5.4.2. Quadratic functions

In this section, we will investigate quadratic functions. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Graph and general form of a quadratic function

The graph of a quadratic function is a U-shaped curve called a **parabola**. Important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens upward, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 5.16.

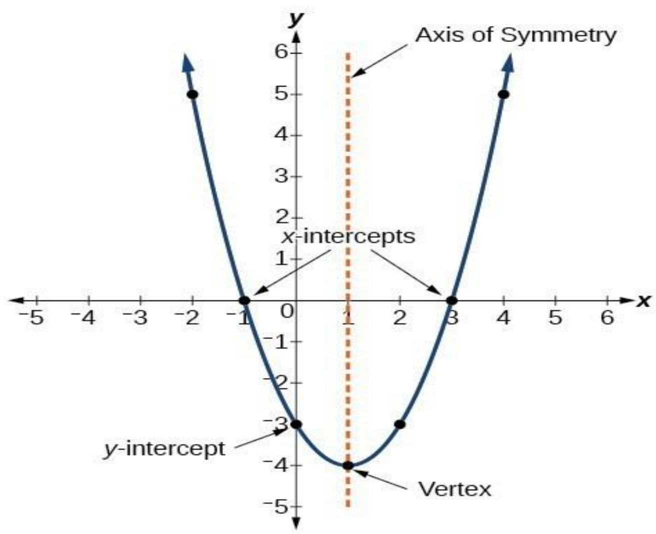

Figure 5.16 The graph of a quadratic function

The $\mathbf{y}$-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the ***zeros***, or ***roots*** of the quadratic function i.e., the values of x at which $y = 0$.

The general form of a quadratic function is

$$f(x) = ax^2 + bx + c$$

Where a, b, and $\mathbf{c}$ are real numbers and $a \neq 0$.

If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. We can use the general form of a parabola to find the equation for the axis of symmetry.

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The axis of symmetry is defined by $x = -\frac{b}{2a}$.

If we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of xhalfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 5.17 shows the graph of the quadratic function written in general form as

$y = x^2 + 4x + 3$. In this form, $a = 1,\ b = 4\ ,\ c = 3$. As $a > 0$, the parabola opens upwards. The axis of symmetry is $x = -\frac{4}{2} = -2$. We can see from the graph that the vertical line

$x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upwards, the vertex occurs at the lowest point on the graph, in this example: $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x –axis, occur at $(-3,0)$ and $(-1,0)$.

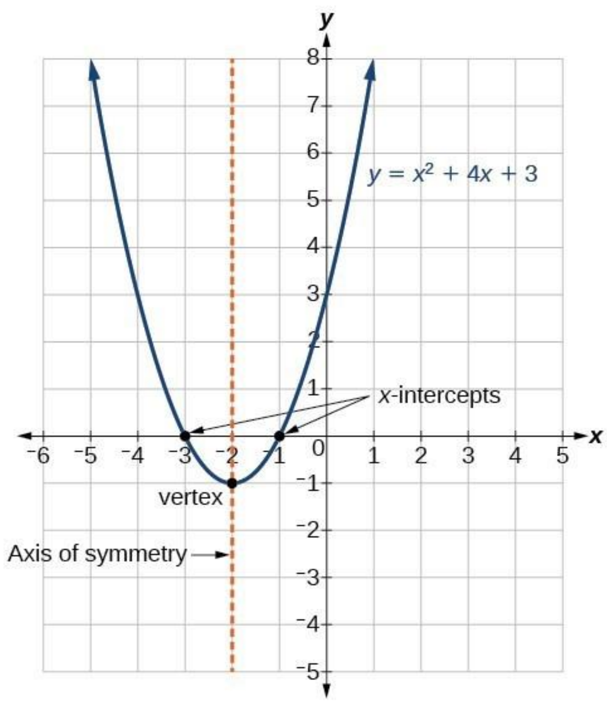

Figure 5.17 The graph of $y = x^2 + 4x + 3$

The **standard form** of a quadratic function is represented by

$$f(x) = a(x-p)^2 + q$$

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where $(p,\ q)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**. The function presented in *Figure 5.17* has the standard form: y= $(x+2)^2-1$.

