5.4.5. Square root function

Square roots are often found in math and science problems. Students can easily understand the rules of square roots and answer any questions involving them, whether they require direct calculation or just simplification.

A square root asks you which number, when multiplied by itself, gives the result after the "$\sqrt{\ }$" symbol. So $\sqrt{4} = 2$ and $\sqrt{25} = 5$.

The "$\sqrt{\ }$" symbol tells you to take the square root of a number and you can find this on most calculators. The symbol "$\sqrt{\ }$" is called the radical and x is called the radicand.

We can factor square roots just like ordinary numbers, so $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ or

$\sqrt{6} = \sqrt{2}\,\sqrt{3}$.

Square root of a negative number

The definition of a square root means that negative numbers should not have a square root (because any number multiplied by itself gives a positive number as a result). The imaginary number i is used to mean the square root of -1,

$$\sqrt{-1} = i, \quad i^2 = -1.$$

Any other negative roots are expressed as multiples of i, $\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \mp 2i$.

Square root function

The square root function is a type of power function $f(x) = x^{\alpha}$, with fractional power as it can be written

$$\boldsymbol{f(x) = x^{\frac{1}{2}},\ f(x) = \sqrt{x}}.$$

Its domain D_f is the set of non-negative real numbers:

$$D_f = [0, \infty) = \mathbb{R}_+ \cup \{0\}.$$

Its range is also the set of non-negative real numbers: $[0, \infty)$.

The graph of the square root function is shown in *Figure 5.36.* with some points.

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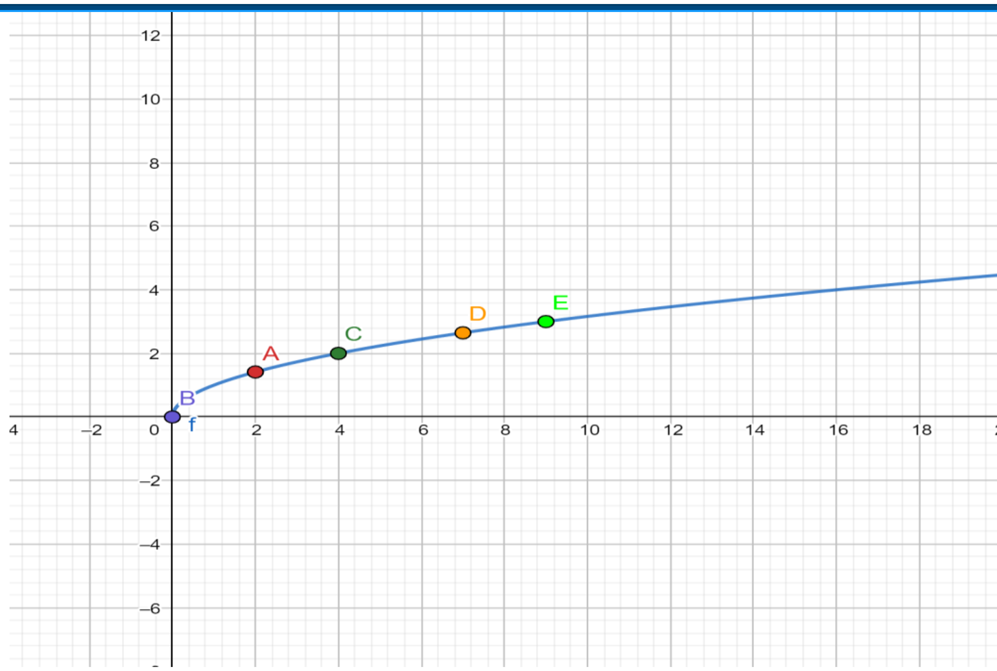

Figure 5.36 The graph of $f(x) = \sqrt{x}$

- **It is wrong** to write $\sqrt{25} = \mp 5$. The radicand is the symbol of the square root function and a function has only one output which as defined above is equal to the positive root.

Correct answer: $\sqrt{25} = 5$

- **It is wrong** to write $\sqrt{x^2} = x$. The output of the square root is non-negative and x in the given expression may be negative, zero or positive.

Correct answer: $\sqrt{x^2} = |x|$

Properties of square root function

Some of the properties of the square root function may be deduced from Fig.5.1.

