5.6.1. Types, graphs, important limits

Trigonometric functions are nowadays used in almost every branch of applied science, from optics, acoustics, electricity to maritime problems (calculating the height of waves and tides) and of course astronomy. Because of their relation to right angle geometry, trigonometry was reseach extensively even in ancient Greece. The first trigonometric tables were complied by Hipparchus of Niecaea (180-125 BC), who is now known as „the father of trigonometry“. The discovery and rapid development of trigonometry was closely related to the discoveries in marine navigation.

To understand trigonometry, it is first crutial to understand the link between the degree and radian measure of an angle.

Definition: Radian

A radian, denoted rad is a unit for measuring angles. It represents the length of an arc on a unit circle corresponding to an angle formed between the point on the unit circle and the positive part of the x axis.

Example 5.43 The following sketch illustrates the definition:

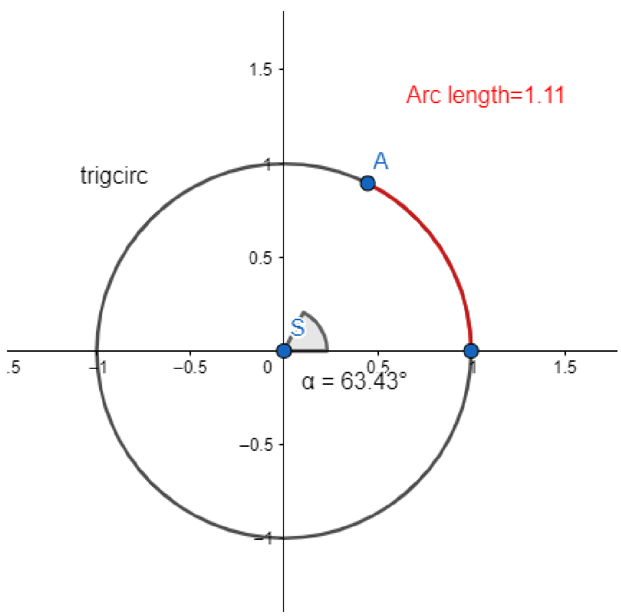

Figure 5.45

The arc length in red is the radian measure corresponding to the angle formed by the positive ray of the x axis, the centre of the circle S, and the point A.

The link below can be used to determine the radian measure of angles less than 360°

https://www.geogebra.org/calculator/mk7cqdxx

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Note that for 360° the radian measure is equal to the **circumference** of the circle, so the radian measure of **360°** is $\boldsymbol{2\pi}$.

Definition: Coterminal angles

For an angle greater than 360° or less than 0°, the **coterminal angle** is an angle in standard form between 0° and 360°. The coterminal angle can be calculated using the following formula:

$$coterminal(\alpha) = \alpha - \left\lfloor \frac{\alpha}{360°} \right\rfloor \cdot 360°$$

where $\lfloor \cdot \rfloor$ is the floor function.

Example 5.44 Determining the coterminal angle

$$coterminal(730°) = 730° - \mathbf{2} \cdot 360° = 730° - 720° = 10°$$

The number **2** is obtained by dividing 730° by 360° and rounding to the nearest **lower** integer.

$$coterminal(1081°) = 1081° - \mathbf{3} \cdot 360° = 1081° - 1080° = 1°$$

The number **3** is obtained by dividing 1080° by 360° and rounding to the nearest **lower** integer.

$$coterminal(-120°) = -120° - (-\mathbf{1}) \cdot 360° = -120° + 360° = 240°$$

The number **(-1)** is obtained by dividing -120° by 360° and rounding to the nearest **lower** integer.

If the angle is greater than 360°, then the radian measure of that angle can be calculated using the coterminal angle.

Example 5.45 Conversion between radians and degrees

For a central angle of 360°, we know that the corresponding radian measure is 2π, because the radian measure is the **circumference.**

For a central angle of 180° (half of 360°) the radian measure would be half of the circumference, so π.