As with the general form, if $a>0$, the parabola opens upwards and the vertex is a minimum. If $a<0$, the parabola opens downwards, and the vertex is a maximum.

Figure 5.18 shows the graph of the quadratic function written in standard form as

$$y=-3(x+2)^2+4\ .$$

Since $x-p=x+2$ in this example, $p=-2$. In this form, $a=-3$, $p=-2$ and $q=4$. Because $a=-3<0$, the parabola opens downward. The vertex is at $(-2,\ 4)$.

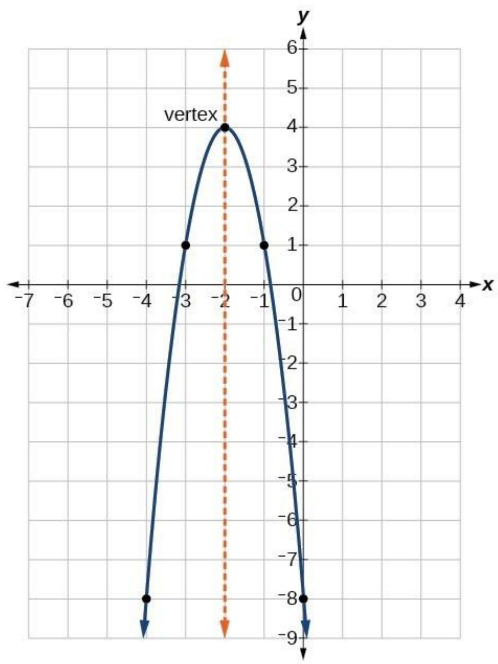

Figure 5.18 The graph of $\boldsymbol{y=-3(x+2)^2+4}$.

The standard form is useful for determining how the graph is transformed from the graph of $y=x^2$. *Figure 5.19* is the graph of this basic function.

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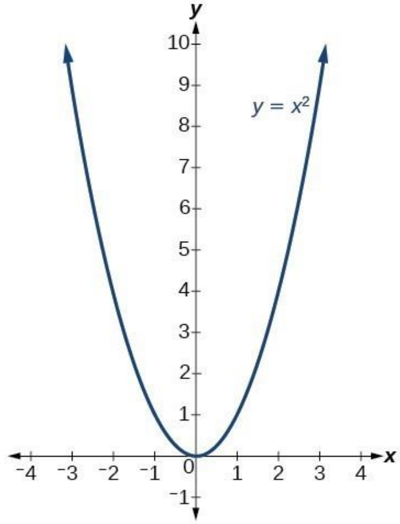

Figure 5.19 Graph of $\boldsymbol{y = x^2}$.

If $\boldsymbol{q > 0}$, the graph shifts upward, whereas if $q < 0$, the graph shifts downward. In Figure 5.18 $\boldsymbol{q > 0}$, so, the graph is shifted 4 units upward. If $p > 0$, the graph shifts toward the right and if $p < 0$, the graph shifts to the left. In Figure 5.18 $p < 0$, so the graph is shifted 2 units to the left.

The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch.

If $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In *Figure 5.18* $|a| > 1$, so, the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x-p)^2 + q = a(x^2 - 2xp + p^2) + q = ax^2 - 2axp + ap^2 + q = ax^2 + bx + c$$

For the quadratic expressions to be equal, the corresponding coefficients must be equal.

$$-2ap = b, \text{ so } \quad \boldsymbol{p = \frac{-b}{2a}}.$$

This gives us the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$ap^2 + q = c \;\; q = c - ap^2 = c - a\left(\frac{-b}{2a}\right)^2 = c - \frac{b^2}{4a} = -\frac{b^2-4ac}{4a}$$

In practice, though, it is usually easier to remember that $\boldsymbol{q}$ is the output value of the function when the input is $\boldsymbol{p}$, so $f(p) = f\left(\frac{-b}{2a}\right) = q$.

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Note

The expression $b^2 - 4ac$ usually denoted as the upper-case Greek letter, Δ , is defined as

$$\Delta = b^2 - 4ac$$

and it is called **discriminant** of a square trinomial $ax^2 + bx + c$.
Quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is a formula for solving quadratic equations in terms of the coefficients.

