5.4.1. Linear function and straight line

Introduction

The linear function is popular in various branches of science (i.e., economics). It is attractive because it is simple and easy to handle mathematically. It has many important applications.

Linear functions are those whose graph is a straight line. The origin of the name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane.

A linear function has the following form

$$\boldsymbol{y} = \boldsymbol{ax} + \boldsymbol{b}, \text{ (slope } \boldsymbol{a}, \ \boldsymbol{y} - \text{intercept } \boldsymbol{b}).$$

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y.
b is the *constant* term or the y ***−intercept***. It is the value of the dependent variable when $x = 0$.

a is the *coefficient* of the independent variable. It is also known as the ***slope*** and gives the *rate of change* of the dependent variable.
The slope of a line is a number that describes both the *direction* and the *steepness* of the line. Slope is often denoted by the letter a . Recall the slop-intercept form of a line, $y = ax + b$. Putting the equation of a line into this form gives you the slope (a) of a line, and its y-intercept (b).

The steepness, or incline, of a line is measured by the absolute value of the slope. A slope with a greater absolute value indicates a steeper line. In other words, a line with a slope of -9 is steeper than a line with a slope of 7.

Slope is calculated by finding the ratio of the "vertical change" to the "horizontal change" between any two distinct points on a line. This ratio is represented by a quotient ("rise over run"), and gives the same number for any two distinct points on the same line. It is represented by $a = \frac{rise}{run}$.

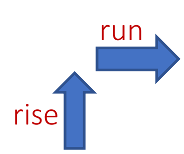

Figure 5.10 Visualization of a slope. The slope of a line is calculated as "rise over run."

Slope describes the direction and steepness of a line, and can be calculated given two points on the line:

$$\boldsymbol{a} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

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When $a > 0$ then the linear function is increasing (y-value increases as the x-value increases - it is easy to see that $y = f(x)$ tends to go up as it goes along - the blue line on Figure 5.10 has a positive slope of $\frac{1}{2}$ and a y-intercept of -3); when $a < 0$ then the linear function is decreasing (y-value decreases as the x-value increases- it is easy to see that $y = f(x)$ tends to go down as it goes along – the red line on Figure 5.11 has a negative slope of -1 and a y-intercept of 5.

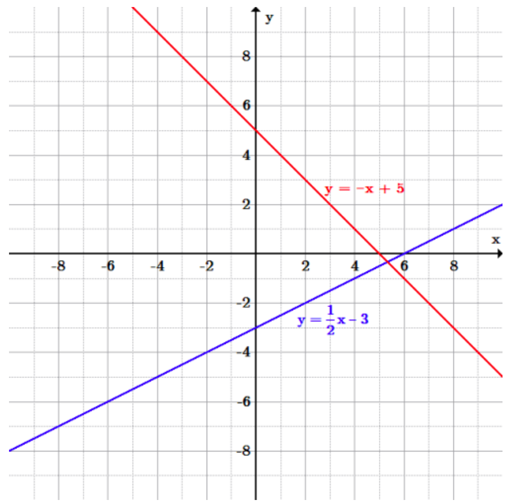

Figure 5.11 Graph of linear functions when $a > 0$ and $a < 0$.

Vertical and Horizontal Lines

Vertical lines have an undefined slope, and cannot be represented in the form $y = ax + b$, but instead as an equation of the form $x = c$ for a constant c, because the vertical line intersects a value on the x-axis, c. For example, the graph of the equation $x = 3$ includes the same input value of 4 for all points on the line, but would have different output values, such as $(3, -3), (3, 0), (3, 1), (3, 6), (3, 2), (3, -1)$, etc. Vertical lines are **NOT functions**, however, since each input is related to more than one output. If a line is vertical the slope is undefined.

Horizontal lines have a slope of zero and they are represented by the form, $y = b$, where b is the y-intercept. A graph of the equation $y = 3$ includes the same output value of 3 for all input values on the line, such as (−1, 3), (0, 3), (2, 3), (3, 3), (6,3) etc. Horizontal lines **ARE functions** because the relation (set of points) has the characteristic that each input is related to exactly one output.