1. x and y intercepts are both at $(0,0)$.
2. the square root function is an increasing function
3. the square root function is a one-to-one function and has an inverse.

Square root functions of the general form:

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$$f(x) = a\sqrt{x-c} + d.$$

Figure 5.37 presents how the graph of $f(x) = a\sqrt{x-c} + d$ looks like when the parameters a, c, d change.

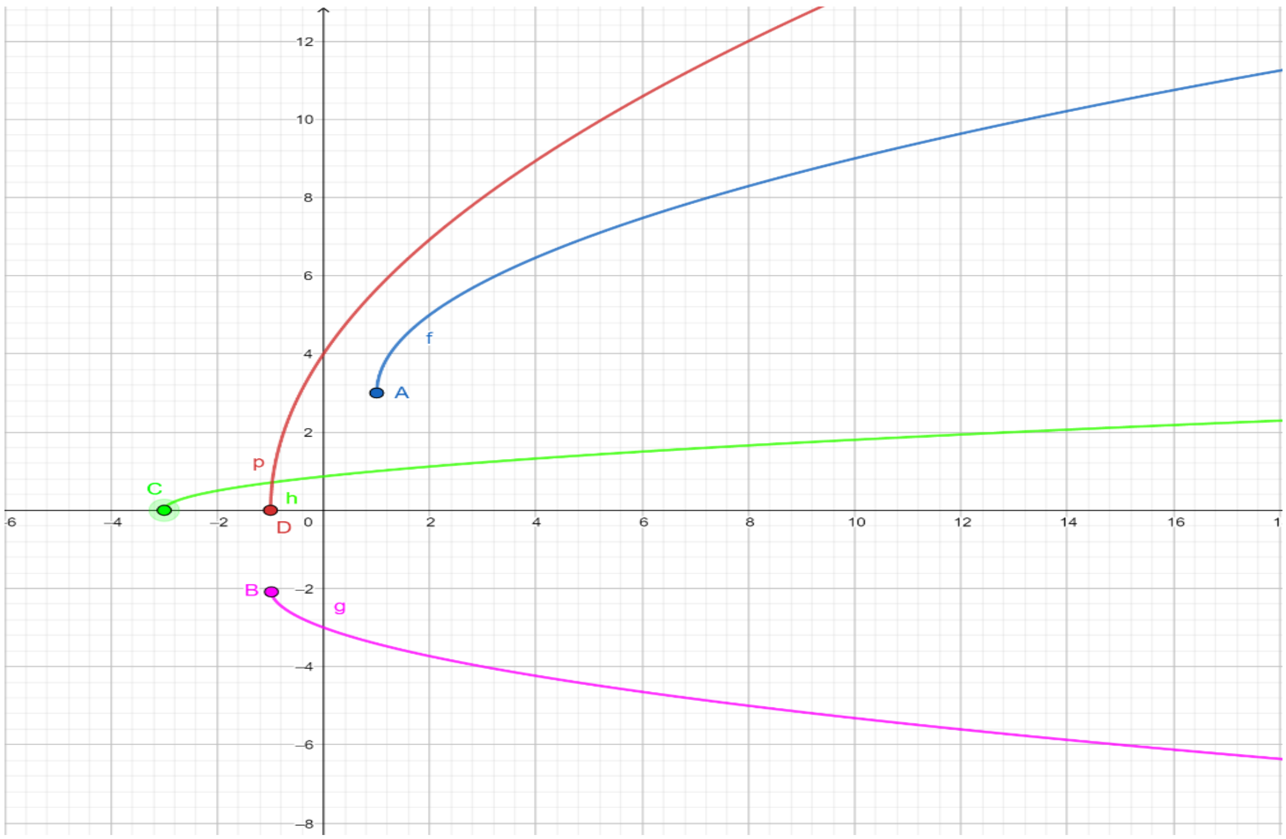

Figure 5.37 Graphs of $f(x) = 2\sqrt{x-1} + 3$, $g(x) = -\sqrt{x+1} - 2$, $h(x) = \frac{1}{2}\sqrt{x+3}$, $p(x) = 4\sqrt{x+1}$.

