5.7. INVERSE TRIGONOMETRIC FUNCTIONS

DETAILED DESCRIPTION:

Inverse trigonometric functions are used in solving trigonometric equations that arise in finding the angles and sides of triangle. The inverse of any function is important - it provides a way to "get back."

AIM:

The students will learn how to interpret and graph an inverse trig. Function and will also learn to solve for an equation with an inverse function.

Learning Outcomes:

1. Understand and use the inverse sine function.
2. Understand and use the inverse cosine function.
3. Understand and use the inverse tangent function.
4. Use a calculator to evaluate inverse trigonometric functions.
5. Find exact values of composite functions with inverse trigonometric functions

Prior knowledge:

If no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function.

If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of the inverse function, denoted f^{-1}. The graph of f^{-1} is a reflection of the graph of about the line y = x.

Relationship to real maritime problems:

Contents:

The Inverse Sine Function

The Inverse Cosine Function

The Inverse Tangent Function

Composition of Functions Involving Inverse Trigonometric Functions

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5.7.1. The inverse sine function

Figure 5.69 shows the graph of $y = \sin x$. Can we see that every horizontal line that can be drawn between -1 and 1 intersects the graph infinitely many times? Thus, the sine function is not one-to-one and has no inverse function.

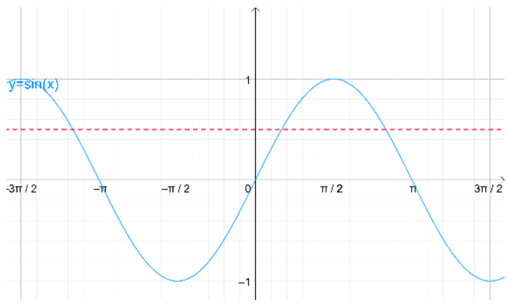

Figure 5.69 The horizontal line test shows that the sine function is not one-to-one and has no inverse function.

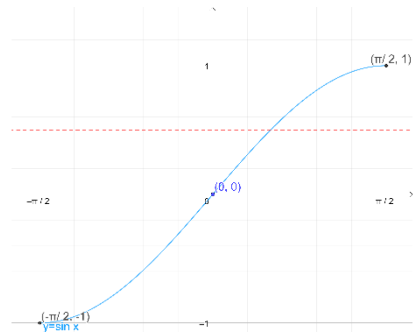

Figure 5.70 The restricted sine function passes the horizontal line test. It is one-to-one and has an inverse function.

In ***Figure 5.70***, we have taken a portion of the sine curve, restricting the domain of the sine function to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. With this restricted domain, every horizontal line that can be drawn between -1 and 1 intersects the graph exactly once. Thus, the restricted function passes the horizontal line test and is one-to-one.

On the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, y = \sin x$ has an inverse function.
The inverse of the restricted sine function is called the inverse sine function. Two notations are commonly used to denote the inverse sine function:

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

We will use $y = \arcsin x$.

Definition:

The inverse sine function, denoted by $\arcsin x$, is the inverse of the restricted sine function $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Thus,

$$y = \arcsin x \quad \text{means} \quad \sin y = x,$$

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-1 \leq x \leq 1$. We read $y = \arcsin x$ as "y equals the inverse sine at x."

One way to graph $y = \arcsin x$ is to take points on the graph of the restricted sine function and reverse the order of the coordinates. For example ***Figure 5.71*** shows that $\left(-\frac{\pi}{2}; -1\right)$, $(0; 0)$ and $\left(\frac{\pi}{2}; 1\right)$ are on the graph of the restricted sine function. Reversing the order of the cordinates gives $\left(1; -\frac{\pi}{2}\right)$,

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$(0;0)$ and $\left(1;\frac{\pi}{2}\right)$. We now use these three point to sketch the inverse sine function. The graph of $y = \arcsin x$ is shown in ***Figure 5.72***.