Graphing a linear function

To graph a linear function, we have to:

1. Find 2 points which satisfy the equation;
2. Plot them;
3. Connect the points with a straight line.

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Example 5.8

$y = 5x + 25$:

Let $x = 1$ then $y = 5\,(1) + 25 = 30$, let $x = 3$ then $y = 5(3) + 25 = 40$

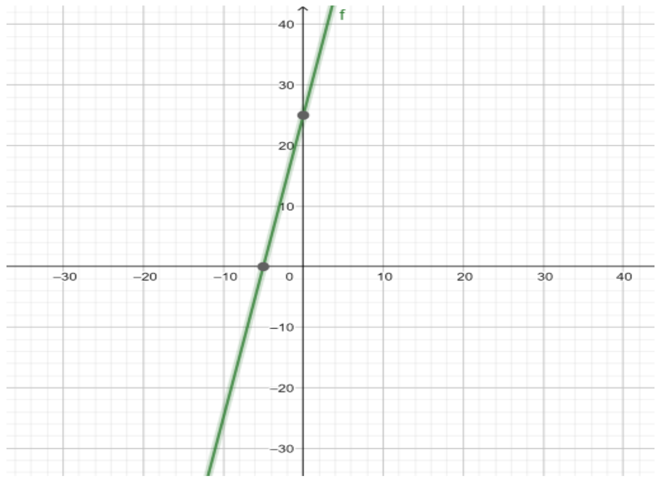

Figure 5.12 Graph of the linear function

In the linear function graphs above (*Figure 5.12*) the constant, $a = 5$, determines the slope or gradient of that line, and the constant term, $b = 25$, determines the point at which the line crosses the y-axis, otherwise known as the y-intercept.

Point-slope form of the equation of a line

The equation of the line through $(x_0,\ y_0)$ with slope a is

$$\boldsymbol{y - y_0 = a(x - x_0)}.$$

The equation of the line through two points: $(x_1,\ y_1),\ (x_2,\ y_2)$ is

$$\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}.$$

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Worked examples and exercises

Example 5.9

The total cost in euros of manufacturing x units of a certain commodity is

$$f(x) = 30x + 1000.$$

a) Compute the cost of manufacturing 10 units.
b) Compute the cost of manufacturing 10th unit.

Solution

a) Substitute $x = 10$ into formula $f(x) = 30x + 1000$: $f(10) = 30\,(10) + 1000 = 1300$ [€].
b) Subtract the cost of manufacturing 9 units from the cost of manufacturing 10 units:

$$f(10) - f(9) = 1300 - 1270 = 30 \text{ [€]}.$$

Example 5.10

Find the slope and y −intercept of the line $3x + 2y = 6$ and draw the graph.

Solution

First solve for y: $y = -\frac{3}{2}x + 3.$

Compare this for the slope-intercept form $\boldsymbol{y = ax + b}$ and conclude that $\boldsymbol{a} = -\frac{3}{2}$ and $b = 3$. To draw the graph, plot the y −intercept $(0, 3)$ and any other convenient point that satisfies the equation, say $(2, 0)$, and draw the line through them.

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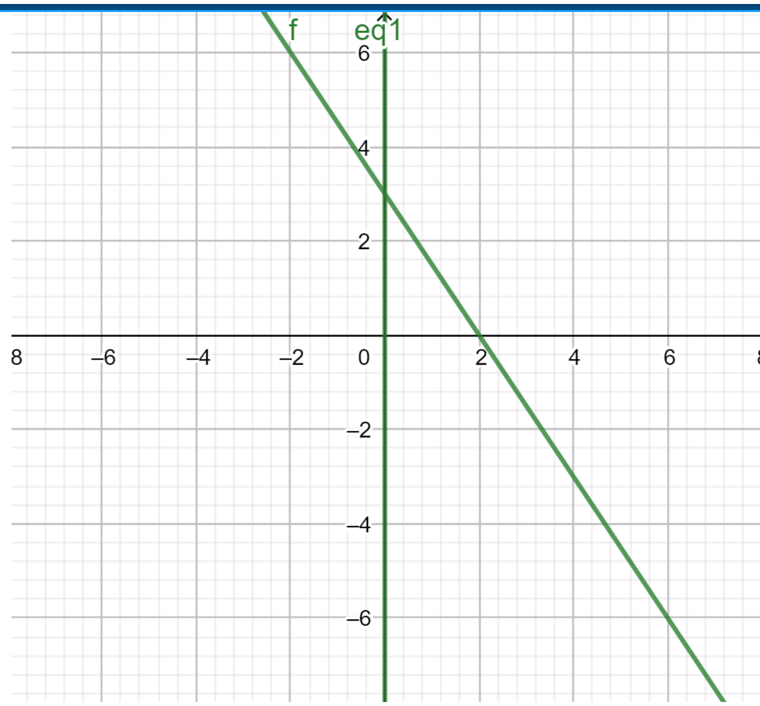

Figure 5.13 ***Graph of the function*** $\mathbf{3x+2y=6}$

Example 5.11

Find the equation of the line through $(2,5)$ and $(1,-2)$.