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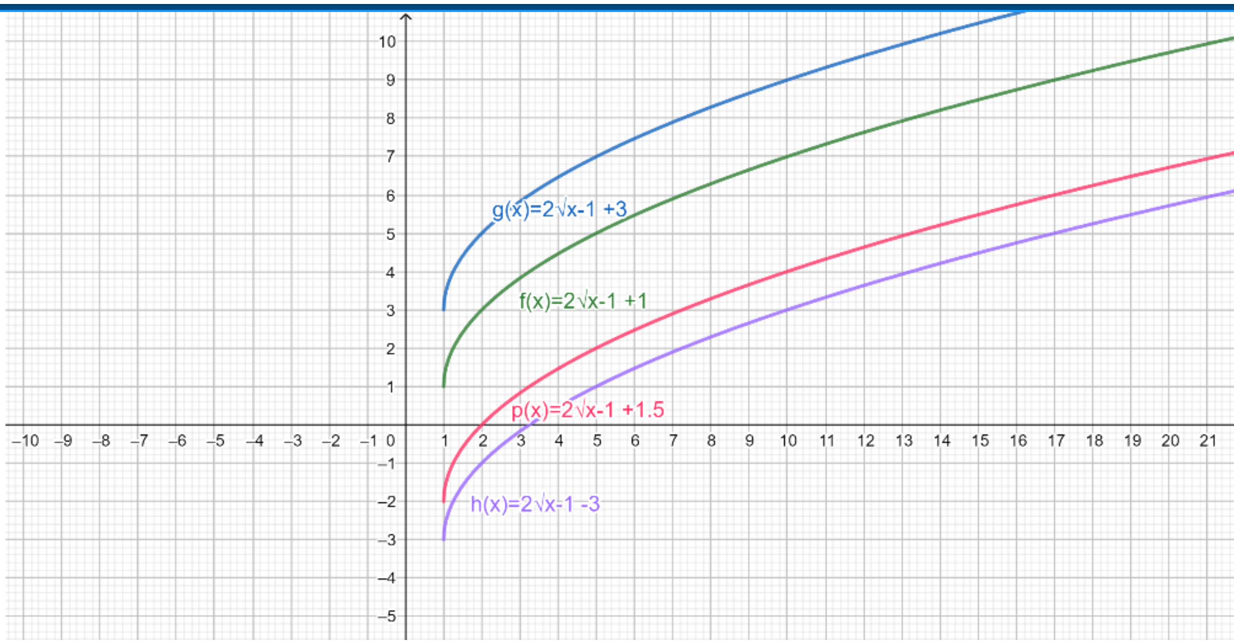

Figure 5.38 What happens to the graph when the value of parameter $\boldsymbol{d}$ changes?

From **Figure 5.37** to **Figure 5.39**.-5.4. we can conclude that

- changes in the parameter d affect the y coordinates of all points on the graph hence the vertical translation or shifting. When d increases, the graph is translated upward and when d decreases the graph is translated downward.

Figure 5.39 What happens to the graph when the value of parameter $\boldsymbol{c}$ changes?

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$p(x) = \sqrt{x+1}$, $h(x) = \sqrt{x-0.5}$, $g(x) = \sqrt{x-1}$, $f(x) = \sqrt{x-3}$.

- When c increases, the graph is translated to the right and when c decreases, the graph is translated to the left. This is also called horizontal shifting.
- Parameter a is a multiplicative factor for the y coordinates of all points on the graph of function $f(x)$.

  If a be greater than zero and larger than 1, the graph stretches (or expands) vertically. If a gets smaller than 1, the graph shrinks vertically. If a changes sign, a reflection of the graph on the x axis occurs.

- Only parameter c affects the domain. The domain of $f(x) = a\sqrt{x-c} + d$ may be found by solving the inequality $x - c \geq 0$. Hence, the domain is the interval $[c, \infty)$.
- Only parameters a and d affect the range. The range of function $f(x)$ may be found as follows: with x in the domain defined by interval $[c, \infty)$, the $\sqrt{x-c}$ is always positive or equal to zero hence $\sqrt{x-c} \geq 0$ and if parameter a is positive then $a\sqrt{x-c} \geq 0$.
  If we add d to both sides, we obtain $a\sqrt{x-c} + d \geq d$. Hence the range of the square root function defined above is the set of all values in the interval $[d, \infty)$.
  If parameter a is negative then $a\sqrt{x-c} \leq 0$. If we add d to both sides we obtain

$$a\sqrt{x-c} + d \leq d$$

  Hence the range of the square root function defined above is the set of all values in the interval $(-\infty, d]$.

