5.4.4. Logarithmic functions

Logarithmic functions are the inverses of exponential functions, and any exponential function can be expressed in logarithmic form. Similarly, all logarithmic functions can be rewritten in exponential form. Logarithms are useful in permitting us to work with very large numbers while manipulating numbers of a much more manageable size.

Definition:

A logarithmic function is a function of the form

$$\boldsymbol{y = \log_a x, \quad x > 0, \quad a > 0, \quad a \neq 1,}$$

which is read " y equals the log of x, base a".

$$\boldsymbol{y = log_a\, x \quad \text{is equivalent to } x = a^y.}$$

There are no restrictions on y.

Example 5.32

Evaluate: a. $\log_3 81$, b. $\log_{25} 5$, c. $\log_{10} 0.001$.

Solution

a) $\log_3 81 = 4$ because $3^4 = 81$;

b) $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$;

c) $\log_{10} 0.001 = -3$ because $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$.

1. $\log_a(a^x) = x$ for every $x \in \mathbb{R}$
2. $a^{\log_a x} = x$ for every $x > 0$.

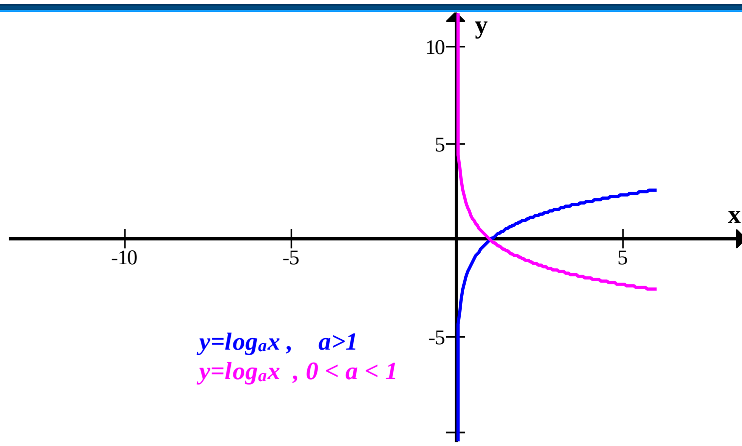

Figure 5.29 Graphs of logarithmic function when the base $a > 1$ *and* $0 < a < 1$.

The logarithmic function $y = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$ and it is continuous since it is the inverse of a continuos function, namely, the exponential function.

As we see in *Figure 5.29* when the base $a > 1$ the logarithmic function is increasing while for $0 < a < 1$ the function is decreasing. *Figure 5.30* shows the graphs of $y = \log_a x$ with various values of the base a. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

