3) We find first and second-order derivatives for Y:

$$Y' = \left((\mathrm{A}\cos x + \mathrm{B}\sin x) \cdot \mathrm{x}\right)' = (\mathrm{A}\cos x + \mathrm{B}\sin x)' \cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x) \cdot (\mathrm{x})' =$$

$$= (-A\sin x + B\cos x) \cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x)$$

$$Y'' = \left((-A\sin x + B\cos x) \cdot \mathrm{x} + (\mathrm{A}\cos x + \mathrm{B}\sin x)\right)' =$$

$$= (-A\sin x + B\cos x)' \cdot \mathrm{x} + (-A\sin x + B\cos x) \cdot \mathrm{x}' + (\mathrm{A}\cos x + \mathrm{B}\sin x)' =$$

$$= (-A\cos x - B\sin x) \cdot \mathrm{x} + (-A\sin x + B\cos x) - A\sin x + B\cos x =$$

$$= (-A\cos x - B\sin x) \cdot \mathrm{x} - 2A\sin x + 2B\cos x$$

4) We substitute Y” and Y into the given nonhomogeneous differential equation:

As a result, we have:

$$(-A\cos x - B\sin x) \cdot \mathrm{x} - 2A\sin x + 2B\cos x + (\mathrm{A}\cos x + \mathrm{B}\sin x) \cdot \mathrm{x} = 3\cos x + 2\sin x$$

We simplify the obtained expression:

$$-2A\sin x + 2B\cos x = 3\cos x + 2\sin x$$

The right and left sides of the equation are equal for every $x \in R.$ It would be possible only if the coefficients at $\sin x$ and $\cos x$ on the right-hand side and left-hand side of the equation are equal:

The coefficients at $\cos x$: $2B = 3 \quad \Longrightarrow \quad B = \frac{3}{2}$

The coefficients at $\sin x$: $-2A = 2 \quad \Longrightarrow \quad A = -1$

We substitute the obtained coefficients into the form of a particular solution Y:

$$Y = \left(-1 \cdot \cos x + \frac{3}{2} \cdot \sin x\right) \cdot \mathrm{x}$$

As a result, the general solution of the given nonhomogeneous equation is:

$$\mathrm{y} = \mathrm{y_c} + \mathrm{Y} = \mathrm{C_1}\cos x + \mathrm{C_2}\sin x + \left(-\cos x + \frac{3}{2}\sin x\right) \cdot \mathrm{x}$$

Superposition Principle

If the right side of a nonhomogeneous equation is the sum of several functions such as

$f(x) = P_n(x)e^{\alpha x}$ and $f(x) = (P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)) \cdot e^{\alpha x}$,

then a particular solution of the differential equation is also the sum of particular solutions constructed separately for each such function on the right-hand side expression.

Example 8.21

Find a general solution of the differential equation:

$$y'' - 2y' + y = e^x + 5\cos 3x.$$

1) The associated homogeneous equation:

$$y'' - 2y' + y = 0$$

The auxiliary equation for this equation is $k^2 - 2k + 1 = 0$

The roots of this equation are real and repeated: $k_1 = k_2 = 1$

Therefore, the general solution of the associated *homogeneous differential* equation is

$$y_c = C_1 e^x + C_2 x e^x$$

2) We see that the right-hand side of the given equation is the sum of two functions $f_1(x) = e^x$ and $f_2(x) = 5\cos 3x$. According to the superposition principle, a particular solution is a sum of particular solutions so that it can be expressed

$$Y = Y_1 + Y_2$$

where Y_1 is a particular solution for the differential equation $y'' - 2y' + y = e^x$

and Y_2 is a particular solution for the equation $y'' - 2y' + y = 5\cos 3x$.

a) First, we determine the function Y_1. In this case $\boldsymbol{f_1(x) = e^x}$ and we will be looking for a solution in the form

$$Y_1 = Ae^{\alpha x} \cdot x^s$$

The power of the exponential function is $\boldsymbol{\alpha = 1}$ and it coincides **with two roots** of the auxiliary (characteristic) equation: $\alpha = k_1 = k_2$, therefore, $s = 2$ and the particular solution Y_1 contains the factor x^2.

