8.4 APPLICATION OF THE LAPLACE TRANSFORM FOR SOLVING DIFFERENTIAL EQUATIONS

In this chapter, we consider the solution of second-order linear nonhomogeneous differential equations by using the Laplace transform. Definition and properties of the Laplace transform also are considered in brief.

8.4.1 *The Laplace transform. Definition and main properties.*

The Laplace transform is one of the most popular solving methods of linear differential equations. It is widely used for solving both ordinary and partial differential equations. For linear ordinary differential equations, the Laplace transform is especially preferred in cases where the right-hand side function f(x) of the equation is not a continuous function of x. This kind of functions often occurs in applications in the electrical circuit theory, automatic control theory, signal theory and etc.

Definition: the Laplace Transform

Suppose that the real argument function f(t) satisfies the following three conditions:

1) $f(t)$ is defined at $t \geq 0$,

2) $f(t)$ is a continuous or piecewise continuous function (it has a finite number of the first-type break points) in the interval $t \in [0, +\infty)$,

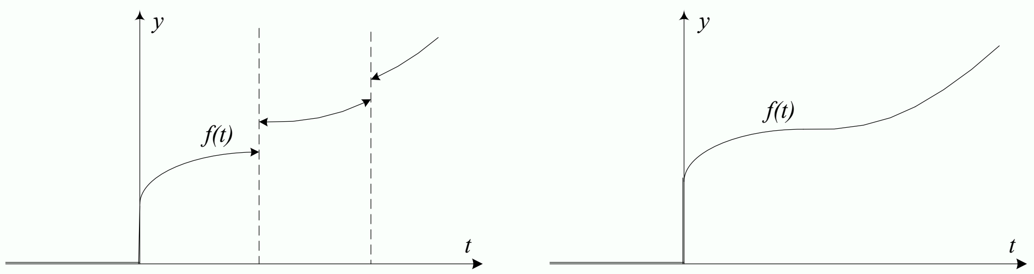

Figure 8.2

3) there exist such positive numbers $M = const$ and $S_0 = const$, that for all $t \geq 0$ holds

$$|f(t)| < Me^{S_0 t}.$$

In this case, the ***Laplace transform*** F(s) of a function f(t) is defined as the improper integral

$$\mathrm{F(s)} = \int_0^{+\infty} \mathrm{f(t)e^{-st}dt}\,,$$

where *s* is a parameter (a complex number $s = \sigma + \omega i$ in the general case).

Co-funded by the
Erasmus+ Programme
of the European Union

The improper integral on the right-hand side is called a ***Laplace integral***.

The function ***f(t)*** is called an ***original*** and the function ***F(s)*** is called a ***transform***.

If the function F(s) is the transform of the function f(t), we use the notation

$$F(s) = L[f(t)] \qquad \text{or} \qquad f(t) \div F(s)$$

Theorem:

If the function satisfies the previously-mentioned conditions, then the Laplace integral exists provided that

$$\sigma = Re(s) > S_o$$

where $\sigma = Re(s)$ is a real part of the complex number $s = \sigma + \omega i$.

In the general case, the parameter *s* is a complex number, but here we assume that s is real.

In applications on solving physical problems by the Laplace transform method it is usually assumed that the function *f(t)* is equal to zero for *t <0*:

$$f(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

This assumption means that processes starting at the moment t = 0 are considered. This kind functions can also be defined by using the Heaviside function *H(t)*:

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

as a product of two functions *f(t)* and *H(t)*: $f(t) \cdot H(t)$

For example,

$$\sin(t) \cdot H(t) = \begin{cases} \text{sint}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Let us consider an example on finding the Laplace transform of the function *f(t)=1* $(t \geq 0)$, using the definition of the Laplace transform.

Example 8.22

Laplace transform of the function ***f(t)=1*** $(t \geq 0)$ is found as

$$L[1] = \mathrm{F(s)} = \int_0^{+\infty} \mathbf{1} \cdot \mathrm{e}^{-\mathrm{st}}\mathrm{dt} = \lim_{\mathrm{b}\to+\infty} \int_0^{\mathrm{b}} \mathrm{e}^{-\mathrm{st}}\mathrm{dt} = \lim_{\mathrm{b}\to+\infty} -\frac{1}{\mathrm{s}} \int_0^{\mathrm{b}} \mathrm{e}^{-\mathrm{st}}\mathrm{d}(-\mathrm{s}\cdot\mathrm{t}) =$$

$$= \lim_{\mathrm{b}\to+\infty} -\frac{1}{\mathrm{s}}\mathrm{e}^{-\mathrm{st}}\Big|_0^{\mathrm{b}} = -\frac{1}{\mathrm{s}} \lim_{\mathrm{b}\to+\infty} \left(\mathrm{e}^{-\mathrm{sb}} - \mathrm{e}^0\right) = -\frac{1}{\mathrm{s}}(0-1) = \frac{1}{\mathrm{s}}$$

