### 5.6.2. Equations

#### *Basic equations*

Suppose we want to know for which angles the sine function is equal to 0.5, one would write the equation $\sin(x) = 0.5$.

This represents a trigonometric equation.

There are two ways to solve a trigonometric equation, algebraically and graphically. First, let us demonstrate the algebraic way:

$$\sin(x) = 0.5$$

When calculating the angle, the **inverse trigonometric functions** or **arcus functions** are used. Without going into too much detail, an arcus function is also called an inverse because it eliminates the trigonometric term on the left hand side. We will apply the inverse sine function ( $arcsin$, or $\sin^{-1}$) on both sides of the equation:

$$\arcsin(\sin(x)) = \arcsin(0.5)$$

The inverse trigonometric function cancels out the trigonometric function, so the equation is now:

$$x = \arcsin(0.5)$$

The value of $\arcsin(0.5)$ can be calculated using a calculator, which will provide the solution:

$$x = 30°$$

But is this the only solution?

Let us next examine the graphical method:

We will graph each side of the equation separately and look for intersections of the graphs.

Let us denote $f(x) = \sin(x)$ and $g(x) = 0.5$

Using Geogebra, we can easily plot the functions, which results in the following graph:

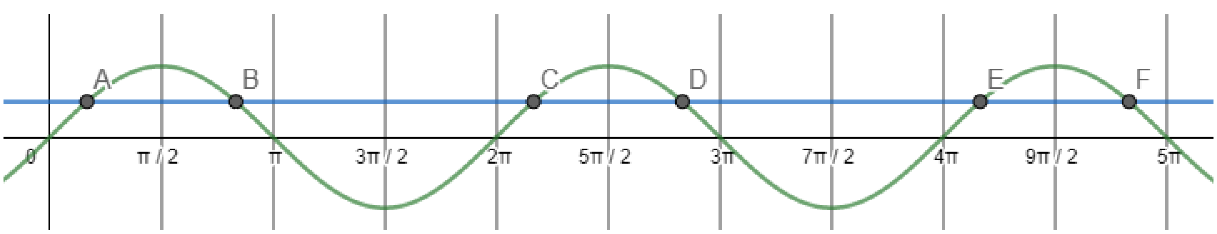


*Figure 5.59*

By observing the graph, we see that there are at least 6 different intersection points between the graphs, so there must be at least 6 solutions of the trigonometric equation.


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Furthermore, we see that the solutions of the equation are repeating periodically in every sine "wave".

Let us also recall that the sine of a number is the $y$ coordinate on the point on the unit circle.

Using the trigonometric unit circle, we can plot the line $y = 0.5$ and obtain all points that intersect the circle, like in the picture below:

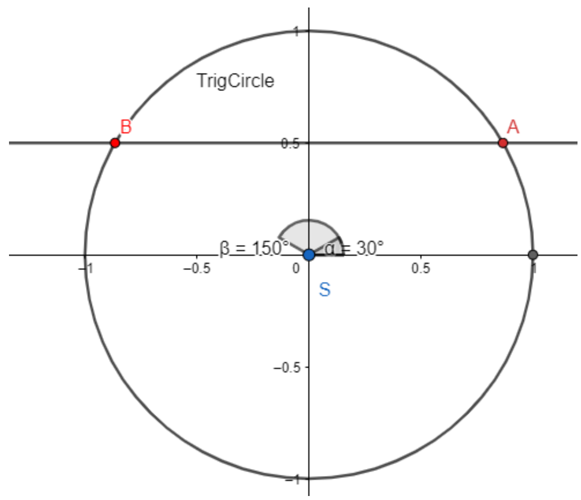


*Figure 5.60*

Hence, we see that the equation $\sin(x) = 0.5$ has two solutions in the interval from $0$ to $2\pi$. One solution is our calculated value $x = 30°$, but another solution is $x = 150°$. All other solutions repeat periodically, with each new "wave" of the sine function. So, the solutions of the equation are also $x = 390°, x = 510°, \text{etc.}$
All solutions of the equation can be obtained by adding a multiple of $360°$ to the two calculated solution ($30°$ and $150°$).

The general solutions can be written as $x = 30° + k \cdot 360°$, or $x = 150° + k \cdot 360°$, where $k$ represents any integer.

The solutions of trigonometric equations are generally preferred in radians. We leave the conversion as an exercise for the students, and write the solutions in radian form:

$$x_1 = \frac{\pi}{6} + 2k\pi, \quad x_2 = \frac{5\pi}{6} + 2k\pi$$

The procedure for solving simple trigonometric equations can be summarized as follows:

1) Find the first value by calculating the arcus function
2) Graph the trigonometric circle and read the second solution or
   Use the following identities:

$$\sin(\pi - x) = \sin(x)$$
$$\cos(2\pi - x) = \cos(x)$$
$$\tan(\pi + x) = \tan(x)$$



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to obtain the second solution

3) Write all remaining solutions using the period of sine/cosine ($2\pi$) or tangent ($\pi$).