Note that a central angle of 120° would close an arc of exactly 1/3 of the total circumference, so the values change **proportionally**. The arc length in relation to the circumference is equal to the angle measure in degrees in relation to 360°.

The following ratio can be used to determine the radian and degree measure of an angle:

$$\alpha° : 360° = \alpha_{rad} : 2\pi$$

or in fractional notation as:

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$$\frac{\alpha^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Example 5.46 Determining the radian measure

$$\alpha^\circ = 45^\circ, \alpha_{rad} = ?$$

Using the equation provided above:

$$\frac{\alpha^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Substituting $\alpha^\circ = 45^\circ$ into the formula:

$$\frac{45^\circ}{360^\circ} = \frac{\alpha_{rad}}{2\pi}$$

Reducing the left side by 45°:

$$\frac{1}{8} = \frac{\alpha_{rad}}{2\pi}$$

Cross-multiplying:

$$2\pi = 8\,\alpha_{rad}$$

Solving:

$$\alpha_{rad} = \frac{2\pi}{8} = \frac{\pi}{4}$$

We defined the radian measure of an angle using the **unit circle** (radius equal to 1).

Each point on the circle also has an x and y coordinate.

One such illustrative point is shown in the picture below.

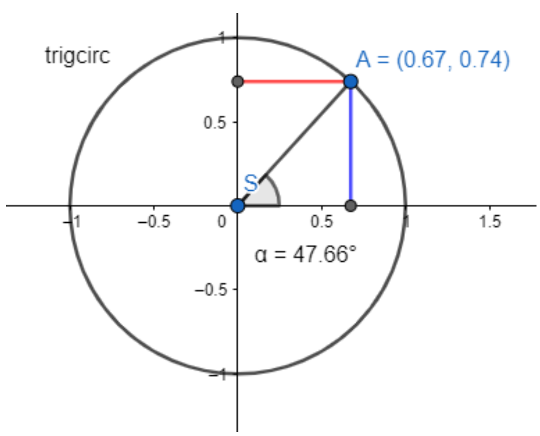

Figure 5.46

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The x coordinate, shown in red, can be read using the x axis.

The y coordinate, shown in blue, can be read using the y axis.

Let us also note that the length of the line segment AS is equal to the radius, so $|AS| = 1$ because the circle is a unit circle.

Definition: *The sine function*

The **sine function** associates each central angle to the y coordinate of the corresponding point on a unit circle.

For example, $\sin(0°) = 0$, because for 0° the corresponding point on the unit circle is (1,0). The y coordinate of that point is 0.

Since we can determine the radian measure for each degree measure, the argument of the sine function can also be a **radian measure**.

Example 5.47 Determining the sine value of angles

Let us determine the value of $\sin(\frac{\pi}{2})$.

As an exercise, try to prove that the radian measure of $\frac{\pi}{2}$ is equal to an central angle of $90°$. Now, we can determine the point on the trigonometric circle for which the corresponding angle is 90°:

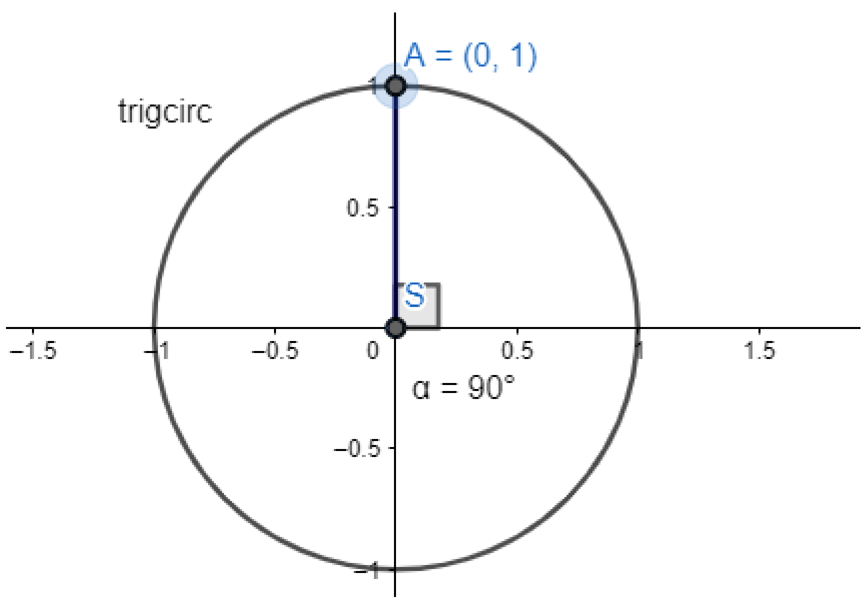

Figure 5.47

The y coordinate of the point represents the sine value, so $\sin\left(\frac{\pi}{2}\right) = 1.$

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Definition: *The cosine function*

The **cosine function** associates each central angle to the x coordinate of the corresponding point on a unit circle.

Example 5.48 Determining the cosine values of angles

Let us determine the value of $\cos(\frac{\pi}{2})$.

Since the radian measure of $\frac{\pi}{2}$ is equal to $90°$, the corresponding point is given in the figure from the previous example.

The x coordinate of the point represents the cosine value, so $\cos\left(\frac{\pi}{2}\right) = 1.$

Definition: *The tangent function*

For every central angle, the tangent function is the quotient of the sine and cosine functions of that angle, $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$.

Definition: The cotangent function

For every central angle, the cotangent function is the reciprocal of the tangent function, $\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{1}{\frac{\sin(\alpha)}{\cos(\alpha)}} = \frac{\cos(\alpha)}{\sin(\alpha)}$.

Trigonometric functions were historically discovered and defined using the right triangle.

In a right triangle, the trigonometric functions are defined as the following ratio:

$$\sin(\text{angle}) = \frac{\text{length of the opposite leg}}{\text{length of the hypotenuse}}.$$

$$\cos(\text{angle}) = \frac{\text{length of the adjacent leg}}{\text{length of the hypotenuse}}.$$

$$\tan(\text{angle}) = \frac{\text{length of the opposite leg}}{\text{length of the adjacent leg}}.$$

Example 5.49 Determining the sine and cosine values in a right triangle

Let us try to demonstrate that these definitions fit in with the unit circle definitions given above. The image below represents a part of the unit circle, with a point A chosen.

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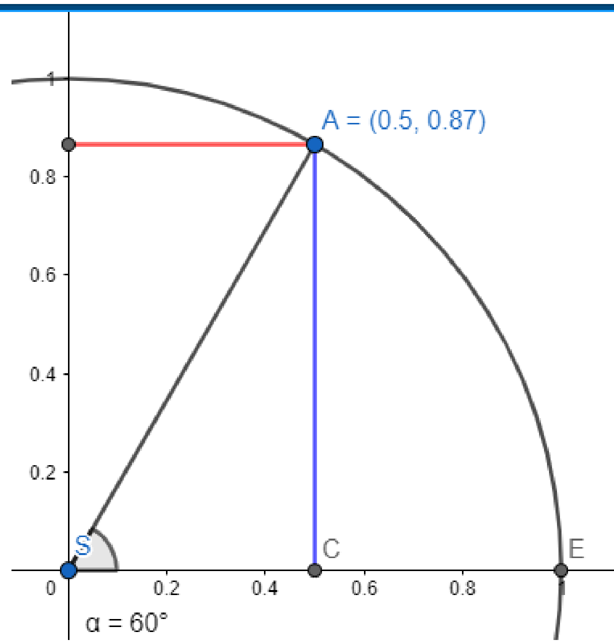

Figure 5.48

Firstly, note that the radius of the circle is 1, so $|SA| = |SE| = 1$.

Second, the triangle SAC is a right triangle, the right angle being in the vertex C.

It is easy to prove now that the triangle ASE is equilateral.