The number of $\mathrm{x}-$intercepts of a quadratic function depends on the sign od the discriminant $\Delta = b^2 - 4ac$:

- If $\Delta < 0$ then there is no $\mathrm{x}-$intercepts of a quadratic function;
- If $\Delta = 0$ then there is one $\mathrm{x}-$intercept of a quadratic function and $x_0 = \frac{-b}{2a}$;
- If $\Delta > 0$ then there are two $\mathrm{x}-$intercepts a quadratic function and

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Example 5.16

Find the vertex of a quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$p = \frac{-b}{2a} = \frac{-(-6)}{2 \cdot 2} = \frac{6}{4} = \frac{3}{2}$$

The vertical coordinate of the vertex will be at

$$q = f(p) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}.$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts, $f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$

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Domain and range of a quadratic function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is the set of all real numbers. As parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $\left[f\left(-\frac{b}{2a}\right), \infty\right)$; the range of a quadratic function written in general form with a negative a value is $\left(-\infty, f\left(\frac{-b}{2a}\right)\right]$.

The range of a ***quadratic function*** written in standard form

$$f(x) = a(x-p)^2 + q$$

with a positive a value is $[q, \infty)$; the range of a quadratic function written in standard form with a negative a value is $(-\infty, q]$.

How to find the domain and range of a given quadratic function:

The domain of any quadratic function is the set of all real numbers.

Determine whether a is positive or negative. If a is positive, the quadratic function has a minimum. If a is negative, the quadratic function has a maximum.

Determine the maximum or minimum value of the quadratic function, q.

If the parabola has a minimum, the range is $[q, \infty)$.

If the parabola has a maximum, the range is $(-\infty, q]$.

Example 5.17

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is a set of all real numbers. Because a is negative, the parabola opens downwards and has a maximum value. We want to determine the maximum value. We start by finding the x −value of the vertex:

$$p = -\frac{b}{2a} = -\frac{9}{-10}.$$

The maximum value q is given by $q = f(p)$:

$$q = f\left(\frac{9}{10}\right) = -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 = \frac{61}{20}.$$

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The range is $\left(-\infty,\ \frac{61}{20}\right]$.

Figure 5.20 presents two parabolas with their extreme values: minimum and maximum.

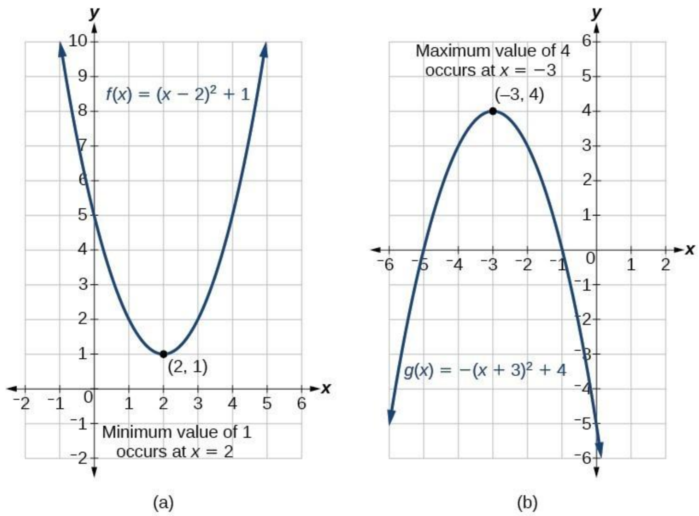

Figure 5.20 Examples of minimum and maximum values of the quadratic functions

Finding the x– and y-intercepts of a quadratic function

We need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x –intercepts at locations where the output is zero. Notice that the number of

x-intercepts can vary depending upon the location of the graph- see *Figure 5.21*.

How to find the $y-$ intercept and $x-$ intercepts of a given a quadratic function $f(x)$.

Evaluate $f(0)$ to find the $y-$ intercept.

a. Solve the quadratic equation $f(x) = 0$ to find the $x-$ intercepts.

