5.4.3. Exponential functions

The exponential functions are mathematical functions used in many real-world situations. They are mainly used to find the exponential decay or exponential growth or to compute investments, model populations, radioactive decay and so on. In this section, you will learn about exponential function formulas, rules, properties, graphs, derivatives, exponential series and examples.

Definition and properties

Let a be given real number auch that, $a > 0,\ \ a \neq 1.$

A function of the form $\boldsymbol{y = a^x}$ is called exponential, x is any number.

a is called the base, while x is called the *exponent*.

The domain of the exponential function is $\mathbb{R}$, while the range is $(0, \infty)$.

The exponential function we define in four stages:

a) If $x = n,$ a positive integer, then

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}};$$

b) If $x = 0,$ then $a^0 = 1;$

c) If $x = -n,$ where n is a positive integer, then $a^{-n} = \frac{1}{a^n};$

d) If x is a rational number, $x = \frac{p}{q},$ where p, q are inegers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}\,.$$

Figure 5.34 presents the graphs of the exponential functions $y = 2^x$, $y = \left(\frac{1}{2}\right)^x$,

$y = 4^x$, $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{3}{2}\right)^x$, $y = \left(\frac{2}{3}\right)^x$. Notice that both graphs pass through the same point $(0, 1)$, because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential functions grows more rapidly (for $x > 0$). When the base $a > 1$ $\left(a = 1,\ a = 4,\ a = \frac{3}{2}\right)$:

- the exponential function is increasing;
- the graph is asymptotic to the x-axis as x approaches negative infinity;
- the function increases without bound as x approaches positive infinity;
- the graph is continuous;
- the graph is smooth.

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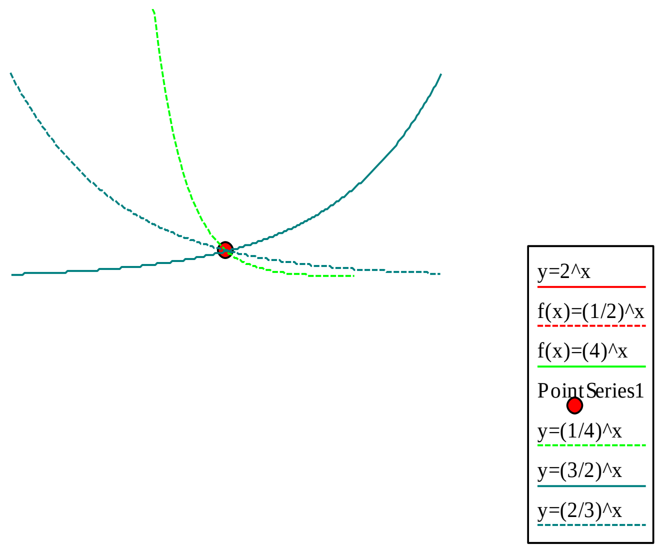

Figure 5.24 $y = 2^x$, $y = \left(\frac{1}{2}\right)^x$, $y = 4^x$, $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{3}{2}\right)^x$, $y = \left(\frac{2}{3}\right)^x$

The properties of the exponential functions and their graphs when the base $0 < a < 1$ are given.

- they form decreasing graphs
- the lines in the graph above are asymptotic to the x −axis as x approaches positive infinity
- the lines increase without bound as x approaches negative infinity
- they are continuous graphs
- they form smooth graphs.

Among the infinity number of exponential functions, two of them play the most important role: $y = 10^x$, $y = e^x$, where the number

$$e = 2{,}71828182845904523536028747135266249775724\ldots$$

is irrational (i.e. it cannot be represented as ratio of integers) and transcendental (i.e. it is not a root of any non-zero polynomial with rational coefficients). It is enough to admit that

$$e \approx 2{,}72.$$

The number e is sometimes called Euler's number, after the Swiss mathematician Leonhard Euler or as Napier's constant. However, Euler's choice of the symbol e is said to have been retained in his

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honor. The constant was discovered by the Swiss mathematician Jacob Bernoulli while studying compound interest.

Figure 5.25 shows the graph of $y = 10^x$, while in *Figure 5.26* you can see the graph of $y = e^x$.