Therefore, the altitude AC bisects the side SE, so $|SC| = \frac{1}{2}$.

Using the right triangle definition, we can calculate the cosine of the angle in vertex A:

$$\cos(\alpha) = \frac{\text{length of the adjacent leg}}{\text{length of the hypotenuse}} = \frac{|SC|}{|SA|} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

By inspecting the $\mathbf{x}$ coordinate of point A, we see that the results are the same. Let us now try to calculate $\sin(\alpha)$. Using the Pythagorean theorem, we can calculate the remaining side, that is the altitude length from point A.

$$1^2 = \left(\frac{1}{2}\right)^2 + a^2, \qquad \text{where } a \text{ is the altitude length}$$

The expression is simplified into:

$$a^2 = \frac{3}{4}$$

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Since the side length cannot be negative, by applying the square root the solution is:

$$a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

Therefore, $|SA| = \frac{\sqrt{3}}{2}$.

Using the right triangle definition, we can now calculate the sine of the angle in vertex A:

$$\cos(\alpha) = \frac{\text{length of the opposite leg}}{\text{length of the hypotenuse}} = \frac{|SA|}{|SA|} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

By inspecting the y coordinate of point A, we see that the value given in the figure is 0.87. This is because the value is only a two decimal approximation of the true value of the point. Hence, $\frac{\sqrt{3}}{2} \approx 0.87$.

Finally, note that the point corresponding to the angle greater than 360° is equivalent to the point represented by the coterminal angle.

This fact allows us to calculate the sine and cosine value of **every** angle, by converting it to a coterminal angle, and then calculating the value of the coterminal angle.

For instance:

$$\cos(450°) = \cos(360° + 90°) = [\text{coterminal angle of } 450° \text{ is } 90°] = \cos(90°) = 0\,.$$

The sine and cosine values of specific points are shown in the following figure:

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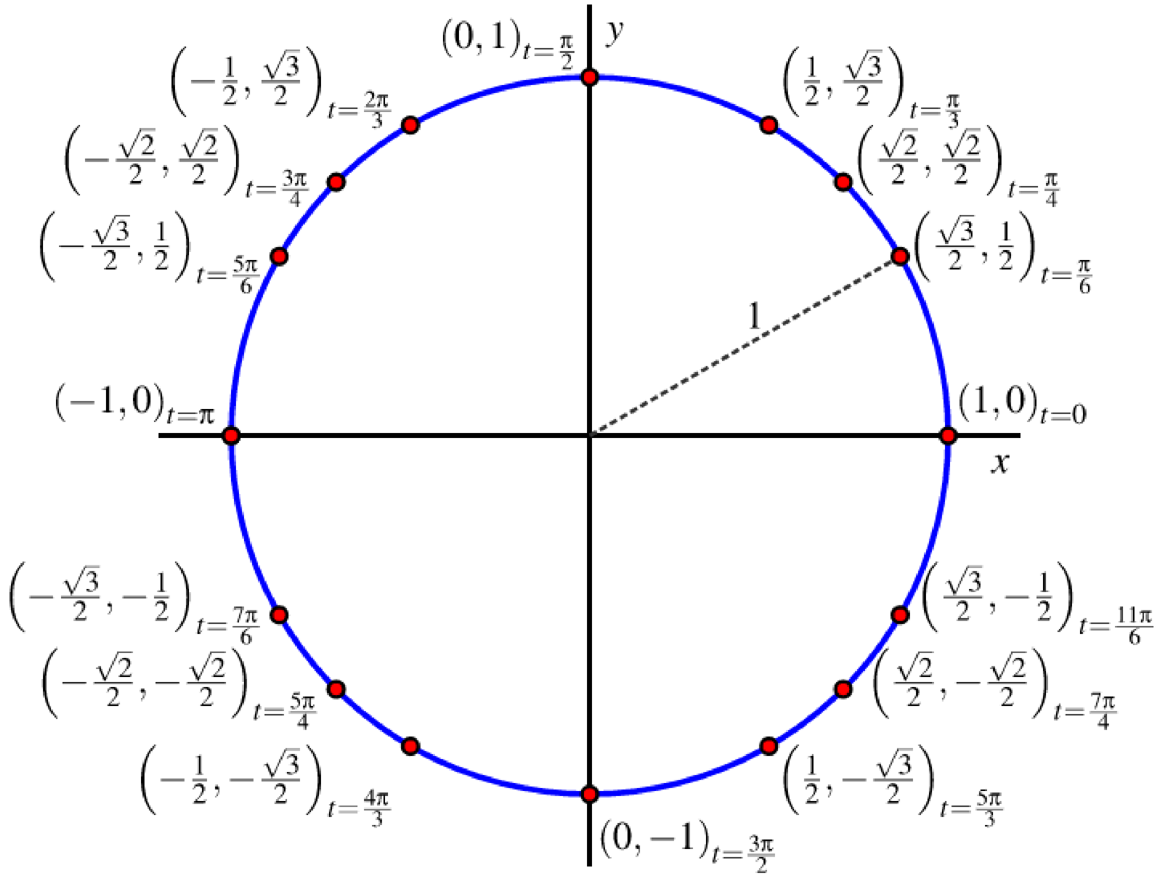