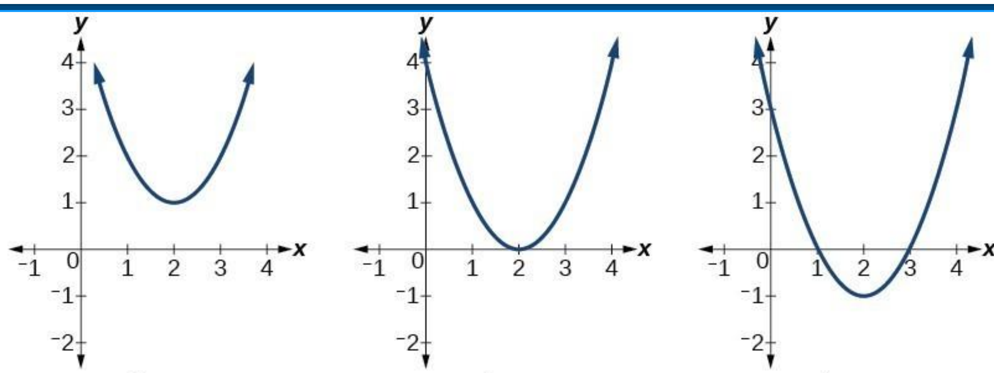

Figure 5.21 Number of x –intercepts of a parabola.

Example 5.18

Find the $y-$ and x –intercepts of a parabola $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y –intercept by evaluating $f(0)$:

$$f(0) = 3 \cdot 0^2 + 5 \cdot 0 - 2 = -2$$

So, the y –intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$3x^2 + 5x - 2 = 0.$$

In this case, using the quadratic formula , $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$, as

$\Delta = b^2 - 4ac = 5^2 - 4 \cdot 3 \cdot (-2) = 49$, $\sqrt{\Delta} = \sqrt{49} = 7$ thus

$x_1 = \frac{-5-7}{6} = -2$ and $x_2 = \frac{-5+7}{6} = \frac{1}{3}$.

So, the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2,0)$.

By graphing the function, we can confirm that the graph crosses the y –axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at at $\left(\frac{1}{3}, 0\right)$ and $(-2,0)$.

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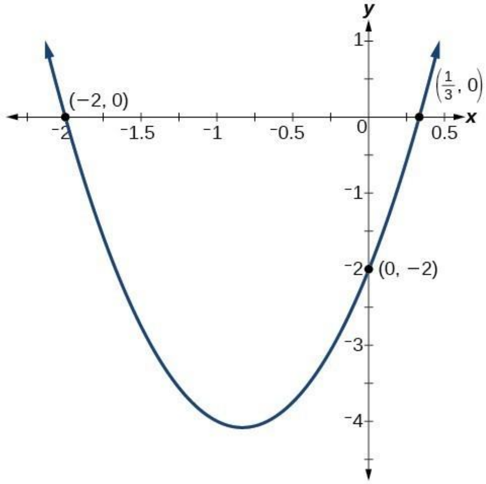

Figure 5.22 The $\boldsymbol{y}-$ and $\boldsymbol{x}-$intercepts of a parabola $\boldsymbol{f(x) = 3x^2 + 5x - 2}$

Vieta's formulas

Vieta's formulas give a simple relation between the roots of a polynomial and its coefficients. In the case of the quadratic equation, they take the following form:

$$\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$$

$$\boldsymbol{x_1 \cdot x_2 = \frac{c}{a}}.$$

These results follow immediately from the relation:

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0,$$

which can be compared term by term with

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0.$$

The first formula above yields a convenient expression when graphing a quadratic function. Since the graph is symmetric with respect to a vertical line through the vertex, when there are two real roots the vertex's x-coordinate is located at the average of the roots (or intercepts). Thus the x-coordinate of the vertex is given by the expression

$$p = \frac{x_1 + x_2}{2} = -\frac{b}{2a}.$$

The y-coordinate can be obtained by substituting the above result into the given quadratic equation, giving

$$q = -\frac{b^2}{4a} + c = -\frac{b^2 - 4ac}{4a}.$$

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Solving quadratic equations

The Quadratic Formula

As for $ax^2 + bx + c = 0$, the values of x which are the solutions of the equation are given by:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

or

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b-\sqrt{\Delta}}{2a},$$

we can also solve a quadratic equation using factorization:

$$ax^2 + bx + c = a(x-x_1)(x-x_2),$$

then $ax^2 + bx + c = a(x-x_1)(x-x_2) = 0,\ (x-x_1) = 0$ and $(x-x_2) = 0$,

so $x = x_1,\ x = x_2$.