Thus,

$$L[1] = \frac{1}{s}$$

or we can also write

$$1 \div \frac{1}{s}$$

Example 8.23

Let us find Laplace transform for the function $f(t) = e^t \quad (t \geq 0)$:

$$L[e^t] = F(s) = \int_0^{+\infty} e^t \cdot e^{-st} dt = \lim_{b \to +\infty} \int_0^b e^{-(s-1)t} dt =$$

$$\lim_{b \to +\infty} -\frac{1}{s-1} \int_0^b e^{-(s-1)t} d(-(s-1)t) = \lim_{b \to +\infty} -\frac{1}{s-1} e^{-(s-1)t} \Big|_0^b =$$

$$= -\frac{1}{s-1} \lim_{b \to +\infty} \left(e^{-(s-1)b} - e^0\right) = -\frac{1}{s-1}(0-1) = \frac{1}{s-1}$$

Thus,

$$L[e^t] = \frac{1}{s-1}$$

In a similar way, the transforms for other elementary functions have been determined and summarized in special tables. Part of such a table is presented below.

f(t)	*F(s)*
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
sin(*at*)	$\frac{a}{s^2+a^2}$
cos(*at*)	$\frac{s}{s^2+a^2}$
$e^{\lambda t}\sin(at)$	$\frac{a}{(s-\lambda)^2+a^2}$

Co-funded by the
Erasmus+ Programme
of the European Union

$e^{\lambda t}\cos(at)$	$\frac{s-\lambda}{(s-\lambda)^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{\lambda t}\sinh(at)$	$\frac{a}{(s-\lambda)^2-a^2}$
$e^{\lambda t}\cosh(at)$	$\frac{s-\lambda}{(s-\lambda)^2-a^2}$

In applications, exactly those summarized tables of elementary Laplace transforms and properties of Laplace transform are used in order to find Laplace transforms of necessary functions.

Properties of the Laplace transform

Let us consider only properties which are necessary for solving differential equations.

1) ***Linearity theorem*** (*C1, C2=const*):

$$L[C_1 f_1(t) \pm C_2 f_2(t)] = C_1 L[f_1(t)] \pm C_2 L[f_2(t)]$$

2) ***Theorem on a derivative of the original***

If $L[f(t)] = F(s)$, then

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

....

$$L\left[f^{(n)}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

where C1 and C2 are constants.

Example 23

Find the Laplace transform of the function $f(t) = 2 - 3\sin 5t + 4e^{2t} + t^2$.

$$F(s) = L[f(t)] = L[2 - 3\sin 5t + 4e^{2t} + t^2] = 2L[1] - 3L[\sin 5t] + 4L[e^{2t}] + L[t^2] =$$

$$= 2\cdot\frac{1}{s} - 3\cdot\frac{5}{s^2+25} + 4\cdot\frac{1}{s-2} + \frac{2!}{s^3}$$

Co-funded by the Erasmus+ Programme of the European Union

As a result,

$$L[f(t)] = \frac{2}{s} - \frac{15}{s^2+1} + \frac{4}{s-2} + \frac{2}{s^3}$$

Definition: the Inverse Laplace Transform

If $L[f(t)] = F(s)$, then the inverse Laplace transform $f(t)$ of the function $F(s)$ is defined as the improper integral

$$f(t) = \int_0^{+\infty} F(t)e^{st}ds$$

It is often written as

$$f(t) = L^{-1}[F(s)]$$

It is to be noted that usually in applications the summarized tables of elementary Laplace transforms and properties of the Laplace transform are used in order to find originals.

In many practical problems the Laplace transform has the form of a rational fraction. In this case, the method of partial fractions can be useful in producing an expression; for those, the inverse Laplace transform can be easily found.