Solving square roots equations and inequalities

We will use the following theorem:

Theorem 5.1.

If $a \geq 0$, $b \geq 0, n \in N$, then

- $a = b \iff a^n = b^n$
- $a < b \iff a^n < b^n$
- $a \leq b \iff a^n \leq b^n$.

Example 5.41

1. Solve the equation $\sqrt{10x+6} = 9 - x$.

Solution

Assume that $10x + 6 \geq 0$, $x \geq -\frac{3}{5}$.

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The left side of the equation is non-negative, so to be sure that the equation is not contradictory we have to assume additionally that $9 - x \geq 0$.

Finally, we get the assumption: $x \in \left[-\frac{3}{5}, 9\right]$.

Then using Theorem 5.1 we have

$$\sqrt{10x+6} = 9 - x \Leftrightarrow 10x + 6 = (9-x)^2 \Leftrightarrow x^2 - 28x + 75 = 0.$$

$$\Delta = 28^2 - 4 \cdot 75 = 484, \quad \sqrt{\Delta} = 22, \quad x_1 = \frac{28-22}{2} = 3, \quad x_2 = \frac{28+22}{2} = 25.$$

Due to the assumption $x \in \left[-\frac{3}{5}, 9\right]$, $x_1 = 3$ is the solution of the equation.

Example 5.42 Solve the inequalities:

a. $\sqrt{x+3} < -2$
b. $\sqrt{2-x} > -5$
c. $\sqrt{5-x} < 3$
d. $\sqrt{11-x} > x - 9$
e. $\sqrt{3 - 2x - x^2} < 2x^2 + 4x - 3$

Solution

a. Assume that $x + 3 \geq 0$, $i.e.$ $x \geq -3$.

Left side of $\sqrt{x+3} < -2$ is non-negative while the right one is negative, so we conclude that the inequality is a contradiction. It means that the inequality has no solutions.

b. Assume that $2 - x \geq 0$ $i.e.$ $x \leq 2$.

Left side of $\sqrt{2-x} > -5$ is non-negative while the right one is negative, so we conclude that the inequality is satisfied for all $x \leq 2$. Hence $x \in (-\infty, 2]$.

c. Assume that $5 - x \geq 0$, $i.e.$ $x \leq 5$.

Both sides of $\sqrt{5-x} < 3$ are non-negative, so according to Theorem 5.1., for $x \leq 5$ we have

$$\sqrt{5-x} < 3 \Leftrightarrow \left(\sqrt{5-x}\right)^2 < 3^2 \Leftrightarrow 5 - x < 9 \Leftrightarrow x > -4.$$

Having regard to the assumption $x \leq 5$ we get $x \in (-4, 5]$.

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d. Assume that $11 - x \geq 0,\ i.e.\ x \leq 11.$

As both sides of $\sqrt{11-x} > x - 9$ do not have the permanent sign at $(-\infty,\ 11]$ we have to consider two cases:

1. $x < 9,$ then the left side of the inequality is non-negative while the right one is negative, so the inequality holds for all $x < 9$;

2. $9 \leq x \leq 11,$ then both sides are non-negative, so when $x \in [9,\ 11]$ we have

$$\sqrt{11-x} > x - 9 \ \Leftrightarrow\ \left(\sqrt{11-x}\right)^2 > (x-9)^2 \ \Leftrightarrow$$

$$\Leftrightarrow\ 11 - x > x^2 - 18x + 81 \ \Leftrightarrow\ x^2 - 17x + 70 < 0$$

$$\Leftrightarrow\ \left|\ \Delta = 289 - 280 = 9,\quad \sqrt{\Delta} = 3,\ x_1 = \frac{17-3}{2} = 7, x_2 = \frac{17+3}{2} = 10\ \right|,$$