## *Quadratic trigonometric equations:*

Let us briefly explain how to solve the quadratic trigonometric equations in the following form:

$$2\,(\sin(x))^2 + 3\sin(x) - 2 = 0$$

When solving trigonometric equations including quadratic terms, first try substituting the term with $t$In this case, $t = \sin(x)$.

The equation is then transformed into:

$$2t^2 + 3t - 2 = 0$$

We can now solve this quadratic equation by factoring:

$$2\left(t - \frac{1}{2}\right)(t + 2) = 0$$

The solutions of the equation above are:

$$t_1 = \frac{1}{2}, t_2 = -2$$

Returning the substituted value, we obtain:

$$\sin(x) = \frac{1}{2}, \qquad \sin(x) = -2$$

Since the sine function values are always between -1 and 1, the equation $\sin(x) = -2$ has no solution.

Therefore, the solutions of the starting equation are solutions of the equation $\sin(x) = \frac{1}{2}$. This is a simple trigonometric equation solved above.

The final solutions are $x_1 = \frac{\pi}{6} + 2k\pi, \quad x_2 = \frac{5\pi}{6} + 2k\pi, k \in Z$.



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## *Complex trigonometric equations:*

Some trigonometric equations cannot be solved algebraically, but the number of solutions can be obtained graphically.

One such equation is $\sin(x) = 0.5x$

To count the number of solutions, graph each side of the equation above separately using Geogebra:

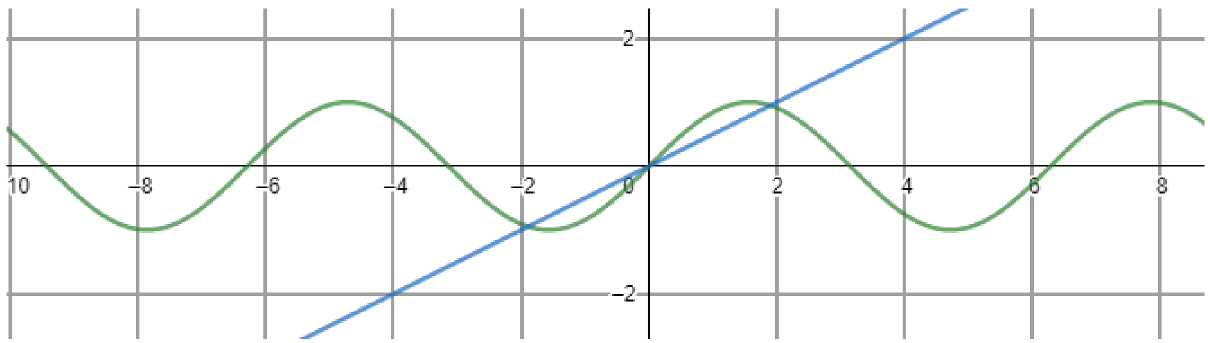

*Figure 5.61*

The left side is the sine function represented by the green graph, and the right side is the blue line.

From the graph it is obvious that there are 3 solutions of the equation.

Furthermore, in Geogebra, we can find the solutions by using the Intersection tool from the left side panel. This procedure gives us the approximate coordinates of the solutions.

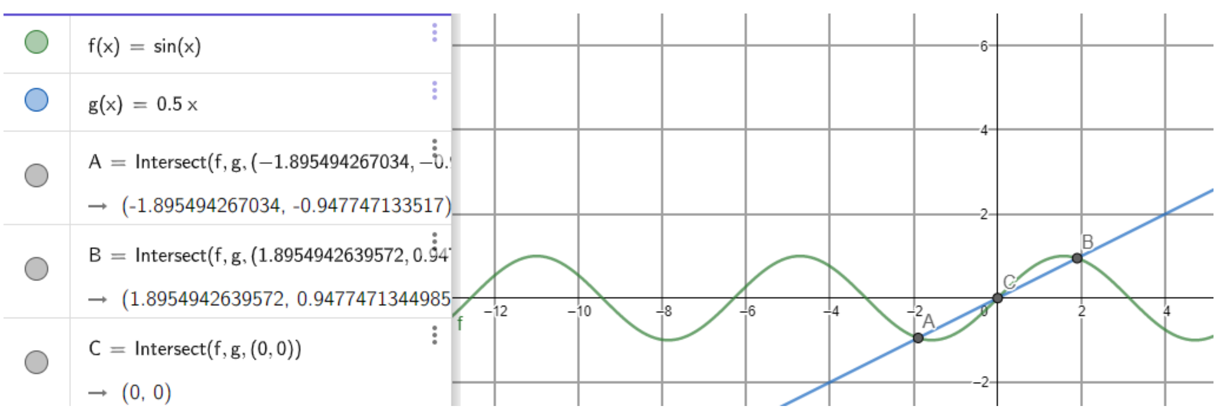


*Figure 5.62*

Task 5.35 Calculate all solutions of the following equations:

a) $\cos(x) = -\frac{\sqrt{3}}{2}$
b) $\sin(x) = 0$
c) $\cos(x) = -1$
d) $\tan(x) = 1$

Solution:

a) $x_1 = \frac{5\pi}{6} + 2k\pi,\ x_2 = \frac{7\pi}{6} + 2k\pi, k \in Z$
b) $x_1 = 0 + 2k\pi, x_2 = \pi + 2k\pi, k \in Z$
c) $x = \pi + 2k\pi, k \in Z$
d) $x = \frac{\pi}{4} + k\pi,\ k \in Z$

Task 5.36 .

Calculate all solutions of the following equations:

a) $\sin(x)^2 - 7x + 3 = 0$
b) $(\cos(x) - 5)(\cos(x) + 1) = 0$
c) $(\tan(x) - 1)(\tan(x)) = 0$

Solution:

a) $x_1 = \frac{5\pi}{6} + 2k\pi,\ x_2 = \frac{7\pi}{6} + 2k\pi, k \in Z$
b) $x_1 = 0 + 2k\pi, x_2 = \pi + 2k\pi, k \in Z$
c) $x = \pi + 2k\pi, k \in Z$

Task 5.37 Graph the equation, and determine the number of solutions:

a) $\sin(x) = x - 3$
b) $\cos(\mathrm{x}) = \mathrm{x}^2$
c) $\sin(x) = \frac{x}{12}$

Solution:

a) 1 solution
b) 2 solutions
c) 6 solutions



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### 5.6.3. Inequalities

First, remember that the solution set of an inequality is almost always an interval. Let us demonstrate the technique using the following inequality:

$$\sin(x) \geq \frac{1}{2}$$

The first step in solving trigonometric inequality is to solve the corresponding equation:

$$\sin(x) = \frac{1}{2}$$

The solving process of that equation was shown in the previous chapter.

The two found solutions were $x_1 = \frac{\pi}{6} + 2k\pi$, and $x_2 = \frac{5\pi}{6} + 2k\pi,\ k \in Z$. To find all the remaining points which satisfy the inequality, we draw the trigonometric circle, and find all points such that the $y$ coordinate is greater than 0.5.

The red interval in the figure below represents the solution set:

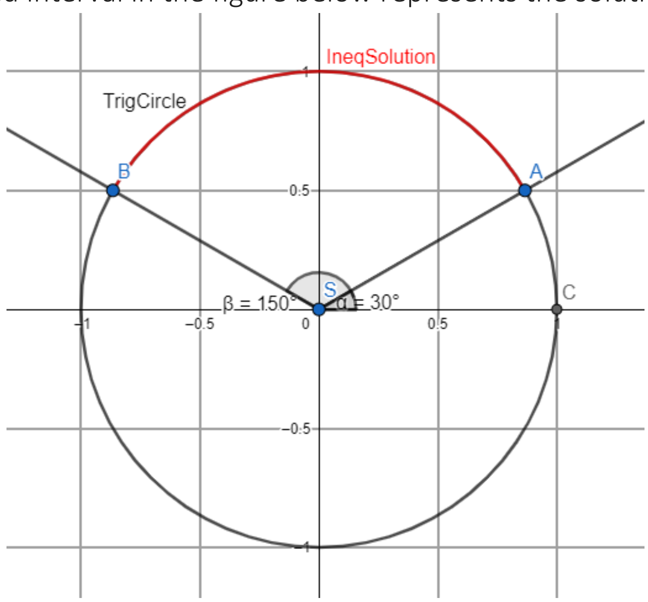


*Figure 5.63*

From the figure, we can see that all points between $\frac{\pi}{6}$, and $\frac{5\pi}{6}$ are included in the interval, including the endpoints. All the other solutions set are periodical in nature, repeating after $2\pi$. Therefore, the solution of the inequality is:

$$x \in \left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right], k \in Z$$

Trigonometric equations and inequalities appear in many everyday tasks, some of which we will demonstrate in the following section.



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