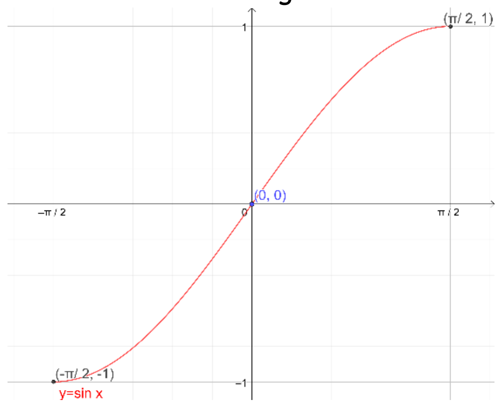

Figure 5.71 The restricted sine function, Domain: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$*, Range:* $[-\mathbf{1},\mathbf{1}]$

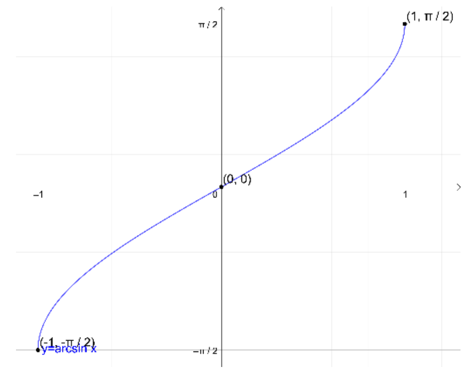

Figure 5.72 The graph of the inverse sine function, Domain $[-\mathbf{1},\mathbf{1}]$*, Range:* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

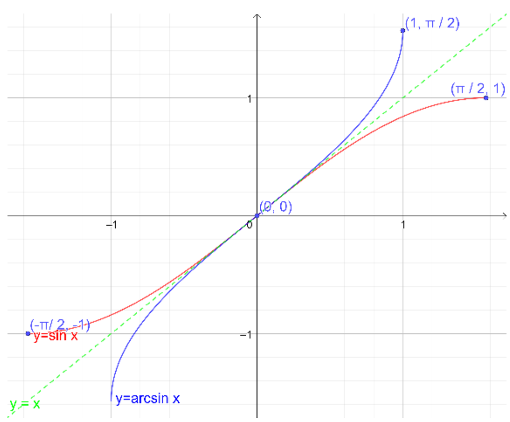

Figure 5.73 Using a reflection to obtain the graph of the inverse sine function.

Another way to obtain the graph o $y = \arcsin x$f is to reflect the graph of the restricted sine function about the line $y = x$, shown in Figure 5.The red graph is the restricted sine function and the blue graph is the graph of $y = \arcsin x.$ Exact values of $\arcsin x$ can be found by thinking of $\arcsin x$ as the angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is x. For example, we can use the two points on the blue graph of the inverse sine function in Figure 5.73 to write

$$\arcsin(-1) = -\frac{\pi}{2};\ \arcsin(1) = \frac{\pi}{2}.$$

Because we are thinking of $\arcsin x$ in terms of an angle, we will represent such an angle by φ.

FINDING EXACT VALUES OF $\boldsymbol{ARCSIN\ x}$

1. Let $\mathbf{arcsin}\,\boldsymbol{x} = \boldsymbol{\varphi}.$
2. Rewrite $\mathbf{arcsin}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{sin}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\boldsymbol{\varphi}\boldsymbol{\epsilon}\left[-\frac{\boldsymbol{\pi}}{\mathbf{2}};\frac{\boldsymbol{\pi}}{\mathbf{2}}\right]$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\boldsymbol{\pi}}{\mathbf{2}};\frac{\boldsymbol{\pi}}{\mathbf{2}}\right]$ that satisfies $\mathbf{sin}\,\boldsymbol{\varphi} = \boldsymbol{x}.$

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$\boldsymbol{\varphi}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\mathbf{sin}\,\boldsymbol{\varphi}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1 Exact values for $\sin\varphi$, $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$

Example 5.51 Finding the exact value of an inverse sine function

Find the exact value of $\arcsin\frac{\sqrt{2}}{2}$.

Solution:

1. Let $\boldsymbol{\arcsin x=\varphi}$. Thus, $\arcsin\frac{\sqrt{2}}{2}=\varphi$. We must find the angle $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, whose sine equals $\frac{\sqrt{2}}{2}$.
2. Rewrite $\boldsymbol{\arcsin x=\varphi}$ as $\boldsymbol{\sin\varphi=x}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Using the definition of the inverse sine function, we rewrite $\arcsin\frac{\sqrt{2}}{2}=\varphi$ as $\sin\varphi=\frac{\sqrt{2}}{2}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\boldsymbol{\sin\varphi=x}$. Table 1 shows that the only angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\varphi=\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Thus, $\varphi=\frac{\pi}{4}$. Because φ in step 1, represents $\arcsin\frac{\sqrt{2}}{2}$, we conclude that
$$\arcsin\frac{\sqrt{2}}{2}=\frac{\pi}{4}.$$