Solution

First compute the slope: $a=\frac{y_2-y_1}{x_2-x_1}=\frac{-2-5}{1-2}=7$.

Then use one of the given points, say (2, 5) as $(x_0,\ y_0)$ in the point-slope formula

$$y-y_0=a(x-x_0):$$

$$y-5=7(x-2)\quad =>\quad y=7x-9.$$

Example 5.12

Since the beginning of the month, a local reservoir has been losing water at constant rate. On the 12th of the month, the reservoir held 400 million liters of water and on the 22th, it held 250 million liters.

a) Express the amount of water in the reservoir as a function of time.
b) How much water was in the reservoir on the eighth of the month?

Solution

a) Let

$x=$ number of days since the first of the month;

$y =$ amount of water in the reservoir (in million litres units).

Since the rate of change of the water is constant, y is a linear function of x. To find this function, use the point-slope form to get the equation of the line through the points $(12, 400)$ and $(22, 250)$:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{250 - 400}{22 - 12} = \frac{-150}{10} = -15.$$

$$y - 400 = -15(x - 12) \quad => \quad y = -15x + 580.$$

b) To calculate the amount of water in the reservoir on the 8th of the month, substitute $x = 8$ into the formula of y : $y = -15(8) + 580 = 460$ [million liters].

Example 5.13

Find the point of intersection of the lines: $y = -x + 5$ and $y = 4\,x + 3$.

Solution

Set two expressions for y equal to each other and solve for x:

$$-x + 5 = 4x + 3$$

$$-5x = -2 \quad => \quad x = 0.4.$$

To find y, substitute $x = 0.4$ into one of the original equations, say $y = -x + 5$:

$y = -0.4 + 5 = 4.6.$ Thus, the point of intersection is $(0.4;\ 4.6)$.

Example 5.14

Membership in a private fitness club costs 800€ per year and entitles the member to use the courts for a fee of 3€ per hour. At a competing club, membership costs 650€ per year and the charge for the use of the gym is 5€ per hour. If only financial considerations are to be taken into account, how should a client choose which club to join?

Solution

Let

$x =$ number of hours of using the gym during a year;

$f_1(x) =$ total cost at the first club

$f_2(x) =$ total cost at the second club.

Then,

$$f_1(x) = 800 + 3x$$

while

$$f_2(x) = 650 + 5x.$$

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To find the point of intersection, set $f_1(x) = f_2(x)$ and solve