Example 5.19

Solve the equation: $x^2 - 16 = 0$.

Solution

Factorize $x^2 - 16 = 0,\ (x-4)(x+4) = 0$.

We have two solutions $(x-4) = 0$ and $(x+4) = 0$, so $x = 4$, $x = -4$ respectively.

Example 5.20

Solve the equation: $x^2 + 6x = 0$.

Solution

Factorize $x^2 + 6x = 0,\ x(x+6) = 0$.

We have two solutions $x = 0$ and $x + 6 = 0$, so $x = 0$, $x = -6$ respectively.

Example 5.21

Solve the equation $2x^2 - 8x + 6 = 0$.

Solution

First, we divide the entire equation by 2 as a common factor of the coefficients, so

$2x^2 - 8x + 6 = 0$ can be written as $x^2 - 4x + 3 = 0$. Now we compute the discriminant

$$\Delta = (-4)^2 - 4\cdot 1\cdot 3 = 4 > 0, \quad \sqrt{\Delta} = 2,$$

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then

$$x_1 = \frac{4+2}{2} = 3\,, \quad x_2 = \frac{4-2}{2} = 1.$$

Example 5.22

Solve the equation $x^2 + 6x + 9 = 0$.

Solution

After factorizing $x^2 + 6x + 9 = 0$ we have $(x+3)^2 = 0$, so we have double root of the equation $x_1 = x_2 = x_0 = -3$.

Example 5.23

Give the example of a quadratic equation with integer coefficients which has two roots:

$\left(5-2\sqrt{3}\right)^{-1}$ and $\left(5+2\sqrt{3}\right)^{-1}$.

Solution

There are given the roots $x_1 = \frac{1}{5-2\sqrt{3}}$ and $x_2 = \frac{1}{5+2\sqrt{3}}$.

As

$$(x-x_1)(x-x_2) = 0 \quad \Leftrightarrow \quad x^2 - (x_1+x_2)x + x_1 \cdot x_2 = 0,$$

so, we compute

$$x_1 + x_2 = \frac{1}{5-2\sqrt{3}} + \frac{1}{5+2\sqrt{3}} = \frac{5+2\sqrt{3}+5-2\sqrt{3}}{(5+2\sqrt{3})(5-2\sqrt{3})} = \frac{10}{25-12} = \frac{10}{13},$$

$$x_1 \cdot x_2 = \frac{1}{5-2\sqrt{3}} \cdot \frac{1}{5+2\sqrt{3}} = \frac{1}{13}.$$

Hence, we obtain:

$$x^2 - \frac{10}{13}x + \frac{1}{13} = 0 \text{ and } 13x^2 - 10x + 1 = 0\,.$$

Example 5.24

Solve the equation $4x^2 + 2x + 1 = 0$.

As $\Delta = (2)^2 - 4\cdot 4\cdot 1 = -12$, so this equation has no real roots.

But we can find two complex roots knowing that $\sqrt{-1} = i$. Then $\sqrt{\Delta} = i\sqrt{12} = 2i\sqrt{3}$ and

$$x_1 = \frac{-2-2i\sqrt{3}}{8} = \frac{-1-i\sqrt{3}}{4}, \quad x_2 = \frac{-1+i\sqrt{3}}{4}.$$

Solving quadratic inequalities

Quadratic inequalities can be of the following forms:

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$$ax^2 + bx + c > 0,$$

$$ax^2 + bx + c \geq 0,$$

$$ax^2 + bx + c < 0,$$

$$ax^2 + bx + c \leq 0.$$

To solve a quadratic inequality, we must determine which part of the graph of a quadratic function lies above or below the x-axis. An inequality can therefore be solved graphically using a graph or algebraically using a table of signs to determine where the function is positive and negative.