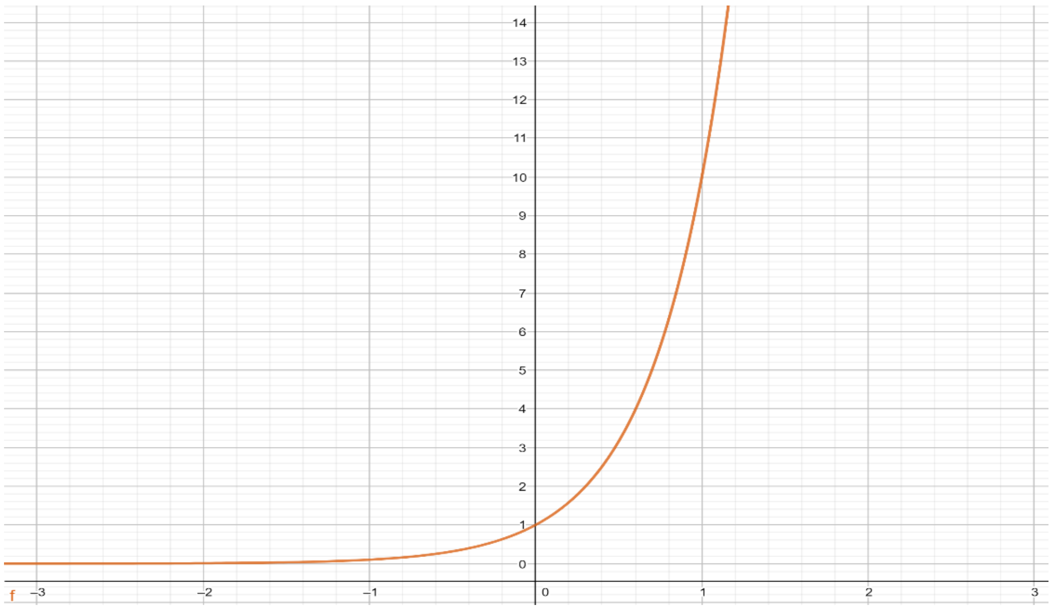

Figure 5.25 The graph of $y = 10^x$

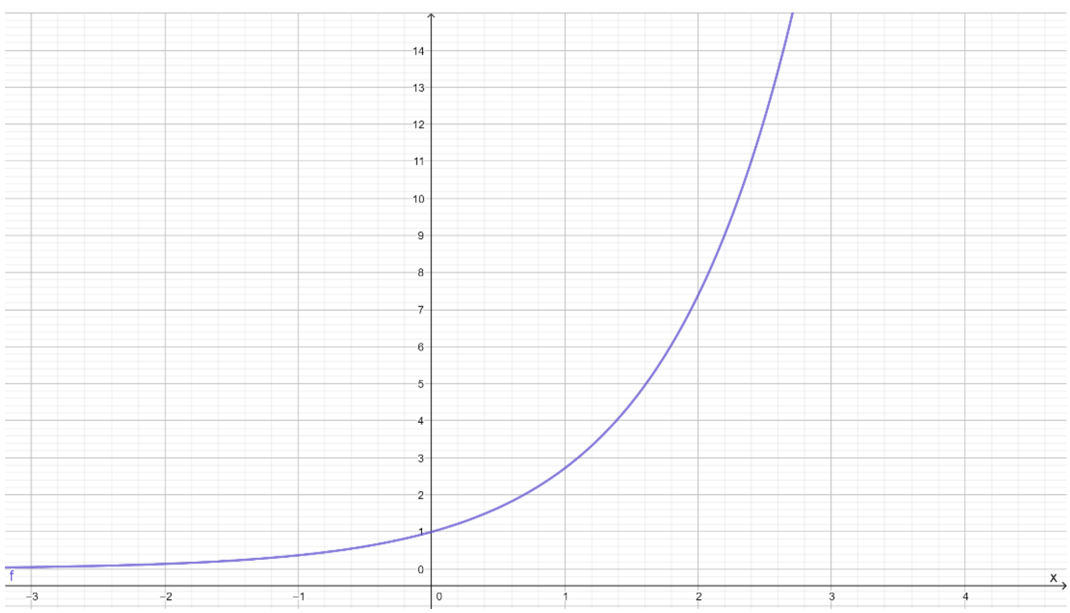

Figure 5.26 The graph of $y = e^x$.