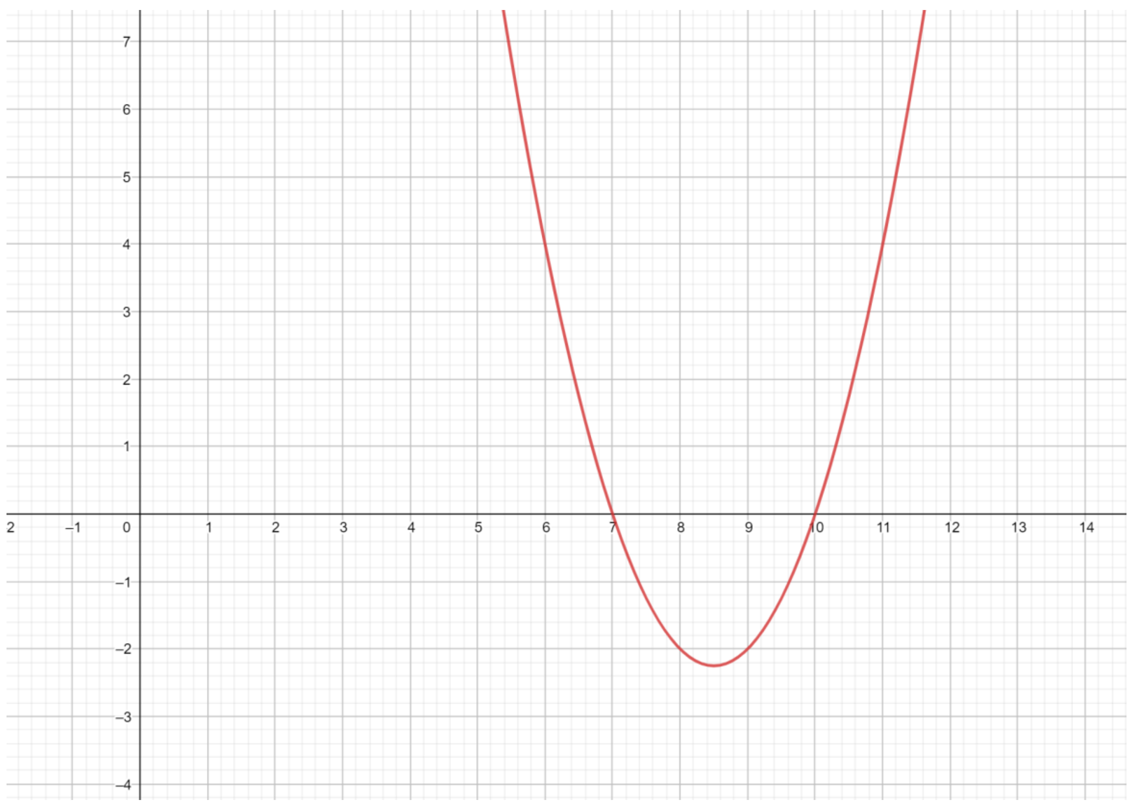

Figure 5.40

hence as we see in *Figure 5.40*. $x \in (7,\ 10)$. Regarding the assumption

$x \in [9,\ 11]$ we finally obtain $x \in [9, 10)$.

Considering both cases we get $x \in (-\infty,\ 9) \cup [9, 10) = (-\infty,\ 10)$.

e. Assume

$$3 - 2x - x^2 \geq 0 \;\Leftrightarrow\; x^2 + 2x - 3 \leq 0 \;\Leftrightarrow\; x^2 + 2x + 1 \leq 4 \;\Leftrightarrow$$

$$\Leftrightarrow\; (x+1)^2 \leq 4 \;\Leftrightarrow\; \sqrt{(x+1)^2} \leq \sqrt{4} \;\Leftrightarrow\; |x+1| \leq 2 \;\Leftrightarrow$$

$$\Leftrightarrow\; -2 \leq x + 1 \leq 2 \;\Leftrightarrow$$

$$\Leftrightarrow\; -3 \leq x \leq 1 \text{ or } x \in [-3,\ 1].$$

Notice that the right side of the inequality $\sqrt{3 - 2x - x^2} < 2x^2 + 4x - 3$ can

expressed by the trinomial $3 - 2x - x^2$ i.e.

$$2x^2 + 4x - 3 = 2(x^2 + 2x) - 3 = -2(-2x - x^2) - 3 =$$

$$-2(3 - 2x - x^2) + 3.$$

Then we can substitute $\sqrt{3 - 2x - x^2} = t$,

hence $3 - 2x - x^2 = t^2$ and

$$2x^2 + 4x - 3 = -2(3 - 2x - x^2) + 3 = -2t^2 + 3.$$

Now the inequality takes the following form:

$$t < -2t^2 + 3 \;\Leftrightarrow\; 2t^2 + t - 3 < 0.$$

After computing $\Delta = 1 + 24 = 25$, $\sqrt{\Delta} = 5$, we get $t_1 = -\frac{3}{2}, t_2 = 1$.