Example 5.25

Solve for x: $x^2 - 5x + 6 \geq 0$.

Solution

Let us factorize the quadratic: $(x-3)(x-2) \geq 0$,

then we determine the critical values of x.

From the factorized quadratic we see that the values for which the inequality is equal to zero are $x = 3$ and $x = 2$. These are called the critical values of the inequality and they are used to complete a table of signs. To do we must determine where each factor of the inequality is positive and negative on the number line:

to the left (in the negative direction) of the critical value

equal to the critical value

to the right (in the positive direction) of the critical value

In the final row of the table, we determine where the inequality is positive and negative by finding the product of the factors and their respective signs.

Critical values		$x = 2$		$x = 3$	
$x - 3$	$-$	$-$	$-$	0	$+$
$x - 2$	$-$	0	$+$	$+$	$+$
$f(x) = (x-3)(x-2)$	$+$	0	$-$	0	$+$

From the table we see that $f(x)$ is greater than or equal to zero for $x \leq 2$ or $x \geq 3$.

The graph in *Figure 5.23* does not form part of the answer and is included for illustration purposes. A graph of the quadratic helps us determine the answer to the inequality. We can find the answer graphically by seeing where the graph lies above or below the x-axis.

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From the standard form, $x^2 - 5x + 6$, $a > 0$ and therefore the parabola opens up and has a minimum turning point.

From the factorized form, $(x-3)(x-2)$, we know the x-intercepts are $(2;0)$ and $(3;0)$.

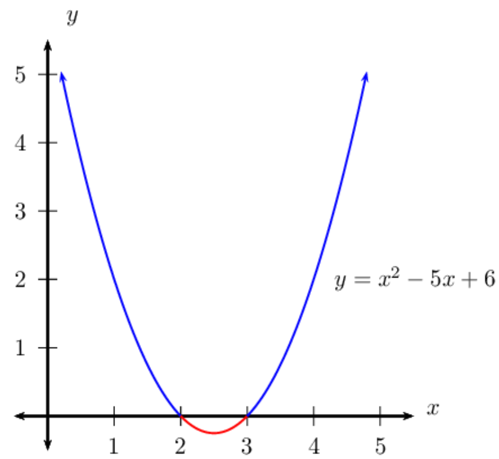
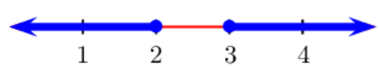


Figure 5.23 Graph of $\boldsymbol{f(x) = x^2 - 5x + 6}$.

The parabola is above or on the x −axis for $x \leq 2$ or $x \geq 3$.

Final answer and presentation on a number line is:

$$x^2 - 5x + 6 \geq 0 \text{ for } x \in (-\infty, 2] \cup [3, \infty)$$



Example 5.26

Suppose that you head out on a river boat cruise that takes 4 hours to go 20 km upstream and then turn around and go 20 km back downstream. When you get back, you notice that the speedometer of the boat wasn't working during the cruise, so you want to calculate the boat's speed. The river has a current of 3 kilometers per hour.

Solution

Assume that v_b-speed of the boat. As $s = v \cdot t$, $t = \frac{s}{v}$ we can create an equation:

$$\frac{20}{v_b - 3} + \frac{20}{v_b + 3} = 4.$$

Multiply both sides by $(v_b - 3)(v_b + 3)$ and get

$$20(v_b - 3) + 20(v_b + 3) = 4(v_b - 3)(v_b + 3)$$

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then

$$40v_b = 4{v_b}^2 - 36 \quad , \quad 4{v_b}^2 - 40v_b - 36 = 0$$

$$\Delta = (-40)^2 - 4(4)(-36) = 2176$$

$$\sqrt{\Delta} = \sqrt{2176}$$

$$x{v_b}_1 = \frac{40 - \sqrt{2176}}{8} \approx -0.83 \quad {v_b}_2 = \frac{40 + \sqrt{2176}}{8} \approx 10.83$$

We see that $v_b = -0.83$ or $v_b = 10.83$. Since we are talking about a speed, the negative answer makes no sense, so the answer is $v_b = 10.83$. In other words, the boat was traveling at a speed of 10.83 km/h.

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