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Basic rules for exponentials

Rule or special case	Formula $a > 0, \quad x \in \mathbb{R}, y \in \mathbb{R}$	Example
Product	$a^x a^y = a^{x+y}$	$2^3 2^4 = 2^7$
Quotient	$\frac{a^x}{a^y} = a^{x-y}$	$\frac{2^4}{2^3} = 2^{4-3} = 2$
Power of power	$(a^x)^y = a^{x\cdot y}$	$(2^3)^2 = 2^{2\cdot 3} = 2^6 = 64$
Power of a product	$(a\cdot b)^x = a^x b^x$	$(3\cdot 4)^2 = 3^2\cdot 4^2 = 9\cdot 16 = 144$
Power of one	$a^1 = a$	$2^1 = 2$
Power of zero	$a^0 = 1$	$2^0 = 1$
Power of negative one	$a^{-1} = \frac{1}{a}$	$2^{-1} = \frac{1}{2}$
Change sign of exponents	$a^{-x} = \frac{1}{a^x}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
Fractional exponents	$a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$	$4^{\frac{3}{2}} = \sqrt{4^3} = (\sqrt{4})^3 = 2^3 = 8$

Solving exponential equations and inequalities

Example 5.27

Solve the following equations and inequalities.

a) $\mathbf{2^x + 2^{x+1} + 2^{x+2} = 6^x + 6^{x+1}}$

Solution

Assumption: $x \in \mathbb{R}$. Using the rules of exponentials we have:

$$2^x + 2\cdot 2^x + 2^2\cdot 2^x = 6^x + 6\cdot 6^x \iff 2^x(1+2+4) = 6^x(1+6) \iff$$
$$2^x = 6^x \iff 1 = \frac{6^x}{2^x} \iff 1 = \left(\frac{6}{2}\right)^x \iff 1 = 3^x \iff x = 0.$$

b) $\left(\frac{7}{11}\right)^{7x-11} \geq \left(\frac{11}{7}\right)^{11x-7}$

Solution

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Assumption: $x \in \mathbb{R}$. Using the rules of exponentials we have:

$$\left[\left(\frac{11}{7}\right)^{-1}\right]^{7x-11} \geq \left(\frac{11}{7}\right)^{11x-7} \quad \Leftrightarrow \quad \left(\frac{11}{7}\right)^{11-7x} \geq \left(\frac{11}{7}\right)^{11x-7}$$

As the base $a = \frac{11}{7} > 1,$ so the last inequality is equivalent to the following inequality:

$$11 - 7x \geq 11x - 7 \quad \Leftrightarrow \quad -18x \geq -18 \quad \Leftrightarrow \quad x \leq 1.$$

c) $\mathbf{6 \cdot 9^x + 5 \cdot 6^x - 6 \cdot 4^x \leq 0}$.

Solution

Assumption: $x \in \mathbb{R}$. Dividing the inequality by positive 4^x we obtain

$$\mathbf{6} \cdot \frac{9^x}{4^x} + \mathbf{5} \cdot \frac{6^x}{4^x} - \mathbf{6} \leq \mathbf{0} \quad \Leftrightarrow \quad 6 \cdot \left[\left(\frac{3}{2}\right)^2\right]^x + 5 \cdot \left(\frac{3}{2}\right)^x - 6 \leq 0 \quad \Leftrightarrow$$

$$6 \cdot \left[\left(\frac{3}{2}\right)^x\right]^2 + 5 \cdot \left(\frac{3}{2}\right)^x - 6 \leq 0\,.$$

Now we substitute $\left(\frac{3}{2}\right)^x = t, \quad t > 0.$ Then the last inequality takes the form:

$$6t^2 + 5t - 6 \leq 0\,.$$

As $\Delta = 25 + 144 = 169, \ \sqrt{\Delta} = 13\,, \ t_1 = \frac{-5-13}{12} = -\frac{3}{2}, \ t_2 = \frac{-5+13}{12} = \frac{2}{3}.$

Therefore $6t^2 + 5t - 6 \leq 0 \ \Leftrightarrow \ -\frac{3}{2} \leq t \leq \frac{2}{3} \ \Leftrightarrow \ -\frac{3}{2} \leq \left(\frac{3}{2}\right)^x \leq \frac{2}{3}.$

Hence $\left(\frac{3}{2}\right)^x \leq \frac{2}{3} \quad \Leftrightarrow \quad \left(\frac{3}{2}\right)^x \leq \left(\frac{3}{2}\right)^{-1} \quad \Leftrightarrow \quad x \leq -1.$

d) $\mathbf{2^{2x} \leq 3 \cdot 2^{x+\sqrt{x}} + 4 \cdot 2^{2\sqrt{x}}}$.