From *Figure 5.41* we conclude $-\frac{3}{2} < t < 1$. Therefore for $x \in [-3, 1]$ we have:

$$-\frac{3}{2} < \sqrt{3 - 2x - x^2} < 1 \;\Leftrightarrow\; \sqrt{3 - 2x - x^2} < 1 \;\Leftrightarrow\; \left(\sqrt{3 - 2x - x^2}\right)^2 < 1 \;\Leftrightarrow$$

$$\Leftrightarrow\; 3 - 2x - x^2 < 1 \;\Leftrightarrow\; x^2 + 2x + 1 > 3 \;\Leftrightarrow\; (x+1)^2 > 3 \;\Leftrightarrow$$

$$\Leftrightarrow\; |x+1| > \sqrt{3} \;\Leftrightarrow\; x + 1 < -\sqrt{3} \;\vee\; x + 1 > \sqrt{3} \;\Leftrightarrow$$

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$$\Leftrightarrow \; x < -1-\sqrt{3} \quad \vee \quad x > -1+\sqrt{3} \; .$$

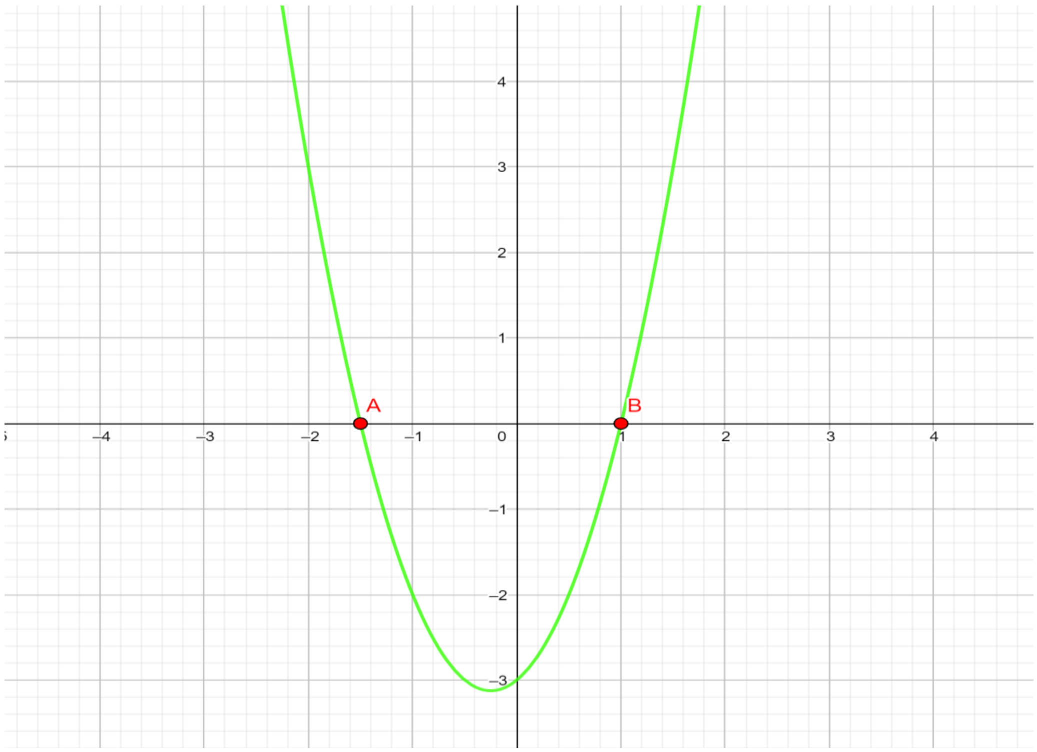

Figure 5.41

Finally, included the assumption $[-3, 1]$ the solution is

$$x \in [-3, \; -1-\sqrt{3}) \cup \left(-1+\sqrt{3}, \; 1\right].$$

3. Solve the equation:

$$\sqrt{11x+3} + \sqrt{3x-6} = \sqrt{4-2x} + \sqrt{12x+1}.$$

<u>Solution</u>

Assume that $11x+3 \geq 0, \; 3x-6 \geq 0, \; 4-2x \geq 0, \; 12x+1 \geq 0.$

Solving all those inequalities at the same time we notice that the equation all adds up only for $x = 2.$ It is easy to verify that $x = 2$ satisfies the equation, so it is its solution.