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Thus, the particular solution Y_1 of the first differential equation has the form

$$Y_1 = Ae^x \cdot x^2 = Ax^2e^x$$

3) We find first- and second-order derivatives for Y_1:

$$Y_1{}' = (Ax^2e^x)' = (Ax^2)' \cdot e^x + (Ax^2) \cdot (e^x)' = 2Ax \cdot e^x + (Ax^2) \cdot e^x = (2Ax + Ax^2) \cdot e^x$$

$$Y_1{}'' = \left((2Ax + Ax^2) \cdot e^x\right)' = (2Ax + Ax^2)' \cdot e^x + (2Ax + Ax^2) \cdot (e^x)' =$$

$$= (2A + 2Ax) \cdot e^x + (2Ax + Ax^2) \cdot e^x = (2A + 4Ax + Ax^2) \cdot e^x$$

Substitute $Y_1{}'$, $Y_1{}''$ and Y_1 into the corresponding nonhomogeneous differential equation:

$$y'' - 2y' + y = e^x,$$

we have

$$(2A + 4Ax + Ax^2) \cdot e^x - 2(2Ax + Ax^2) \cdot e^x + Ax^2e^x = e^x$$

We simplify the obtained expression:

$$(2A + 4Ax + Ax^2 - 4Ax - 2Ax^2 + Ax^2)e^x = e^x$$

and get: $2A = 1$ ➔ $A = 1/2$.

Then

$$Y_1 = \frac{1}{2}x^2e^x$$

b) <u>We determine the function Y_2.</u>

Due to the form of the function $\boldsymbol{f_2(x) = 5\cos 3x = e^{0x}(5 \cdot \cos 3x + 0 \cdot \sin 3x)}$, we seek for a solution in the form

$$Y_2 = e^{ox}(C \cdot \cos 3x + D \cdot \sin 3x) \cdot x^s$$

The power of the exponential function is $\alpha = 0$ and the coefficient before x in the argument of cosine and sine is $\boldsymbol{\beta = 3}$. The number $\boldsymbol{\alpha + i\beta = 0 + 3i = 3i}$ does not coincide with any root of the auxiliary equation, therefore, $s = 0$.

$$Y_2 = C \cdot \cos 3x + D \cdot \sin 3x$$

We find first- and second-order derivatives for Y_2:

$$Y_2{}' = (C\cos 3x + D\sin 3x)' = -3C\sin 3x + 3D\cos 3x$$

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$Y_2" = (-3C\sin 3x + 3D\cos 3x)' = -9C\cos 3x - 9D\sin 3x$.

After substituting Y_2', Y_2'' and Y_2 into the corresponding nonhomogeneous differential equation:

$$y'' - 2y' + y = 5\cos 3x,$$

We have:

$$-9C\cos 3x - 9D\sin 3x - 2(-3C\sin 3x + 3D\cos 3x) + C\cos 3x + D\sin 3x = 5\cos 3x$$

$$(-8C - 6D)\cos 3x + (-8D + 6C)\sin 3x = 5\cos 3x$$

The coefficients at $\cos 3x$: $-8C - 6D = 5$

The coefficients at $\sin 3x$: $-8D + 6C = 0 \Longrightarrow C = \frac{4D}{3}$

It follows from the first equation: $-8 \cdot \left(\frac{4D}{3}\right) - 6D = 5 \Rightarrow \frac{-50D}{3} = 5 \Rightarrow D = -\frac{3}{10}$

and $$C = \frac{4D}{3} = -\frac{4\cdot 3}{3\cdot 10} = -\frac{2}{5}$$

As a result,

$$Y_2 = -\frac{2}{5}\cdot\cos 3x - \frac{3}{10}\cdot\sin 3x$$

As a result, the general solution of the given nonhomogeneous equation is:

$$y = y_c + Y = y_c + Y_1 + Y_2$$

Then the general solution of the given differential equation is:

$$y = C_1e^x + C_2xe^x + \frac{1}{2}x^2e^x - \frac{2}{5}\cos 3x - \frac{3}{10}\cdot\sin 3x$$

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8.3.5 *Exercises*

Exercise 8.7.