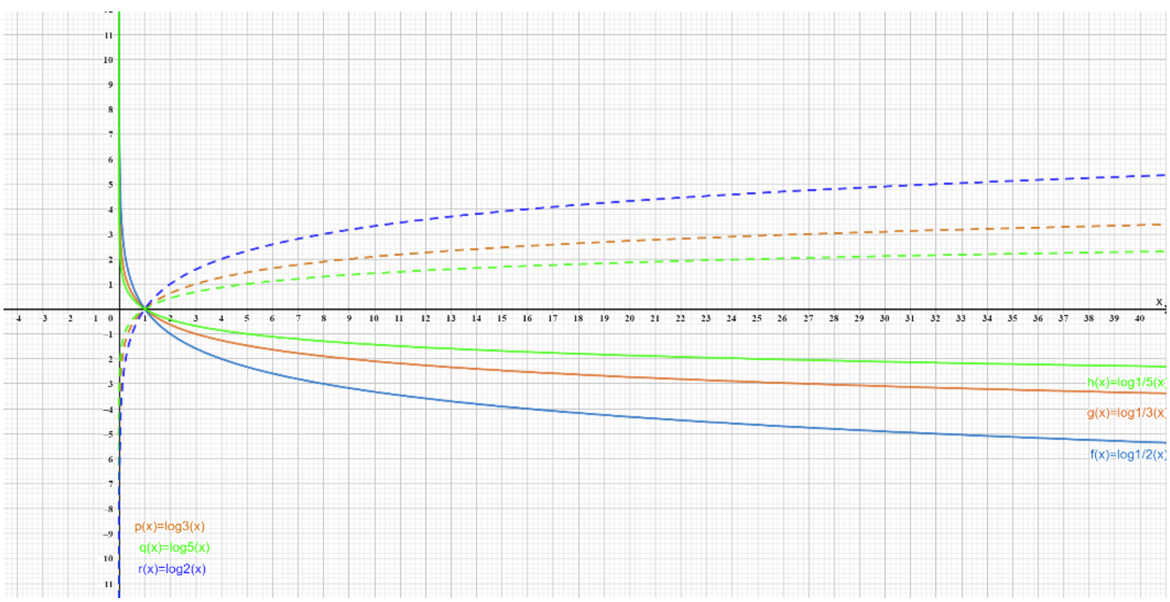

Figure 5.30 The graphs of $y = log_a x$ *with various values of the base* a.

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We need to introduce some special logarithms that occur on a very regular basis. They are the **common logarithm** and the **natural logarithm**. Here are the definitions and notations that we will be using for these two logarithms:

- common logarithm: $\log_{10} x = \log x$, the base $a = 10$;
- natural logarithm: $\log_e x = \ln x$, where e is Euler's number.

So, as we see the common logarithm is simply the log base 10, except we drop the “base 10” part of the notation. Similarly, the natural logarithm is simply the log base e with a different notation and where e is the same number that we saw in the previous section and is defined to be $e = 2.718281828\ldots$.*Figure 5.31* presents the graphs of the logarithms.

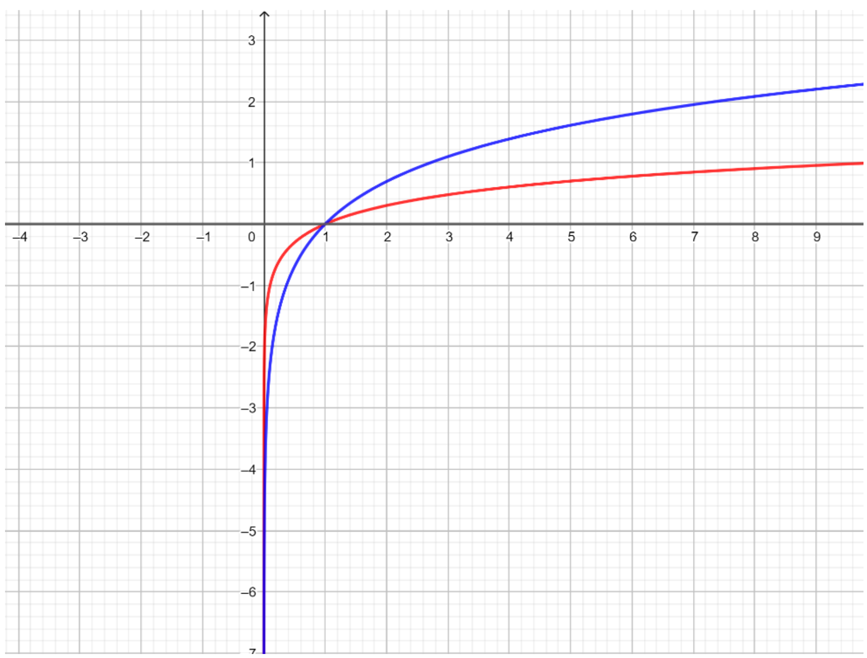

Figure 5.31 Graphs of $y = \log x$, $y = \ln x$

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Remember

If we put: $a = 10$ and $\log_{10} x = \log x$, $a = e$, $\log_e x = \ln x$, then the **defined properties** of the common logarithm and the natural logarithm functions become

$$y = \log x \quad \Leftrightarrow \quad 10^y = x \ ,$$

$$y = \ln x \quad \Leftrightarrow \quad e^y = x \ ,$$

and also

$$\log(10^x) = x \, , \quad x \in \mathbb{R} \ ,$$

$$10^{\log x} = x, \qquad x > 0,$$

$$\ln(e^x) = x \, , \quad x \in \mathbb{R} \ ,$$

$$e^{\ln x} = x, \qquad x > 0.$$

In particular, if we set $x = 10$, $x = e$, we get

$$\log 10 = 1, \ \ln e = 1.$$

Example 5.33

a) Find x if $\ln x = 5$.