4. Solve the inequality:

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$$\sqrt{2x-1}+\sqrt{3x-2}<\sqrt{4x-3}+\sqrt{5x-4}.$$

Solution

Assume that $2x-1\geq 0$ and $3x-2\geq 0$ and $4x-3\geq 0$ and $5x-4\geq 0.$ Hence

$$x\geq\frac{4}{5}.$$

Let us transform the inequality, namely:

$$\left(\sqrt{4x-3}-\sqrt{2x-1}\right)+\left(\sqrt{5x-4}-\sqrt{3x-2}\right)>0$$

$$\left|\quad \sqrt{a}-\sqrt{b}=\frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}}=\frac{a-b}{\sqrt{a}+\sqrt{b}}\quad\right|$$

$$\frac{(4x-3)-(2x-1)}{\sqrt{4x-3}+\sqrt{2x-1}}+\frac{(5x-4)-(3x-2)}{\sqrt{5x-4}+\sqrt{3x-2}}>0$$

$$2(x-1)\left(\frac{1}{\sqrt{4x-3}+\sqrt{2x-1}}+\frac{1}{\sqrt{5x-4}+\sqrt{3x-2}}\right)>0.$$

As the expression in parentheses is positive, so the last inequality is satisfied for $x-1>0.$ Each solution $x>1$ satisfies the assumption $x\geq\frac{4}{5}$. Finally we obtain $x\in(1,\infty)$.

5. At what temperature, the veocity distribution function for the oxygen molecules will have maximum value at the speed $400\,\frac{m}{s}$?

Solution

The maximum speed for any gas occurs when it is at most probable temerature $v_{mp}=\sqrt{\frac{2RT}{m}}$, where R is the gas constant, T is the absolute temperature, m is the molar mass of the gass. (Maxwell-Boltzman Distribution).

Given $v_{mp}=400\frac{m}{s}$, $m=32\cdot 10^{-3}kg$ for 1 mole of oxygen molecules, $R=8{,}31$.
Then

$$v_{mp}^2=\frac{2RT}{m},$$

$$T=\frac{m\cdot v_{mp}^2}{2R}$$

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$$T = \frac{400^2 \cdot 32 \cdot 10^{-3}}{2 \cdot 8{,}31} \approx 308°\text{C}.$$

6. The temperature of the gas is raised from $27°\text{C}$ to $927°\text{C}$. What is the root mean square velocity?

Solution

The **root-mean-square velocity** is the measure of the of particles in a gas, defined as the **square root** of the **average velocity-squared** of the molecules in a gas. The **root-mean-square velocity** takes into account both molecular weight and temperature, two factors that directly affect the kinetic energy of a material.

T in Kelvin= °C + 273

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

Change °C into Kelvin:

$$T_1 = 27°\text{C} + 273 = 300$$
$$T_2 = 927°\text{C} + 273 = 1200$$
$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \quad => \quad v_2 = v_1\sqrt{\frac{T_2}{T_1}}$$

and $v_2 = v_1\sqrt{\frac{1200}{300}} = 2v_1$

therefore root mean square velocity will be doubled.

7. Boat builders share an old rule of thumb for sailboats. The maximum speed K in knots is 1.35 times the square root of length L in feet of the boat's waterline. A customer is planning to order a sailboat with a maximum speed of 8 knots. How long should the waterline be?

Solution

The *knot* (kn) is *a* unit of speed equal to *one* nautical mile per hour, exactly 1.852 km/h The feet (ft') is a unit of length in the British imperial and United State customary systems of measurement, exactly $0{,}3048$ m.

$$K = 1.35\sqrt{L}$$
$$8 = 1.35\sqrt{L}$$
$$\sqrt{L} = \frac{8}{1.35} \approx 5.926 \quad => \quad L = 35.12$$

The waterline should be 35.12 feet long.

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