Find a general and particular solution of the differential equation:

$$y'' - 4y' + 13y = 0\,, \quad y(0) = 6, \quad y'(0) = 1$$

Solution:

The auxiliary equation for the given differential equation is

$$k^2 - 4k + 13 = 0$$

The discriminant of the quadratic equation is $D = -36 < 0$, therefore, the roots are complex and conjugated:

$$k_1 = \frac{4 + \sqrt{-36}}{2} = 2 + 3 \cdot i \quad \text{and} \quad k_2 = \frac{4 - \sqrt{-36}}{2} = 2 - 3 \cdot i$$

It means that the general solution of the given differential equation is

$$y = C_1 e^{2 \cdot x} \cos(3x) + C_2 e^{2 \cdot x} \sin(3x)$$

In order to find the particular solution that satisfies the given initial conditions,

1) We substitute $x = 0$ and $y = 6$ (i.e., the initial condition) into the general solution:

$$6 = C_1 e^{2 \cdot 0} \cos(0) + C_2 e^{2 \cdot 0} \sin(0)$$

$$6 = C_1 \cdot 1 + C_2 \cdot 0$$

$$C_1 = 6$$

2) We find a y'(x) derivative of the general solution y (x).

$$y' = \left(e^{2 \cdot x} \cdot (C_1 \cos 3x + C_2 \sin 3x)\right)' = (e^{2 \cdot x})'(C_1 \cos 3x + C_2 \sin 3x) + \; e^{2 \cdot x}(C_1 \cos 3x + C_2 \sin 3x)' =$$

$$= 2e^{2 \cdot x}(C_1 \cos 3x + C_2 \sin 3x) + \; e^{2 \cdot x}(-3C_1 \sin 3x + 3C_2 \cos 3x) \; =$$

$$= e^{2 \cdot x}(2C_1 \cos 3x + 2C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x)$$

Thus, the derivative of the general solution is

$$y' = e^{2 \cdot x}(2C_1 \cos 3x + 2C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x)$$

We substitute $x = 0$ and $y' = 1$ from the initial conditions into the obtained expression:

$$1 = e^{2 \cdot 0}(2C_1 \cos 0 + 2C_2 \sin 0 - 3C_1 \sin 0 + 3C_2 \cos 0) = 2C_1 + 0 - \; 0 + 3C_2$$

$$1 = 2C_1 + 3C_2$$

We substitute $C_1 = 6$ into the obtained expression, then

$$1 = 12 + 3C_2$$

$$C_2 = -\frac{11}{3}$$

We substitute the obtained constants into the general solution. As a result, the particular solution of the given differential equation is

$$y = 6e^{2x}\cos(3x) - \frac{11}{3}e^{2x}\sin(3x)$$

Exercise 8.8.

Solve the equation:

$$y'' + 2y' + y = 3e^{-x}\sqrt{x+1}$$

Solution:

We use the Method of Variation of Constants

1) We solve the associated homogeneous equation:

$$y'' + 2y' + y = 0$$

Its characteristic (auxiliary) equation is

$$k^2 + 2k + 1 = 0$$

$$(k+1)^2 = 0$$

The roots of this equation are real and repeated

$$k_1 = k_2 = -1$$

The general solution of the associated homogeneous differential equation is

$$y_0 = C_1e^{-x} + C_2xe^{-x}$$

where C_1 and C_2 are constants.

