8.5 CONNECTIONS AND APPLICATIONS

Example 1:

Ship stability is a maritime safety issue that needs to be explored even at the design stage. Rolling and pitching of the ship in the water are extremely important factors affecting the stability of a is the

1) Rolling of a vessel from one side to the another one, occurring in calm water without resistance is described by the following second-order differential equation:

$$\theta'' + {n_\theta}^2\theta = 0$$

where $\theta = \theta(t)$ is the rolling amplitude (Fig.3).

n_θ is the circular frequency of free (natural) vibrations during the rolling **without resistance**.

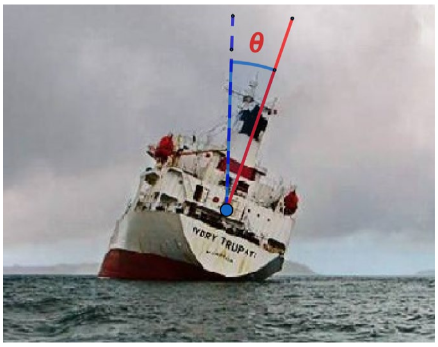

Figure 8.3 *Rolling of a vessel*

This equation is a second-order linear homogeneous differential equation with constants coefficients. Let us solve this equation.

The auxiliary equation is

$$k^2 + {n_\theta}^2 = 0,$$

whose roots are complex numbers

$$k_1 = n_\theta i \ , \qquad k_2 = -n_\theta i$$

The general solution of the equation is

$$\theta(t) = C_1\cos(n_\theta t) + C_2\sin(n_\theta t)$$

2) Taking into account the resistance during the rolling in calm water, the equation of motion of a vessel takes the form

$$\theta'' + 2\mu_\theta\theta' + {n_\theta}^2\theta = 0$$

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where μ_θ is the relative coefficient of resistance.

The corresponding auxiliary equation is

$$k^2 + 2\mu_\theta k + {n_\theta}^2 = 0,$$

whose roots are

$$k_1 = -\mu_\theta + i\sqrt{{\mu_\theta}^2 - {n_\theta}^2}\ , \qquad k_2 = -\mu_\theta - i\sqrt{{\mu_\theta}^2 - {n_\theta}^2}$$

$$\theta(t) = C_1\, \mathrm{e}^{-\mu_\theta t}\cos(\omega_\theta \cdot t) + C_2\, \mathrm{e}^{-\mu_\theta t}\sin(\omega_\theta \cdot t)$$

where

$\omega_\theta = \sqrt{{\mu_\theta}^2 - {n_\theta}^2}$ is a natural (their own) frequency during the rolling with resistance.

It should be noted that similar differential equations describe also pitching and heaving motions of a vessel.

Example 2:

Any modern vessel is not complete without electrical and electro-mechanical systems. An alternating-current electrical circuit is a component of any such system. Transition processes in such electrical circuits that occur in a short period of time after switching on or off (after connecting the circuit to voltage or after disconnecting the circuit from voltage), as well as when the capacitive element is turned on or off, are described by the ordinary differential equations. As an example, we can consider one of the easiest electrical circuits: a resistor-inductor-capacitor circuit (RLC).

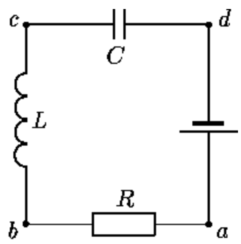

Figure 8.4 A resistor-inductor-capacitor circuit

1) For example, in the case of source of unchanging voltage, the following second-order differential equation describes the transition processes in RLC circuit:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

where

t is the time,

i(t) is the current admitted through the circuit,

R is the effective resistance of the combined load, source, and components,

L is the inductance of the inductor component,

C is the capacitance of the capacitor component.

This is a homogeneous second-order ordinary differential equation whose *characteristic equation is*

$$Lk^2 + Rk + \frac{1}{C} = 0$$

or

$$k^2 + \frac{R}{L}k + \frac{1}{LC} = 0$$

The roots are

$$k_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad and \quad k_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The solution of the differential equation has the form

$$i(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

where C_1 and C_2 are terms of amplitude.

2) If a RL circuit with constant resistance R and inductance L at time t = 0 is connected to voltage U_0 (for example, battery), then the transition process within a short time period after switching on is described by the following 1st order linear inhomogeneous differential equation with constant coefficients

$$L\frac{di}{dt} + R \cdot i = U_0$$

Example 3:

Ships often carry containers with various liquids so that liquid leakage problems are essential. In this connection, we consider the problem of the liquid flowing out of a cylindrical tank of radius *R* through a small hole of radius *r* at the bottom of the container.

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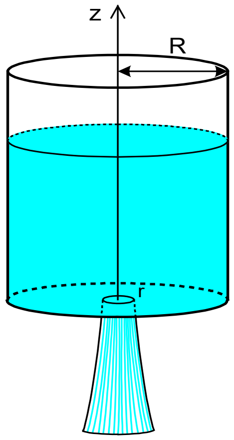

Figure 8.5

Liquid level in the tank at time moment *t* is a function of time which is described by the following differential equation:

$$R^2\frac{dz}{dt}+r^2k\sqrt{2gz}=0$$

where

t is time,

g is the gravitational acceleration (g=9.80665 m/s^2),

k is coefficient of the flow rate that depends on the viscosity of the liquid,

z=z(t) is the liquid level above the hole at time moment *t* .