Solution

$\ln x = 5$ means $e^5 = x$.
Therefore $x = e^5$.

b) Solve the equation $e^{5-3x} = 10$.

Solution :

We take natural logarithm of both sides of the equation:

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \frac{1}{3}(5 - \ln 10).$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution to four decimal places: $x \approx 0.8991$.

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The following theorem summarizes the properties of logarithmic functions.

Theorem 4.1.

If $a > 1$, the function $f(x) = \log_a x$ is one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$, then

1) $\mathbf{\log_a(xy) = \log_a x + \log_a y}$
2) $\mathbf{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$
3) $\mathbf{\log_a(x^y) = y\log_a x}$

Example 5.34

a) Evaluate $\log_4 2 + \log_4 32$.
b) Evaluate $\log_2 80 - \log_2 5$.
c) Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

Solution:

a) Using Property 1 in Theorem 4.1., we have

$$\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3,$$

since $4^3 = 64$.

b) Using Property 2 we have

$$\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = 4,$$

since $2^4 = 16$.

c) Using Properties 3 and 1 of logarithms, we have

$$\ln a + \frac{1}{2}\ln b = \ln a + \ln b^{\frac{1}{2}} = \ln a + \ln\sqrt{b} = \ln\left(a\sqrt{b}\right).$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

For any positive number a, $a \neq 1$, we have

4) $\mathbf{\log_a x = \frac{\ln x}{\ln a}}$.

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Generally, if we need to change the base a for another, let say b , we can do it as follows

$$5)\ \log_a x = \frac{\log_b x}{\log_b a}.$$

Example 5.35 *Evaluate* $log_8 5 = \frac{ln\, 5}{ln\, 8}$.

Solution

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976.$$

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 4.4.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is a continuous, increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote.

Example 5.36

Determine which of the numbers are greater: $\log_3 222$ or $\log_2 33$.

Solution:

As the function $f(x) = \log_a x$, $a > 1$ is increasing, we have

$$\log_3 222 < \log_3 243 = \log_3 3^5 = 5 = \log_2 2^5 = \log_2 32 < \log_2 33.$$

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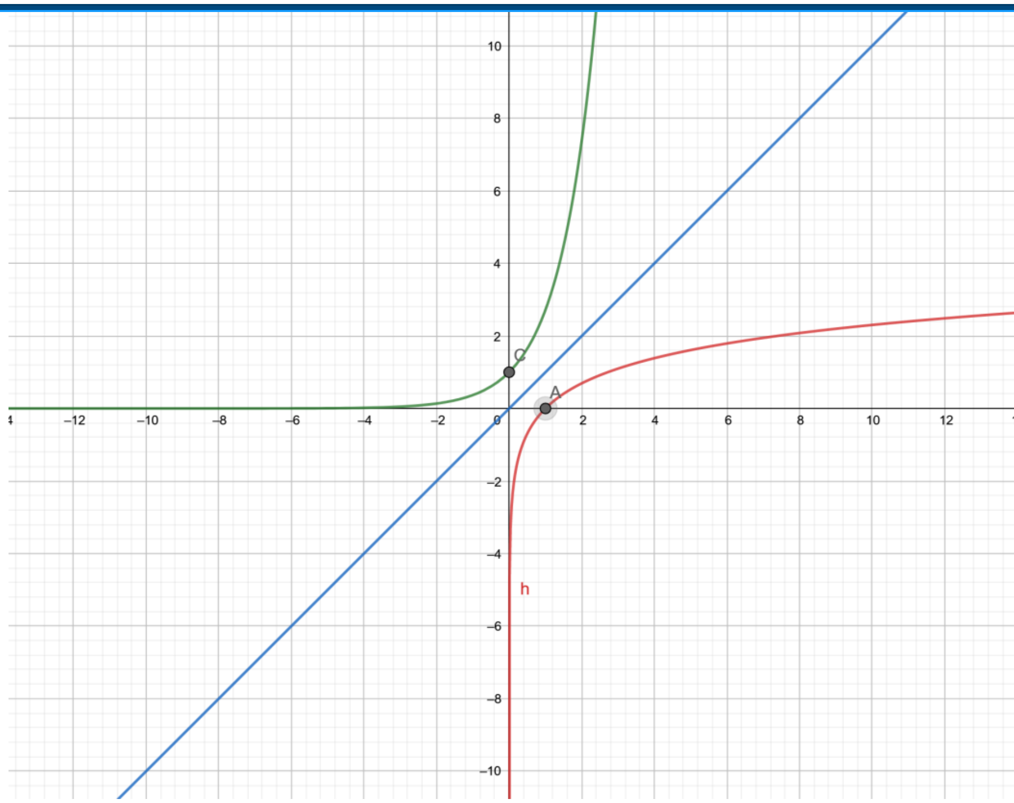