$$800 + 3x = 650 + 5x$$

$$150 = 2x \Rightarrow \quad x = 75.$$

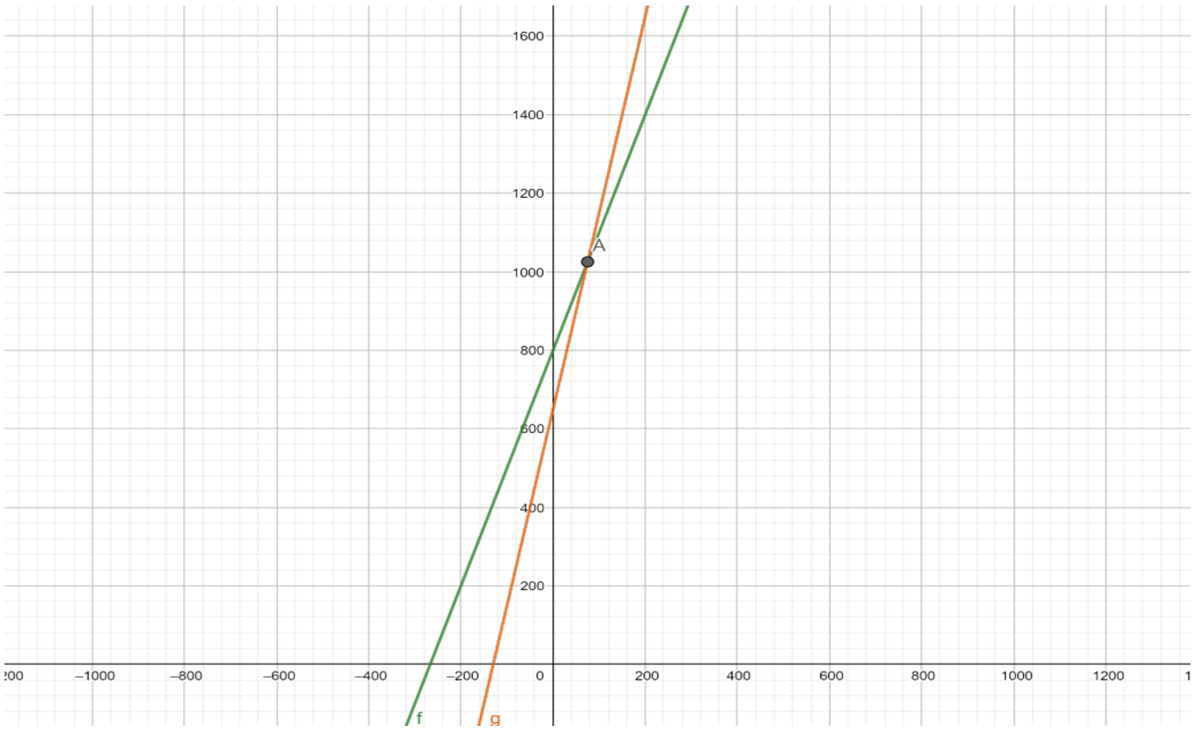

Figure 5.14

Conclude that if $x < 75$, the client should join the second club and if $x > 75$ the client should join the first club.

Example 5.15

A river ship with a drive does not stop along its way along the Vistula River from the city of Włocławek in Poland to the city of Gdańsk also in Poland for three days, while from Gdańsk to Włocławek for four days. How many days will the raft sail (without a drive) from Włocławek to Gdańsk?

Solution

Let v_1 be a speed of the ship, v_2 be a speed of the raft. The ship while travels downstream has the speed

$$v_1 + v_2$$

and while travels upstream has the speed:

$$v_1 - v_2.$$

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When we compare the distance s, defined by

$$s = v \cdot t$$

from Włocławek to Gdańsk and from Gdańsk to Włocławek we get the equation:

$$3(v_1 + v_2) = 4(v_1 - v_2).$$

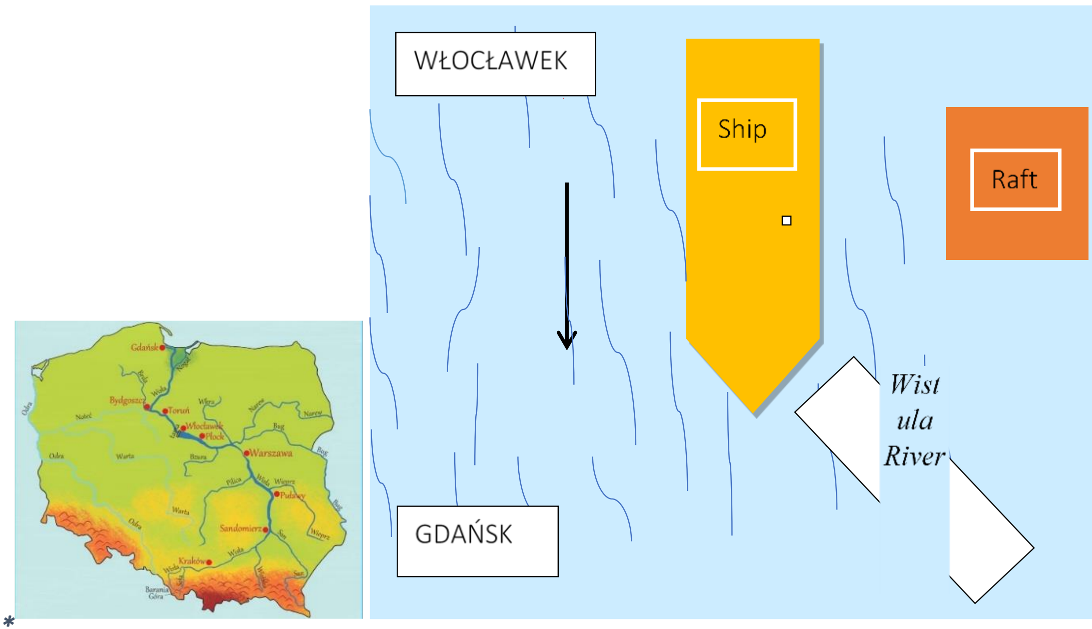

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Figure 5.15

Hence

$$v_1 = 7v_2.$$

The speed of the raft is the same as the river current i.e., v_2.

Therefore, the number of days the raft will pass the distance

$$s = 3(v_1 + v_2) = 3(7v_2 + v_2) = 24v_2$$

will be

$t = \frac{24v_2}{v_2} = 24.$

The raft will sail from Włocławek to Gdańsk 24 days.

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