Example 5.52 Finding the exact value of an inverse sine function

Find the exact value of $\arcsin\left(-\frac{1}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arcsin x=\varphi}$. Thus, $\arcsin\left(-\frac{1}{2}\right)=\varphi$. We must find the angle $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, whose sine equals $\left(-\frac{1}{2}\right)$.
2. Rewrite $\boldsymbol{\arcsin x=\varphi}$ as $\boldsymbol{\sin\varphi=x}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Using the definition of the inverse sine function, we rewrite $\arcsin\left(-\frac{1}{2}\right)=\varphi$ as $\sin\varphi=-\frac{1}{2}$, where $\varphi\epsilon\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.
3. Use the exact values in Table 1 to find the value of $\boldsymbol{\varphi}$ in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\boldsymbol{\sin\varphi=x}$. Table 1 shows that the only angle in the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ that satisfies $\sin\varphi=-\frac{1}{2}$ is $\left(-\frac{\pi}{6}\right)$. Thus, $\varphi=-\frac{\pi}{6}$. Because φ in step 1, represents $\arcsin\left(-\frac{1}{2}\right)$, we conclude that
$$\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}.$$

NB! Some inverse sine expressions cannot be evaluated. Because the domain of the inverse sine function is $[-1;1]$ it is only possible to evaluate for values of x in this domain. Thus, $\arcsin(3)$ cannot be evaluated. There is no angle whose sine is 3.

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5.7.2. The inverse cosine function

Figure 5.74 shows how we restrict the domain of the cosine function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $[0;\pi]$, shown by the light green graph. Over this interval, the restricted cosine function passes the horizontal line test and has an inverse function.

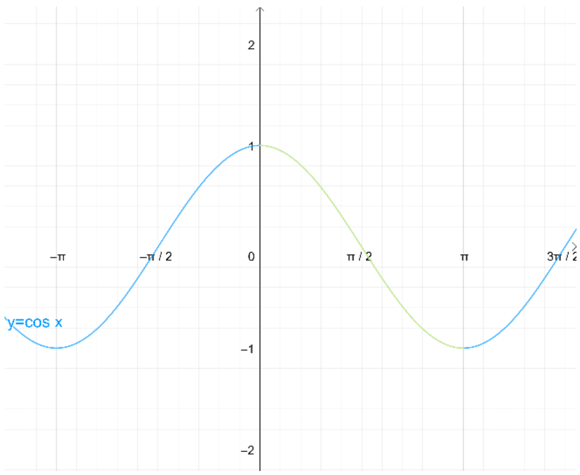

Figure 5.74 y=cos x is one-to one on interval $(\mathbf{0};\ \boldsymbol{\pi})$

Definition: The inverse cosine function, denoted by $\arccos x$, is the inverse of the restricted cosine function $y=\cos x,\ 0\le x\le\pi$. Thus,

$$y=\arccos x \quad \text{means} \quad \cos y = x,$$

where $0\le y\le\pi$ and $-1\le x\le 1$.

One way to graph $y=\arccos x$ is to take points on the graph of the restricted cosine function and reverse the order of the coordinates. For example, *Figure 5.75* shows that $(0,1)$, $\left(\frac{\pi}{2},0\right)$ and $(\pi,-1)$ are on the graph of the restricted cosine function. Reversing the order of the coordinates gives $(1,0)$, $\left(0,\frac{\pi}{2}\right)$ and $(-1,\pi)$.

We now use these three points to sketch the inverse cosine function. The graph of $y=\arccos x$ is shown in *Figure 5.76*. You can also obtain this graph by reflecting the graph of the restricted cosine function about the line $y = x$.

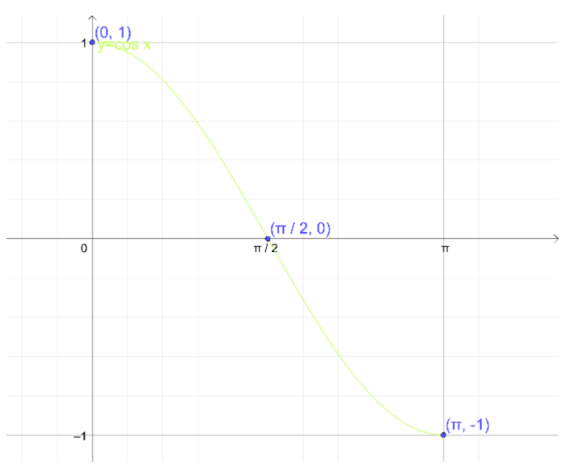

Figure 5.75 The restricted cosine function, Domain: $[\mathbf{0};\ \boldsymbol{\pi}]$*, Range:* $[-\mathbf{1},\mathbf{1}]$.