Figure 5.32 Graphs of $y = \ln x$, $y = e^x$ are symmetric with regard to $y = x$.

Example 5.37 Solve the equations

a. $\log(3-x)\,(x-5) = \log(x-3) + \log(5-x)$.
b. $\log_3(4 \cdot 3^{x-1} - 1) = 2x - 1$.
c. $\log(5x^2 + 2x - 1) - \log(x+2) = 1$.

Solutions

a. As $(3-x)(x-5) = (x-3)(5-x)$, so we can write

$$\log(x-3)\,(5-x) = \log(x-3) + \log(5-x).$$

So the equation is satisfied if and only if

$$(x - 3 > 0 \ \text{ and } \ 5 - x > 0\,) \Leftrightarrow (x > 3 \ \text{ and } \ x < 5) \ \Leftrightarrow \ 3 < x < 5.$$

b. Assumption: $4 \cdot 3^{x-1} - 1 > 0$, $3^{x-1} > \frac{1}{4}$ and we take $\log$ base 3 of both sides of the inequality and we get

$$x - 1 > \log_3 \tfrac{1}{4},$$
$$x - 1 > -\log_3 4,$$

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$$x > 1 - \log_3 4,$$
$$x > \log_3 3 - \log_3 4,$$
$$x > \log_3 \tfrac{3}{4}.$$

Now:

$$\text{Log}_3(4 \cdot 3^{x-1} - 1) = \log_3(3^{2x-1})$$

$$4 \cdot 3^{x-1} - 1 = 3^{2x-1} \ |\cdot 3$$

$$4 \cdot 3^{x} - 3 = 3^{2x} \ \Leftrightarrow \ (3^x)^2 - 4 \cdot 3^x + 3 = 0.$$

Let $3^x = t, \ t > 0$, then

$$t^2 - 4t + 3 = 0 \ \Leftrightarrow \ (t-1)(t-3) = 0 \ \Leftrightarrow \ t = 1 \text{ or } t = 3.$$

Hence $3^x = 1$, $3^x = 3 \ \Leftrightarrow \ x = 0, \ x = 1$.

Both $x = 0, \ x = 1$ satisfy the condition $x > \log_3 \frac{3}{4}$.

c. Assumption 1: $5x^2 + 2x - 1 > 0$.
Assumption 2: $x + 2 > 0$.

The equation can be written as follows:

$$\log(5x^2 + 2x - 1) = \log(x+2) + 1 \ \Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log(x+2) + \log(10) \ \Leftrightarrow$$

$$\log(5x^2 + 2x - 1) = \log[10(x+2)] \ \Leftrightarrow$$

$5x^2 + 2x - 1 = 10(x+2)$ for all x satisfying Assumptions: 1 and 2.

Notice that if x_0 satisfies the equation and the Assumption 2 then x_0 satisfies also Assumption 1. Therefore, after solving the equation, it is enough to verify if its solutions satisfy Assumption 2 – it is much easier.

Now we solve the quadratic equation:

$$5x^2 - 8x - 21 = 0:$$
$$\Delta = 64 + 420 = 484, \ \sqrt{\Delta} = 22, \ x_1 = \frac{8-22}{10} = -\frac{7}{5}, \ x_2 = \frac{8+22}{10} = 3.$$

It is easy to check that both solutions satisfy Assumption 2, thereby Assumption 1 which means that

$$x_1 = -\frac{7}{5}, \ x_2 = 3$$

are solutions of the equation

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$$\log(5x^2 + 2x - 1) - \log(x + 2) = 1.$$

If we need to solve the logarithmic inequalities we will use the following facts:

- If $a > 1$, $g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \iff f(x) \geq g(x)$.
- If $0 < a < 1$, $f(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \iff f(x) \leq g(x)$.