Example 8.24

Find the original of the function

$$F(s) = \frac{s+3}{s(s+1)}$$

i.e.

$$f(t) = L^{-1}\left[\frac{s+3}{s(s+1)}\right] = ?$$

We expand the given rational function into elementary fractions with undefined coefficients:

$$F(s) = \frac{s+3}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

In order to find the unknown coefficients, we find the least common denominator and equate the numerators of the functions on the right-hand side and left-hand side of the obtained expression:

Co-funded by the
Erasmus+ Programme
of the European Union

$$\frac{s+3}{s(s+1)} = \frac{A(s+1)+Bs}{s(s+1)}$$

$$s+3 = A(s+1)+Bs$$

$$s+3 = As+A+Bs$$

$$1\cdot s+3 = (A+B)s+A$$

The coefficients at s: 1=A+B,

The coefficient at s^0: $3 = A$ ➔ $A = 3$

It follows from the first equation that $B = 1 - A = 1 - 3 = -2$

As a result, we have

$$F(s) = \frac{3}{s} + \frac{-2}{s+1} = 3\cdot\frac{1}{s} - 2\cdot\frac{1}{s+1}$$

Using the linearity theorem and the table of Laplace transforms, we have

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[3\cdot\frac{1}{s} - 2\cdot\frac{1}{s+1}\right] = 3L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s+1}\right] = 3\cdot 1 - 2e^{-t}$$

So,

$$f(t) = L^{-1}[F(s)] = 3\cdot 1 - 2e^{-t}.$$

8.4.2 *Application of the Laplace transform for solving differential equations*

As was mentioned above, the Laplace transform is one of the most popular methods for solving differential equations. Here we consider the application of the Laplace transform for second-order linear differential equations with constant coefficients.
The Laplace transform can be only used for solving differential equations with given initial conditions at the point t=0, i.e. only for solving Cauchy problems.
Let us consider the linear differential equation with constant coefficients:

$$ay" + by' + cy = f(t)$$

with the initial conditions $y(0) = y_0$ and $y'(0) = y_1$,

where a, b, c are constants, $y = y(t)$ is a function of t and $a \neq 0$.

The algorithm of solving a Cauchy problem by the Laplace transform is:

1) Apply the Laplace transform to both sides of the differential equation

Co-funded by the Erasmus+ Programme of the European Union

$$L[ay" + by' + cy] = L[f(t)]$$

2) Use the linearity theorem together with the *Theorem on a derivative of the original*

$$aL[y"] + bL[y'] + cL[y] = L[f(t)]$$

Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the *Theorem on a derivative of the original,* it yields

$$\mathrm{L}[y'(\mathrm{t})] = \mathrm{sY(s)} - y(0) = \mathrm{sY(s)} - y_0$$

$$\mathrm{L}[y''(\mathrm{t})] = \mathrm{s^2Y(s)} - sy(0) - y'(0) = \mathrm{s^2Y(s)} - sy_0 - y_1$$

On applying the Laplace transform to the given differential equation, we have got the algebraic equation for the unknown function Y(s):

$$\mathrm{a} \cdot (\mathrm{s^2 Y(s)} - \mathrm{s}y_0 - y_1) + b \cdot (\mathrm{sY(s)} - y_0) + c \cdot Y(s) = F(s)$$

where $F(s) = L[f(t)]$ is the Laplace transform of the right-hand side function.

3) Solve the obtained ***algebraic equation*** for the function Y(s):

$$\mathrm{Y(s)}(\mathrm{as^2} + bs + c) = F(s) + asy_0 + by_0 + ay_1$$

$$\mathrm{Y(s)} = \frac{F(s) + asy_0 + by_0 + ay_1}{\mathrm{as^2} + bs + c}$$

4) ***Find the original*** y(t) of the function Y(s) using properties of the Laplace transform and the table of Laplace transforms as

$$\mathrm{y(t)} = \mathrm{L^{-1}[Y(s)]}$$

Example 0.25

Solve the Cauchy problem

$$y" + 9y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 2$$

1) We apply the Laplace transform to both sides of the given differential equation

$$\mathrm{L}[y" + 9y] = \mathrm{L}[e^{2t}]$$

$$\mathrm{L}[y"] + 9L[y] = \mathrm{L}[e^{2t}]$$

Co-funded by the
Erasmus+ Programme
of the European Union

Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the Theorem on a derivative of original, it yields

$$L[y'(t)] = sY(s) - y(0) = sY(s) - 1$$

$$L[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s \cdot 1 - 2$$

The Laplace transform of the right-hand side function is

$$L[e^{2t}] = \frac{1}{s-2}$$

After substituting $L[y''(t)]$, $L[y(t)]$ and $L[e^{2t}]$, we obtain the algebraic equation for the unknown function Y(s):

$$s^2 \cdot Y(s) - s - 2 + 9 \cdot Y(s) = \frac{1}{s-2}$$

3) We solve the obtained algebraic equation for the function Y(s):

$$Y(s) \cdot (s^2 + 9) = \frac{1}{s-2} + s + 2$$

$$Y(s) = \frac{1}{(s-2)(s^2+9)} + \frac{s}{(s^2+9)} + \frac{2}{(s^2+9)}$$

4) We find the original y(t) for the function Y(s).