2) We replace the constants C_1 and C_2 with arbitrary functions $C_1(x)$ and $C_2(x)$ and find the general solution of the given nonhomogeneous differential equation in the form

$$y = C_1e^{-x} + C_2xe^{-x}$$

3) To determine the unknown functions $C_1(x)$ and $C_2(x)$, we write a system of equations for derivatives of the unknown functions

$$\begin{cases} C'_1(x)\cdot e^{-x} + C'_2(x)\cdot xe^{-x} = 0 \\ C'_1(x)\cdot (e^{-x})' + C'_2(x)\cdot (xe^{-x})' = 3e^{-x}\sqrt{x+1} \end{cases}$$

After finding derivatives, we have

$$\begin{cases} C'_1(x)\cdot e^{-x} + C'_2(x)\cdot xe^{-x} = 0 \\ C'_1(x)\cdot (-e^{-x}) + C'_2(x)\cdot \left(1\cdot e^{-x} + x(-e^{-x})\right) = 3e^{-x}\sqrt{x+1} \end{cases}$$

The system can be written in the form

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$$\begin{cases} (C'_1(x) + C'_2(x) \cdot x)e^{-x} = 0 \\ \left(-C'_1(x) + C'_2(x) \cdot (1-x)\right) \cdot e^{-x} = 3e^{-x}\sqrt{x+1} \end{cases}$$

Let us simplify the system to the form

$$\begin{cases} C'_1(x) + C'_2(x) \cdot x = 0 \\ -C'_1(x) + C'_2(x) \cdot (1-x) = 3\sqrt{x+1} \end{cases}$$

It follows from the first equation of the system:

$$C'_1(x) = -C'_2(x) \cdot x$$

Substituting the obtained $C'_1(x)$ into the second equation of the system, it yields:

$$C'_2(x) \cdot x + C'_2(x) \cdot (1-x) = 3\sqrt{x+1}$$

As a result, we obtain $C'_2(x)$:

$$C'_2(x) = 3\sqrt{x+1}$$

Taking into account the expression for $C'_1(x)$, we have

$$C'_1(x) = -C'_2(x) \cdot x = -3x\sqrt{x+1}$$

4) We find the unknown functions $C_1(x)$ and $C_2(x)$ using integration

$$C_1(x) = \int C'_1(x)\,dx = \int -3x\sqrt{x+1}\,dx$$

To find this integral, we use the substitution $x + 1 = t^2$.

Then $x = t^2 - 1$ and $dx = (t^2 - 1)'dt = 2tdt$

$$\int -3x\sqrt{x+1}\,dx = -3\int (t^2-1)t \cdot 2tdt = -6\int (t^4 - t^2)dt = -6\left(\frac{t^5}{5} - \frac{t^3}{3}\right) + C_1$$
$$= -\frac{6}{5}\left(\sqrt{x+1}\right)^5 + 2\left(\sqrt{x+1}\right)^3 + C_1$$

As a result,

$$C_1(x) = -\frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + C_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int 3\sqrt{x+1}\ \ dx = \ 3\int (x+1)^{\frac{1}{2}}\ d(x+1) = 2(x+1)^{\frac{3}{2}} + C_2$$

As a result,

$$C_1(x) = -\frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + C_1 \quad \text{and} \quad C_2(x) = 2(x+1)^{\frac{3}{2}} + C_2$$

where C_1 and C_2 are constants.

6) We insert the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution:

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$$y = \left(-\frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + C_1\right)e^{-x} + \left(2(x+1)^{\frac{3}{2}} + C_2\right)xe^{-x}$$

Let us simplify the obtained expression:

$$y = e^{-x}\left(C_1 + xC_2 - \frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}} + 2x(x+1)^{\frac{3}{2}}\right) =$$

$$= e^{-x}\left(C_1 + xC_2 - \frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{3}{2}}(1+x)\right) = e^{-x}\left(C_1 + xC_2 - \frac{6}{5}(x+1)^{\frac{5}{2}} + 2(x+1)^{\frac{5}{2}}\right)$$

As a result, the general solution of the given nonhomogenous differential equation is:

$$y = e^{-x}\left(C_1 + xC_2 + \frac{4}{5}(x+1)^{\frac{5}{2}}\right)$$

Exercise 8.9.

Solve the equation

$$y'' - 2y' = \frac{4e^{2x}}{1+e^{2x}}$$

Solution:

For this equation we use the Method of Variation of Constants, since the function on the right-hand side does not have a special form.