Example 5.38 Solve the inequalities

a. $\log(x-4) + \log x \leq \log 21$.
b. $\log(2^x + x - 13) > x - x\log 5$.
c. $3^{(\log_3 x)^2} + x^{\log_3 x} \leq 162$.
d. $\log_{(x-2)} \frac{x-1}{x-3} \geq 1$.

Solutions

a. Assumption: $(x - 4 > 0\,,\ x > 0) \iff x > 4$.
For $x > 4$ we have

$$\log(x-4) + \log x \leq \log 21 \iff \log x(x-4) \leq \log 21 \iff x(x-4) \leq 21$$

$$\iff x^2 - 4x - 21 \leq 0.$$

Compute $\Delta = 16 + 84 = 100$, $\sqrt{\Delta} = 10$, $x_1 = -3$, $x_2 = 7$.

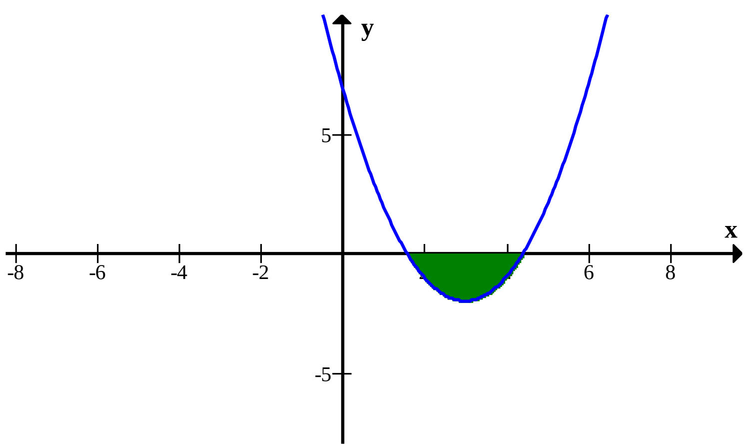

Figure 5.33

Thus $x \in [-3, 7]$ and taking included the assumption $x > 4$, finally we obtain

$$x \in (4,\ 7].$$

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b. We write the right side of the inequality as common logarithm, namely

$$x - x\log 5 = x(1-\log 5) = x(\log 10 - \log 5) = x\log 2 = \log 2^x.$$

Now the inequality b. can be presented as

$$\log(2^x + x - 13) > \log 2^x.$$

We know that

if $a > 1,\quad g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \;\Leftrightarrow\; f(x) \geq g(x)$.

We have $a = 10,\;\; g(x) = 2^x > 0$ for $x \in \mathbb{R}$, so

$$\log(2^x + x - 13) > \log 2^x \;\Leftrightarrow\; 2^x + x - 13 > 2^x \;\Leftrightarrow\; x > 13.$$

Then $x \in (13, \infty)$.

c. Assumption: $x > 0$.

As $3^{(\log_3 x)^2} = \left(3^{\log_3 x}\right)^{\log_3 x} = x^{\log_3 x}$, so the inequality takes the form of

$$2 \cdot x^{\log_3 x} \leq 162 \;\Leftrightarrow\; x^{\log_3 x} \leq 81.$$

Both sides of the inequality are positive, so take both sides log base 3 we get

$$\log_3 x^{\log_3 x} \leq \log_3 81 \;\Leftrightarrow\; (\log_3 x)^2 \leq 4 \;\Leftrightarrow\; |\log_3 x| \leq 2 \;\Leftrightarrow$$

$$\Leftrightarrow -2 \leq \log_3 x \leq 2 \;\Leftrightarrow\; \log_3 3^{-2} \leq \log_3 x \leq \log_3 3^2 \;\Leftrightarrow$$

$$\Leftrightarrow 3^{-2} \leq x \leq 3^2 \;\Leftrightarrow\; \tfrac{1}{9} \leq x \leq 9.$$

These numbers satisfy the assumption $x > 0$, so $x \in \left[\frac{1}{9},\; 9\right]$.