a) First, we expand the first term on the right-hand side into elementary fractions with undefined coefficients:

$$\frac{1}{(s-2)(s^2+9)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9) + (Bs+C)(s-2)}{(s-2)(s^2+9)}$$

Thus, we get

$$1 = A(s^2+9) + (Bs+C)(s-2)$$

$$1 = As^2 + 9A + Bs^2 - 2Bs + Cs - 2C$$

$$0 \cdot s^2 + 0 \cdot s + 1 = (A+B)s^2 + (-2B+C)s + 9A - 2C$$

The coefficients at s^2 are $0 = A + B$

The coefficients at s are 0=-2B+C,

coefficients at s^0 are $1 = 9A - 2C$

Solving the system of equations for the unknown coefficients A, B and C, we obtain:

$$A = \frac{1}{13}, \quad B = -\frac{1}{13}, \quad C = -\frac{2}{13}$$

As a result, we have

$$\frac{1}{(s-2)(s^2+9)} = \frac{\frac{1}{13}}{s-2} + \frac{-\frac{1}{13}s - \frac{2}{13}}{s^2+9} = \frac{1}{13}\cdot\frac{1}{s-2} - \frac{1}{13}\cdot\frac{s}{s^2+9} - \frac{2}{13}\cdot\frac{1}{s^2+9}$$

We substitute the obtained expression into the expression for $Y(s)$ instead of the first term:

$$Y(s) = \frac{1}{13}\cdot\frac{1}{s-2} - \frac{1}{13}\cdot\frac{s}{s^2+9} - \frac{2}{13}\cdot\frac{1}{s^2+9} + \frac{s}{s^2+9} + \frac{2}{s^2+9}$$

We simplify the obtained expression as

$$Y(s) = \frac{1}{13}\cdot\frac{1}{s-2} + \frac{12}{13}\cdot\frac{s}{s^2+9} + \frac{24}{13}\cdot\frac{1}{s^2+9}$$

5) We find the original for the function $Y(s)$:

$$y(t) = L^{-1}[Y(\mathbf{s})] = L^{-1}\left[\frac{1}{13}\cdot\frac{1}{s-2} + \frac{12}{13}\cdot\frac{s}{s^2+9} + \frac{24}{13}\cdot\frac{1}{s^2+9}\right] =$$

$$= \frac{1}{13}\cdot L^{-1}\left[\frac{1}{s-2}\right] + \frac{12}{13}\cdot L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{24}{13}\cdot L^{-1}\left[\frac{1}{3}\cdot\frac{3}{s^2+9}\right] =$$

$$= \frac{1}{13}\cdot L^{-1}\left[\frac{1}{s-2}\right] + \frac{12}{13}\cdot L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{24}{13}\cdot\frac{1}{3}\cdot L^{-1}\left[\frac{3}{s^2+9}\right] =$$

$$= \frac{1}{13}e^{2t} + \frac{12}{13}\cos 3t + \frac{24}{39}\cdot\sin 3t$$

Thus, we have obtained the solution of the given Cauchy problem:

$$y(t) = L^{-1}[Y(s)] = \frac{1}{13}e^{2t} + \frac{12}{13}\cos 3t + \frac{24}{39}\sin 3t$$

8.4.3 *Exercises*

Exercise 8.11.

Co-funded by the
Erasmus+ Programme
of the European Union

Solve the Cauchy problem using the Laplace transform

$$y" + 4y' + 5y = 1\ ,\quad y(0) = 0,\quad y'(0) = 1$$

Solution:

1) We apply the Laplace transform to the given differential equation:

$$L[y" + 4y' + 5y] = L[1]$$

$$L[y"] + 4L[y'] + 5L[y] = L[1]$$

Let the Laplace transform of the unknown function y(t) be $L[y] = Y(s)$, then according to the *Theorem on a derivative of original,* it yields

$$\mathrm{L}[y'(\mathrm{t})] = \mathrm{sY(s)} - y(0) = \mathrm{sY(s)} - 0 = \mathrm{sY(s)}$$

$$\mathrm{L}[y''(\mathrm{t})] = \mathrm{s}^2\mathrm{Y(s)} - \mathrm{s}y(0) - y'(0) = \mathrm{s}^2\mathrm{Y(s)} - \mathrm{s}\cdot 0 - 1 = \mathrm{s}^2\mathrm{Y(s)} - 1$$