Figure 5.49

Let us rewrite some of the most important values in a table:

α	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin(\alpha)$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
$\cos(\alpha)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
$\tan(\alpha)$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$UNDEFINED$	0

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Example 5.50 Trigonometric functions in the xy-plane

Since the sine and cosine functions are defined for every angle, and therefore for every radian measure, it is possible to calculate their values for any number.

In the following link, the graphs of the sine and cosine functions are determined and drawn for x values in the interval $[0,12.5]$.

https://www.geogebra.org/m/sfzxqfzp

To see the animation, click on the Play button on the Num slider, or move it manually left/right.

The tangent function can also be shown in green. To show/hide the tangent function click on the green TAN point in the left side panel (scroll down, it is the last point in the panel).

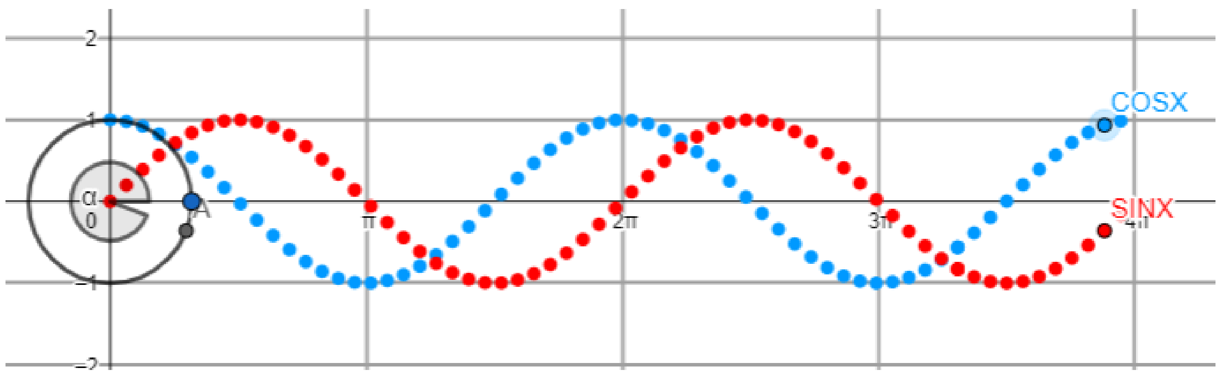

Figure 5.50

We see from the graph that the sine and cosine functions are wave-like functions.

The maximum value of both functions is 1, and the minimum value is -1.

The wave-like property is called **periodicity**, meaning that the sine and cosine functions repeat after a certain number.

In the above graph, it is easy to see that that number is 2π, so the sine and cosine function have a **period of 2π**.

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