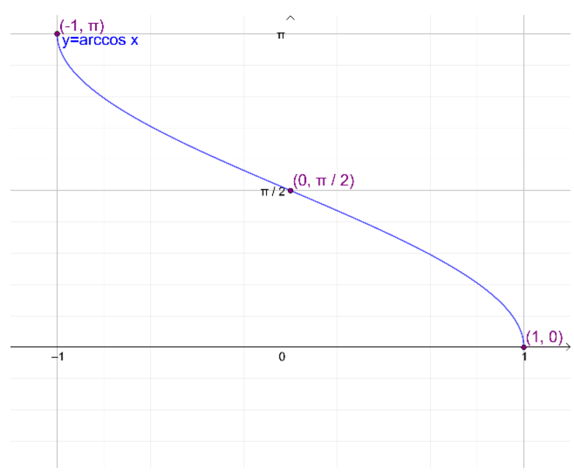

Figure 5.76 The graph of the inverse cosine function

Exact values of $\boldsymbol{y}=\mathbf{arccos}\,\boldsymbol{x}$ can be found by thinking of $\mathbf{arccos}\,\boldsymbol{x}$ as the angle in the interval $[\mathbf{0},\boldsymbol{\pi}]$ whose cosine is $\boldsymbol{x}$.

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FINDING EXACT VALUES OF $\boldsymbol{ARCCOS\,x}$

1. Let $\mathbf{arccos}\,\boldsymbol{x} = \boldsymbol{\varphi}$.
2. Rewrite $\mathbf{arccos}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{cos}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\boldsymbol{\varphi}\boldsymbol{\epsilon}[\mathbf{0};\boldsymbol{\pi}]$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[\mathbf{0};\boldsymbol{\pi}]$ that satisfies $\mathbf{cos}\,\boldsymbol{\varphi} = \boldsymbol{x}$.

$\boldsymbol{\varphi}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\mathbf{cos}\,\boldsymbol{\varphi}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 2 Exact values for $\mathbf{cos}\,\boldsymbol{\varphi}$*,* $\boldsymbol{\varphi}\boldsymbol{\epsilon}[\mathbf{0};\boldsymbol{\pi}]$*.*

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Example 5.53 Finding the exact value of an inverse cosine function

Find the exact value of $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arccos x = \varphi}$. Thus, $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \varphi$. We must find the angle $\varphi\epsilon[0;\pi]$, whose cosine equals $-\frac{\sqrt{3}}{2}$.
2. Rewrite $\boldsymbol{\arccos x = \varphi}$ as $\boldsymbol{\cos\varphi = x}$, where $\varphi\epsilon[0;\pi]$. Using the definition of the inverse cosine function, we rewrite $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \varphi$ as $\cos\varphi = -\frac{\sqrt{3}}{2}$, where $\varphi\epsilon[0;\pi]$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[0;\pi]$ that satisfies $\boldsymbol{\cos\varphi = x}$. Table 2 shows that the only angle in the interval $[0;\pi]$ that satisfies $\cos\varphi = -\frac{\sqrt{3}}{2}$ is $\frac{5\pi}{6}$. Thus, $\varphi = \frac{5\pi}{6}$. Because φ in step 1, represents $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, we conclude that

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Example 5.54 Finding the exact value of an inverse cosine function

Find the exact value of $\arccos\left(-\frac{1}{2}\right)$.

Solution:

1. Let $\boldsymbol{\arccos x = \varphi}$. Thus, $\arccos\left(-\frac{1}{2}\right) = \varphi$. We must find the angle $\varphi\epsilon[0;\pi]$, whose cosine equals $-\frac{1}{2}$.
2. Rewrite $\boldsymbol{\arccos x = \varphi}$ as $\boldsymbol{\cos\varphi = x}$, where $\varphi\epsilon[0;\pi]$. Using the definition of the inverse cosine function, we rewrite $\arccos\left(-\frac{1}{2}\right) = \varphi$ as $\cos\varphi = -\frac{1}{2}$, where $\varphi\epsilon[0;\pi]$.
3. Use the exact values in Table 2 to find the value of $\boldsymbol{\varphi}$ in $[0;\pi]$ that satisfies $\boldsymbol{\cos\varphi = x}$. Table 2 shows that the only angle in the interval $[0;\pi]$ that satisfies $\cos\varphi = -\frac{1}{2}$ is $\frac{2\pi}{3}$. Thus, $\varphi = \frac{2\pi}{3}$. Because φ in step 1, represents $\arccos\left(-\frac{1}{2}\right)$, we conclude that