d. Assumption: $\begin{cases} x-2>0 \\ x-2 \neq 1 \\ \frac{x-1}{x-3} > 0 \end{cases} \;\Leftrightarrow\; \begin{cases} x > 2 \\ x \neq 3 \\ x > 3 \end{cases} \;\Leftrightarrow\; x > 3\,.$

Now we have

$$\log_{(x-2)} \frac{x-1}{x-3} \geq \log_{(x-2)}(x-2).$$

As for $x > 3,\;\; a = x - 2 > 1,\;\; g(x) = x - 2 > 0,$ then on the basis of the fact: "If $a > 1,\;\; g(x) > 0$ then $\log_a f(x) \geq \log_a g(x) \;\Leftrightarrow\; f(x) \geq g(x)$", we obtain

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$$\left(\frac{x-1}{x-3} \geq x-2 \quad , \quad x>3\right) \Leftrightarrow \begin{cases} x>3 \\ (x-2)(x-3) \leq x-1 \end{cases} \Leftrightarrow \begin{cases} x>3 \\ x^2-6x+7 \leq 0 \end{cases}.$$

Now we compute $\Delta = 36-28=8$, $\sqrt{\Delta} = 2\sqrt{2}$, $x_1 = 3-\sqrt{2}$, $x_2 = 3+\sqrt{2}$. From Fig. 4.6. it follows that $x \in \left[3-\sqrt{2}\, ,\ 3+\sqrt{2}\right]$ and $x>3$. Finally

$$x \in (3\, ,\ 3+\sqrt{2}].$$

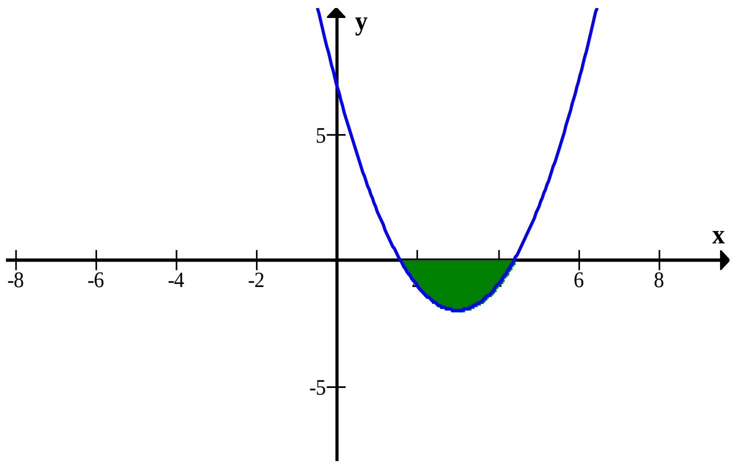

Figure 5.34

Example 5.39 ***Measuring loudness***

Sounds can vary in intensity from the lowest level of hearing (a ticking watch 7 meters away) to the pain threshold (the roar of a jumbo jet). Sound is detected by the ear as changes in air pressure measured in micropascals (μP). The ticking watch is about $20\ (\mu P)$, conversational speech about $20\,000\ (\mu P)$, a jet engine close up about $200\,000\,000\ (\mu P)$, an enormous range of values. A scale was required to compress the range of 20 to $200\,000\,000$ into a more manageable and useful form from 0 to 140. The decibel scale was invented for this purpose. If P is the level of sound intensity to be measured and P_0 is a reference level, then

$$n = 20\log\left(\frac{P}{P_0}\right),$$

where n is the decibel scale level.

If we assume $20\ (\mu P)$ to be the threshold level, then the equation would be:

$$n = 20\log\left(\frac{P}{20}\right)$$

and the graph of the relationship would resemble the one below (Figure 5.35).