The result of application of the Laplace transform to the given differential equation gives us the algebraic equation for the unknown function Y(s):

$$\mathrm{s}^2\mathrm{Y(s)} - 1 + 4\cdot \mathrm{sY(s)} + 5\cdot Y(s) = \frac{1}{s}$$

3) We solve the obtained algebraic equation for the function Y(s):

$$\mathrm{Y(s)}(\mathrm{s}^2 + 4\mathrm{s} + 5) = \frac{1}{s} + 1$$

$$\mathrm{Y(s)}(\mathrm{s}^2 + 4\mathrm{s} + 5) = \frac{1 + s}{s}$$

$$\mathrm{Y(s)} = \frac{1 + s}{s(\mathrm{s}^2 + 4\mathrm{s} + 5)}$$

4) We find the original y(t) for the function Y(s) using properties of the Laplace transform and the table of Laplace transforms, as

$$\mathrm{y(t)} = \mathrm{L}^{-1}[\mathrm{Y(s)}]$$

For this purpose, we expand the function on the right-hand side into elementary fractions with undefined coefficients:

$$\frac{1 + s}{s(\mathrm{s}^2 + 4\mathrm{s} + 5)} = \frac{\mathrm{A}}{\mathrm{s}} + \frac{\mathrm{Bs} + \mathrm{C}}{\mathrm{s}^2 + 4\mathrm{s} + 5} = \frac{\mathrm{A}(\mathrm{s}^2 + 4\mathrm{s} + 5) + (\mathrm{Bs} + \mathrm{C})\mathrm{s}}{\mathrm{s}(\mathrm{s}^2 + 4\mathrm{s} + 5)}$$

Co-funded by the
Erasmus+ Programme
of the European Union

So that

$$s+1=A(s^2+4s+5)+(Bs+C)s$$

$$s+1=As^2+4As+5A+Bs^2+Cs$$

$$0\cdot s^2+1\cdot s+1=(A+B)s^2+(4A+C)s+5A$$

The coefficients at s^2 : $0=A+B$
The coefficients at s: $1=4A+C$,
The coefficients at s^0: $1=5A$

Solving the system of equations for the unknown coefficients, we have:

$$A=\frac{1}{5},\quad B=-\frac{1}{5},\quad C=\frac{1}{5}$$

As a result, we have

$$Y(s)=\frac{1+s}{s(s^2+4s+5)}=\frac{\frac{1}{5}}{s}+\frac{-\frac{1}{5}s+\frac{1}{5}}{s^2+4s+5}=\frac{1}{5}\cdot\frac{1}{s}-\frac{1}{5}\cdot\frac{s-1}{s^2+4s+5}$$

5) We find the original for the function $Y(s)$:

$$y(t)=L^{-1}[Y(s)]=L^{-1}\left[\frac{1}{5}\cdot\frac{1}{s}-\frac{1}{5}\cdot\frac{s-1}{s^2+4s+5}\right]=\frac{1}{5}\cdot L^{-1}\left[\frac{1}{s}\right]-\frac{1}{5}\cdot L^{-1}\left[\frac{s-1}{s^2+4s+5}\right]$$

It follows from the Laplace transform table that $L^{-1}\left[\frac{1}{s}\right]=1$,

However, for finding

$$L^{-1}\left[\frac{s-1}{s^2+4s+5}\right]$$

first, we should transform the fraction

$$\frac{s-1}{s^2+4s+5}=\frac{s-1}{(s+2)^2+1}=\frac{s+2-3}{(s+2)^2+1}=\frac{s+2}{(s+2)^2+1}-\frac{3}{(s+2)^2+1}$$

Then

$$L^{-1}\left[\frac{s+2}{(s+2)^2+1}-\frac{3}{(s+2)^2+1}\right]=L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right]-3\cdot L^{-1}\left[\frac{1}{(s+2)^2+1}\right]=$$

Co-funded by the
Erasmus+ Programme
of the European Union

$$= e^{-2t}\cos t - 3e^{-2t}\sin t$$

As a result, we have

$$\mathrm{y(t)} = \frac{1}{5}\cdot 1 - \frac{1}{5}\cdot\left(e^{-2t}\cos t - 3e^{-2t}\sin t\right) = \frac{1}{5} - \frac{e^{-2t}}{5}\cdot(\cos t - 3\sin t)$$

Co-funded by the
Erasmus+ Programme
of the European Union