Assuming, that in the initial time moment t=0 the liquid level was H, let us find:

a) *unknown function of liquid level in the tank z=z(t) ;*

b) *time T during which the liquid will completely drain out of the tank.*

In order to find unknown function of liquid level in the tank *z(t),* we solve the given differential equation. This is a separable-variables equation.

$$R^2\frac{dz}{dt}=-r^2\sqrt{2g}\cdot\sqrt{z}$$

$$\frac{dz}{\sqrt{z}}=-\frac{r^2}{R^2}\sqrt{2g}dt$$

$$\int\frac{dz}{\sqrt{z}}=-\frac{r^2}{R^2}\sqrt{2g}\int dt$$

$$2\sqrt{z}=-\frac{r^2}{R^2}\sqrt{2g}\cdot t+C$$

Taking into account, that at the initial time moment t=0 the height of the liquid in the container was H, we get

$$2\sqrt{H}=-\frac{r^2}{R^2}\sqrt{2g}\cdot 0+C$$

$$C=2\sqrt{H}$$

$$2\sqrt{z}=-\frac{r^2}{R^2}\sqrt{2g}\cdot t+2\sqrt{H}$$

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As a result, we obtain the function z (t), which describes the liquid level in the tank at time moment t:

$$z(t) = \left(-\frac{r^2}{2R^2}\sqrt{2g}\cdot t + \sqrt{H}\right)^2$$

In order to find the time T during which the liquid will completely drain out of the tank, we take into account, that at the time moment t=T, the level of the liquid in the container will be z=0. Then we obtain the dependence of time on the height of the fluid

$$2\sqrt{0} = -\frac{r^2}{R^2}\sqrt{2g}\cdot T + 2\sqrt{H}$$

$$\frac{r^2}{R^2}\sqrt{2g}\cdot T = 2\sqrt{H}$$

Expressing T, we get the time during which the liquid will completely drain out of the tank.

$$T = \frac{R^2}{r^2}\sqrt{\frac{2H}{g}}$$

Example 4:

There are many marine ecological issues where differential equations are useful. For example, the mathematical modelling of propagation and extinction of fish population that is important for fish catch control.

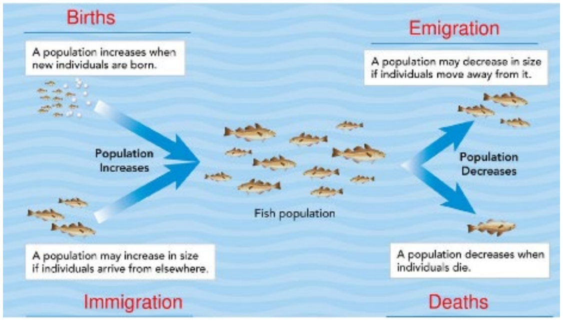

Figure 8.6

Fish population *P(t)* in the lake at the time moment *t* can be described by the first-order differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where

t is time, *k* is the growth parameter,

M is the carrying capacity, representing the largest population that the environment can support.

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If for some reason the population exceeds the carrying capacity, the population will decrease; and otherwise, as long as the population is less than the carrying capacity, the population will increase. This equation is known as the logistic equation.

The population *P(t)* of codfish in a certain marine fishery is modelled by a modified logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - H$$

where *H* is the rate at which fish are harvested.

An important question in this problem is how the fate of the fish population depends on the parameter H.

Example 5:

Differential equations are used in beam theory which is an important tool in the sciences, especially in structural and mechanical engineering. It is also very important in ship design. For example, we consider the Euler–Bernoulli equation which describes the relationship between the beam's deflection and the applied load. A beam is a constructive element capable of withstanding heavy loads in bending.

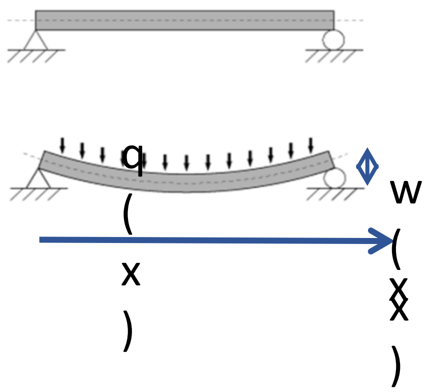

Figure 8.7

In the case of small deflections, the beam shape can be described by a fourth-order linear differential equation

$$E \cdot I \frac{d^4 w}{dx^4} = q(x)$$

where *q(x)* is external load acting on the beam,

E is the modulus of elasticity of the beam,

I is the second moment of area of the beam's cross-section.

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The curve w(x) describes the deflection of the beam in the direction *z* at some position *x*. Often, the product $E \cdot I$ is a constant, known as the flexural rigidity.
This equation under the appropriate boundary conditions determines the deflection of a loaded beam.

Example 6:

Ordinary differential equations are widely used for cooling/heating problems.

For example, consider a process of cooling down of a heated body placed in an environment. The temperature of a hot object decreases with the rate proportional to the difference between its temperature and the temperature of the surrounding environment. If the temperature of the environment is given by E(t), then the following differential equation describes the temperature of the body T(t) as the function of time:

$$\frac{dT}{dt} = -k(T(t) - E(t))$$

where $k>0$ is a physical constant depending on the materials and sizes of the bodies.

If the object, whose temperature is being modelled, contains a source of heat, then the cooling of the body is described by the differential equation

$$\frac{dT}{dt} = -k(T(t) - E(t)) + mH(t)$$

where *m* is a positive constant, inversely proportional to the heat capacity of the object and H(t) denotes the rate that heat is generated within the object. (H(t) would be negative in some cases, such as air conditioning).

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