Decibel scale for loudness of sound

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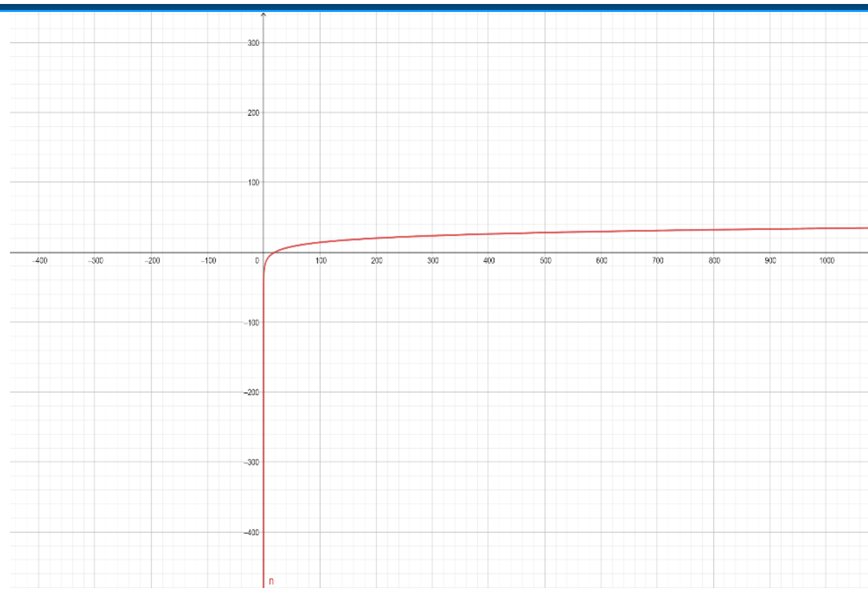

Figure 5.35 Sound level (μP)

As the sound is measured in a logarithmic scale using a unit called a decibel then we can use the following formula:

$$d = 10\log\left(\frac{P}{P_0}\right)$$

where P is the power or intensity of the sound and P_0 is the weakest sound that the human ear can hear.

One hot water pump has a noise rating of 50 decibels. One device in engine room, however, has a noise rating of 62 decibels. How many times is the noise of the device in engine room more intense than the noise of the hot water pump?

Solution

We can't easily compare the two noises using the formula, but we can compare them to P_0. Start by finding the intensity of noise for the hot water pump. Use h for the intensity of the hot water pump's noise:

$50 = 10\log\left(\frac{h}{P_0}\right)$,

$5 = \log\left(\frac{h}{P_0}\right)$,

$10^5 = \frac{h}{P_0}$

$h = 10^5 P_0$,

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then repeat the same process to find the intensity of the noise for the device in engine room

$$62 = 10\log\left(\frac{d}{P_0}\right),$$
$$6{,}2 = \log\left(\frac{d}{P_0}\right),$$
$$10^{6{,}2} = \frac{d}{P_0},$$

$$d = 10^{6{,}2}P_0.$$

To compare d to h, we divide $\frac{d}{h} = \frac{10^{6.2}P_0}{10^5 P_0} = 10^{1.2}$

Answer: The noise of the device in engine room is $10^{1.2}$ (or about 15.85) times more intense of the noise of the hot water pump.

Remark

The conditions of people on board the ships are particularly difficult, because even after working in crew cabins are at risk of being in high-level areas vibration and noise. During the cruise, the crew cannot escape to the forest or park in areas of peace and quiet. The crew (especially the mechanics) is exposed to danger related to hearing loss. In addition, excessive levels of noise and vibration cause others ailments such as cardiovascular and nervous system diseases. Ailments syndrome health related to noise, infrasonic noise and vibrations is called vibroacoustic disease. For example, at room of marine power plant a permissible noise level is 90 decibels ($90\ dB$), at control room, navigation cabin – $65dB$.

Example 5.40

The stellar magnitude of a star is negative logarithmic scale, and the quantity measured is the brightness of the star. If $SM = -\log B$, where SM is stellar magnitude and B is brightness, answer the following questions:

a) What is the stellar magnitude of star A which has a brightness of 0.7943?
b) Star B has a magnitude of 2.1, what is its brightness?
c) Compare the brightness of the two stars.

Solution

a) If the star has a brightness of 0.7943 from the formula $SM = -\log B$ it will have a stellar magnitude of $-\log(0.7943)$ or 0.1. This can be confirmed from the graph.
b) If the magnitude is 2.1 then

$$2.1 = -\log B$$
$$-2.1 = \log B$$

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$B = 10^{-2.1}$
$B \approx 0.007943.$

The brightness is 0.007943.

c) Comparing the star A with the star B : A is about 100 times brighter then B.

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