Solution

Assumption: $x \geq 0.$ Let us divide both sides of the inequality by positive $2^{2\sqrt{x}}$ and obtain

$$2^{2x-2\sqrt{x}} \leq 3 \cdot 2^{x+\sqrt{x}-2\sqrt{x}} + 4 \quad \Leftrightarrow \quad 2^{2(x-\sqrt{x})} \leq 3 \cdot 2^{x-\sqrt{x}} + 4\ .$$

Now we substitute $2^{x-\sqrt{x}} = t, \ t > 0.$ Then we get

$$t^2 \leq 3t + 4 \quad \Leftrightarrow \quad t^2 - 3t - 4 \leq 0 \quad \Leftrightarrow \quad (t-4)(t+1) \leq 0\,.$$

As we see in *Figure 5.27* $-1 \leq t \leq 4 \ \Leftrightarrow \ -1 \leq 2^{x-\sqrt{x}} \leq 4.$

Hence $2^{x-\sqrt{x}} \leq 4 \ \Leftrightarrow \ 2^{x-\sqrt{x}} \leq 2^2 \ \Leftrightarrow \ x - \sqrt{x} \leq 2.$

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Now we use substitution $\sqrt{x} = u,\ u \geq 0$ and obtain

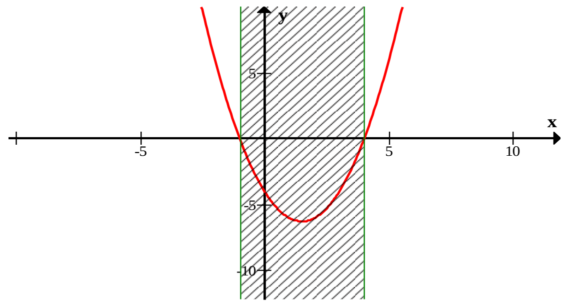

Figure 5.27 The illustration of $\boldsymbol{x^2 - 3x - 4 \leq 0}$.

$$u^2 - u \leq 2 \quad \Leftrightarrow \quad u^2 - u - 2 \leq 0 \quad \Leftrightarrow \quad (u+1)(u-2) \leq 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad -1 \leq u \leq 2 \quad \text{(see } \textit{Figure 5.28}\text{)}$$

$$\Leftrightarrow \quad -1 \leq \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \sqrt{x} \leq 2 \quad \Leftrightarrow \quad \begin{cases} x \geq 0 \\ x \leq 4 \end{cases} \quad \Leftrightarrow \quad x \in [0, 4].$$

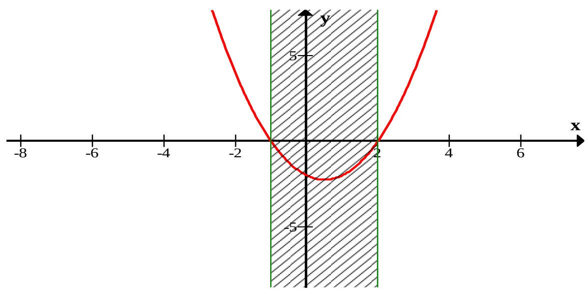

Figure 5.28 The illustration of $\boldsymbol{x^2 - x - 2 \leq 0}$.

Example 5.28 Compound interest

Most people who have a savings account with a bank or other financial institution leave their deposits for a period of time expecting to accrue money as time passes. If the deposits are made in an account carrying **simple interest** (flat rate of interest) the interest received is calculated on the original deposit for the duration of the account.

This would mean that if one invested $1\ 000$ € at a **flat interest rate** of 3.5% then in the first year he or she would have accrued:

$$\textbf{total earned = principal + } \mathbf{3.5\%} \textbf{ of principal over } \mathbf{1} \textbf{ year}$$
$$= 1000 + \frac{3{,}5}{100} \cdot 1000 \cdot 1 = 1000(1 + 0.035) = 1035$$

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Or $1\,035$ €.

We could perform the same calculations over five years, shown in the table below.

YEAR	TOTAL IN ACCOUNTS €	
Year 0	1000	1000
Year 1	$(1+0.035)$ of amount earned in year 0	$1000(1+0.035) = 1035$
Year 2	$(1+0.035)$ of amount earned in year 1	$(1+0.035)1000(1+0.035) = 1071{,}225$
Year 3	$(1+0.035)$ of amount earned in year 2	$= 1000 \cdot (1+0.035)^3 = 1108{,}71788$
Year 4	$(1+0.035)$ of amount earned in year 3	$1000 \cdot (1+0.035)^4 = 1147{,}523$
Year 5	$(1+0.035)$ of amount earned in year 4	$1000 \cdot (1+0.035)^5 = 1187{,}686$

The compound interest formula is as follows

$$A = P\left(1+\frac{r}{100}\right)^n,$$

where

- A is the total amount returned, P is the principal (initial amount)
- r is the rate as a percentage returned in each investment period and n is the number of investment periods.