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

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5.7.3. The inverse tangent function

Figure 5.77 shows how we restrict the domain of the tangent function so that it becomes one-to-one and has an inverse function. Restrict the domain to the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ shown by the solid blue graph. Over this interval, the restricted tangent function passes the horizontal line test and has an inverse function.

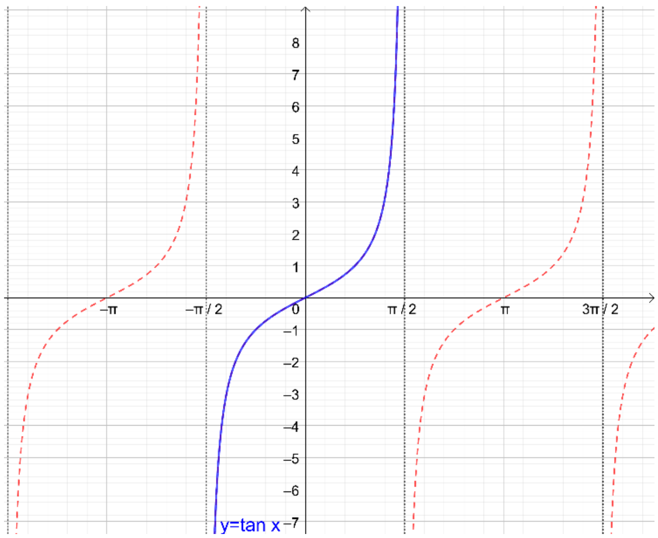

Figure 5.77 $\boldsymbol{y = \tan x}$ *is one-to-one on the interval* $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

Definition:

The inverse sine function, denoted by $\arctan x$, is the inverse of the restricted tangent function $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Thus,

$$y = \arctan x \quad \text{means} \quad \tan y = x,$$

where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $-\infty < x < \infty$. We read $y = \arctan x$ as "y equals the inverse tangent at x."

We graph $y = \arctan x$ by taking points on the graph of the restricted function and reversing the order of the coordinates. *Figure 5.78* shows that $\left(-\frac{\pi}{4};-1\right)$, $(0,0)$, $\left(\frac{\pi}{4};1\right)$ and are on the graph of the restricted tangent function. Reversing the order gives $\left(-1,-\frac{\pi}{4}\right)$, $(0,0)$ and $\left(1,\frac{\pi}{4}\right)$. We now use these three points to graph the inverse tangent function. The graph of $y = \arctan x$ is shown in *Figure 5.79*. Notice that the vertical asymptotes become horizontal asymptotes for the graph of the inverse function.

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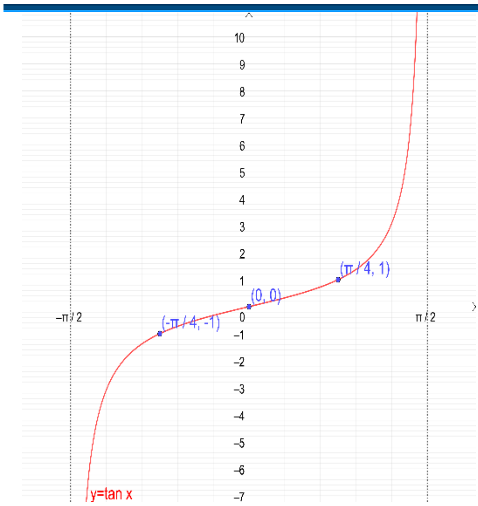

Figure 5.78 The restricted tangent function, Domain: $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$*, Range:* $(-\infty;\infty)$.