1) The associated homogeneous equation is

$$y'' - 2y' = 0$$

The characteristic (auxiliary) equation is

$$k^2 - 2k = 0$$

$$k(k-2) = 0$$

The roots of the characteristic (auxiliary) equation are real and distinct:

$$k_1 = 0 \quad \text{and} \quad k_2 = 2$$

The general solution of the associated homogeneous differential equation is

$$y_c = C_1e^{0\cdot x} + C_2e^{2\cdot x}$$

or

$$y_c = C_1 \cdot 1 + C_2 \cdot e^{2x}$$

where C_1 and C_2 are constants.

2) We replace the constants C_1 and C_2 with the arbitrary (but still unknown) functions $C_1(x)$ and $C_2(x)$ and find the general solution of the given nonhomogeneous differential equation in the form:

$$y = C_1(x) \cdot 1 + C_2(x) \cdot e^{2x}$$

3) To determine the unknown functions $C_1(x)$ and $C_2(x)$, we write a system of equations for derivatives of the unknown functions

$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_1(x)\cdot (1)' + C'_2(x)\cdot (e^{2x})' = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

The system can be written in the form

$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_1(x)\cdot 0 + C'_2(x)\cdot 2\cdot e^{2x} = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

or

$$\begin{cases} C'_1(x)\cdot 1 + C'_2(x)\cdot e^{2x} = 0 \\ C'_2(x)\cdot 2\cdot e^{2x} = \dfrac{4e^{2x}}{1+e^{2x}} \end{cases}$$

4) From the second equation of the system we have:

$$C'_2(x) = \frac{2}{1+e^{2x}}$$

From the first equation of the system, it follows that

$$C'_1(x) = -C'_2(x)\cdot e^{2x} = -\frac{2\cdot e^{2x}}{1+e^{2x}}$$

5) We find the unknown functions $C_1(x)$ and $C_2(x)$ using integration

$$C_1(x) = \int C'_1(x)\,dx \quad \text{and} \quad C_2(x) = \int C'_2(x)\,dx \ .$$

$$C_1(x) = \int C'_1(x)\,dx = -\int \frac{2\cdot e^{2x}}{1+e^{2x}}dx = -\int \frac{d(e^{2x})}{1+e^{2x}} = -\int \frac{d(1+e^{2x})}{1+e^{2x}} = -\ln|1+e^{2x}| + C_1$$

$$C_2(x) = \int C'_2(x)\,dx = \int \frac{2}{1+e^{2x}}dx = 2\int \frac{1+e^{2x}-e^{2x}}{1+e^{2x}}dx = 2\int \frac{1+e^{2x}}{1+e^{2x}}dx - 2\int \frac{e^{2x}}{1+e^{2x}}dx =$$

$$= 2\int 1\,dx - \int \frac{2e^{2x}}{1+e^{2x}}dx = 2x - \ln|1+e^{2x}| + C_2$$

As a result,

$$C_1(x) = -\ln|1+e^{2x}| + C_1 \quad \text{and} \quad C_2(x) = 2x - \ln|1+e^{2x}| + C_2$$

where C_1 and C_2 are constants.

6) Insert the obtained functions $C_1(x)$ and $C_2(x)$ into the form of the general solution:

$$y = (-\ln|1+e^{2x}| + C_1)\cdot 1 + (2x - \ln|1+e^{2x}| + C_2)e^{2x}$$

As the result, the general solution of the given nonhomogeneous differential equation is:

$$y = C_1 + C_2 e^{2x} - \ln|1+e^{2x}| + (2x - \ln|1+e^{2x}|)e^{2x}$$

It can be also written in the form

$$y = C_1 + C_2 e^{2x} + 2xe^{2x} - \ln|1+e^{2x}|\cdot(1+e^{2x})$$

Exercise 8.10.