Example 5.29 Compound interest

Suppose in 2010 a man purchased a yacht Delfia 47S/Y 4 Breeze valued at $\$275\,000$. We know that yacht depreciate at 11.2% each year. What would the value of the yacht be after a period of time? Examine the table below for calculations for 5 years.

Solution

It is a special case of the depreciation formula:

$$D = P\left(1-\frac{r}{100}\right)^n,$$

where D is final value of the asset, P is the initial value of the asset, r is the rate of depreciation per period and n is the number of depreciation periods.

YEAR		
Year 0	275000	275000
Year 1	$(1-0.112) of\ year\ 0$	$(1-0.112)\cdot 275000 == 244200$
Year 2	$(1-0.112) of\ year\ 1$	$275000 \cdot (1-0.112)^2 == 216849.6$

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Year 3	$(1-0.112) of\ year\ 2$	$275000 \cdot (1-0.112)^3 == 192\ 562{,}4$
Year 4	$(1-0.112) of\ year\ 3$	$275000 \cdot (1-0.112)^4 == 170995{,}5$
Year 5	$(1-0.112) of\ year\ 4$	$275000 \cdot (1-0.112)^5 == 151844$

Example 5.30

Exponential decrease can be modeled as:

$$N(t) = N_0 e^{-\lambda t}$$

where N is the quantity, N_0 is the initial quantity, λ is the decay constant (specific for each element), and t is time.
Oftentimes, half-life is used to describe the amount of time required for half of a sample to decay. It can be defined mathematically as:

$$t_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$$

where $t_{\frac{1}{2}}$ is half-life.
Half-life can be inserted into the exponential decay model as such:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}.$$

Find how much carbon and iodine are present after a set period of time (t) given the information provided in the following table.

Element	λ	N_0 (in grams)	t
Carbon	$1{,}203 \cdot 10^{-4}\ years^{-1}$	3	$5760\ years$
Iodine	$0{,}08666\ days^{-1}$	5	$8\ days$

Solution

Using the decay function

$$N(t) = N_0 e^{-\lambda t}$$

for Carbon, the amount left after $t = 5\ 760$ years is

$$N(t) = 3e^{-1{,}203 \cdot 10^{-4} \cdot 5760} = 3e^{-6929} \approx 1{,}5\ g.$$

For Iodine, the amount left after $t = 8$ days is

$$N(t) = 5e^{-0{,}08666 \cdot 8} = 5e^{-0{,}6933} \approx 2{,}5\ g.$$

Notice that in each of the cases above the resultant mass is half of the initial mass of each element. This is an important notion in nuclear research. The time taken for a quantity of a specific element to

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be reduced to one half of its original mass is known as the **half-life** of the element. The half-life of carbon is $5\,760$ years and the half-life of iodine is 8 days.

Remark

- **half-life**: The time it takes for a substance (drug, radioactive nuclide, or other) to lose half of its pharmacological, physiological, biological, or radiological activity.

  Imagine we have 100 kg of a substance with a half-life of 5 years. Then in 5 years half the amount (50 kg) remains. In another 5 years there will be 25 kg remaining. In another 5 years, or 15 years from the beginning, there will be 12.5. The amount by which the substance decreases, is itself slowly decreasing.
- **isotope**: Any of two or more forms of an element where the atoms have the same number of protons, but a different number of neutrons. As a consequence, atoms for the same isotope will have the same atomic number but a different mass number (atomic weight).

Example 5.31

A certain substance decays exponentially over time and is modelled by the function

$$N(t) = 4e^{-\lambda t},$$

where $\lambda = \frac{1}{5771}\ years^{-1}$ and $N(t)$ is measured in grams. Find how much of the substance is present initially and how much is present $4\,000$ years later. Use the findings to comment on the half-life of this particular substance.

Solution

Given the function: $N(t) = 4e^{\left(-\frac{t}{5771}\right)}$,
to find the mass present initially put $t = 0$.

$$N(0) = 4e^{\left(-\frac{0}{5771}\right)} = 4 \cdot e^{0} = 4\ .$$

Initial mass is 4 grams.

When $t = 4000$

$$N(4000) = 4e^{\left(-\frac{4000}{5771}\right)} = 4e^{(-0{,}69312)} = 2{,}000052 \approx 2$$

After 4000 years the mass is 2 grams.
The mass has halved after $4\,000$ years so the half-life of the substance must be $4\,000$ years.

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