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Find the general solution of the differential equation:

$$y'' - 9y = x + 2e^{-3x}$$

Solution:

1) The associated homogeneous equation is

$$y'' - 9y = 0$$

The auxiliary equation for this equation is $k^2 - 9 = 0$

The roots are real and distinct: $k_1 = 3\,,\ k_2 = -3$

Therefore, the general solution of the associated homogeneous differential equation is

$$y_c = C_1e^{3x} + C_2e^{-3x}$$

2) The right-hand side of the given equation is the sum of two functions:

$f_1(x) = x$ and $f_2(x) = 2e^{-3x}$.

According to the superposition principle, a particular solution is expressed by the formula

$$Y = Y_1 + Y_2$$

where Y_1 is a particular solution for the differential equation $y'' - 9y = x$

and Y_2 is a particular solution for the equation $y'' - 9y = 2e^{-3x}$.

a) First, we determine the function Y_1. The function $f_1(x)$ can be written as

$$f_1(x) = x = (x - 0)\cdot e^{0x}$$

In this case we will be looking for a solution in the form

$$Y_1 = (Ax + B)e^{\alpha x}\cdot x^s$$

The coefficient in the argument of the exponential function is $\alpha = 0$. It does not coincide with roots of the auxiliary (characteristic) equation: $k_1 = 3,\ k_2 = -3,$ therefore $s = 0$ and the particular solution Y_1 does not contain any additional factor.

Thus, the particular solution Y_1 of the differential equation has the form

$$Y_1 = (Ax + B)e^{0x}\cdot x^0 = Ax + B$$

3) We find first- and second-order derivatives for Y_1 :

$$Y_1{}' = (Ax + B)' = A$$

$$Y_1{}'' = (A)' = 0$$

We substitute $Y_1{}'$, $Y_1{}''$ and Y_1 into the corresponding nonhomogeneous differential equation

$$y'' - 9y = x,$$

As a result, we have:

$$0 - 9(Ax + B) = x$$

We simplify the obtained expression:

$$-9Ax - 9B = 1 \cdot x$$

The coefficients at x are $-9A = 1 \quad \Longrightarrow \quad A = -1/9$

The coefficients at x^0 are $-9B = 0 \quad \Longrightarrow \quad B = 0$

Then

$$Y_1 = -\frac{1}{9}x$$

b) We determine the function Y_2.

Due to function $f_2(x) = 2e^{-3x}$, we will construct the form of the particular solution as

$$Y_2 = Ce^{-3x} \cdot x^s$$

The coefficient in the argument of the exponential function is α =-3. It coincides with one root $k_2 = -3$ of the auxiliary equation, therefore $s = 1$ and the particular solution contains the factor x^1.

Thus, the particular solution Y_2 of the differential equation has the form:

$$Y_2 = Ce^{-3x} \cdot x$$

Find first- and second-order derivatives for Y:

$$Y_2{}' = (Ce^{-3x} \cdot x)' = -3Ce^{-3x}x + Ce^{-3x} = e^{-3x}(-3Cx + C)$$

$$Y_2" = (e^{-3x} \cdot (-3Cx + C))' = -3e^{-3x} \cdot (-3Cx + C) + e^{-3x} \cdot (-3C) = e^{-3x} \cdot (9Cx - 6C)$$

After substituting $Y_2{}'$, $Y_2{}''$ and Y_2 into the corresponding nonhomogeneous differential equation

$$y'' - 9y = 2e^{-3x},$$

we obtain

$$e^{-3x} \cdot (9Cx - 6C) - 9Ce^{-3x} \cdot x = 2e^{-3x}$$

$$e^{-3x} \cdot (9Cx - 6C - 9Cx) = 2e^{-3x}$$

$$-6C = 2$$

$$C = -1/3$$

As a result,

$$Y_2 = -\frac{1}{3}e^{-3x} \cdot x$$

The general solution of the given nonhomogeneous equation is equal to

$$y = y_c + Y = y_c + Y_1 + Y_2$$

Therefore, the general solution of the given differential equation is:

$$y = C_1e^{3x} + C_2e^{-3x} - \frac{1}{9}x - \frac{1}{3}xe^{-3x}$$

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