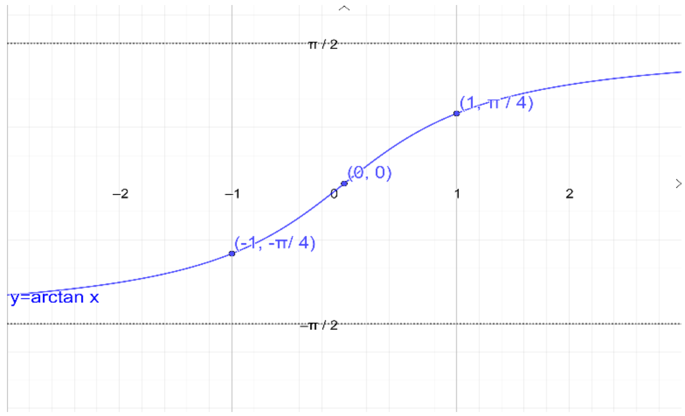

Figure 5.79 The graph of the inverse tangent function Domain: $(-\infty;\infty)$*, Range:* $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

Exact values of $y=\arctan x$ can be found by thinking of $\arctan x$ as **the angle in the interval** $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ **whose tangent is** x.

FINDING EXACT VALUES OF $\boldsymbol{ARCTAN\ x}$

1. Let $\arctan x=\varphi$.
2. Rewrite
3. $\arctan x=\varphi$ as $\tan\varphi=x$, where $\varphi\epsilon\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.
4. Use the exact values in Table 2 to find the value of φ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi=x$.

φ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan\varphi$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

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Example 5.55 Finding the exact value of an inverse tangent function	*Example 5.56 Finding the exact value of an inverse tangent function*
Find the exact value of $\arctan(-\sqrt{3})$. **Solution:** **1.** Let $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$. Thus, $\arctan(-\sqrt{3}) = \boldsymbol{\varphi}$. We must find the angle $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose tangent equals $-\sqrt{3}$. **2.** Rewrite $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. Using the definition of the inverse tangent function, we rewrite $\arctan(-\sqrt{3}) = \varphi$ as $\tan\varphi = -\sqrt{3}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. **3.** Use the exact values in Table 3 to find the value of $\boldsymbol{\varphi}$ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$. Table 3 shows that the only angle in the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi = -\sqrt{3}$ is $-\frac{\pi}{3}$. Thus, $\varphi = -\frac{\pi}{3}$. Because φ in step 1, represents $\arctan(-\sqrt{3})$, we conclude that $$\arctan(-\sqrt{3}) = -\frac{\pi}{3}.$$	Find the exact value of $\arctan(-1)$. **Solution:** **1.** Let $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$. Thus, $\arctan(-1) = \varphi$. We must find the angle $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose cosine equals -1. **2.** Rewrite $\mathbf{arctan}\,\boldsymbol{x} = \boldsymbol{\varphi}$ as $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. Using the definition of the inverse tangent function, we rewrite $\arctan(-1) = \varphi$ as $\tan\varphi = -1$, where $\varphi \epsilon \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. **3.** Use the exact values in Table 3 to find the value of $\boldsymbol{\varphi}$ in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\mathbf{tan}\,\boldsymbol{\varphi} = \boldsymbol{x}$. Table 3 shows that the only angle in the interval $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ that satisfies $\tan\varphi = -1$ is $-\frac{\pi}{4}$. Thus, $\varphi = -\frac{\pi}{4}$. Because φ in step 1, represents $\arctan(-1)$, we conclude that $$\arctan(-1) = -\frac{\pi}{4}.$$

Table 3 Exact values for $\mathbf{tan}\,\boldsymbol{\varphi}$, $\boldsymbol{\varphi \epsilon} \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$.

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5.7.4. COMPOSITION OF FUNCTIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Inverse properties

The Sine Function and Its Inverse	The Cosine Function and Its Inverse	The Tangent Function and Its Inverse
• $\sin(\arcsin x) = x$, for every $x \in [-1,1]$ • $\arcsin(\sin x) = x$, for every $x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$	• $\cos(\arccos x) = x$, for every $x \in [-1,1]$ • $\arccos(\cos x) = x$, for every $x \in [0;\pi]$	• $\tan(\arctan x) = x$, for every $x \in [-\infty,\infty]$ • $\arctan(\tan x) = x$, for every $x \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$

The restrictions on in the inverse properties are a bit tricky. For example,

$$\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

We know that $\arcsin(\sin x) = x$, for every $x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Observe that $\frac{\pi}{4}$ is in interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. But we cannot use $\arcsin(\sin x) = x$ to find the exact value of $\arcsin\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}$, because $\frac{5\pi}{4}$ is not in interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. Thus, to evaluate $\arcsin\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}$, we must first find $\sin\frac{5\pi}{4}$. Value $\frac{5\pi}{4}$ is in quadrant III, where the sine is negative.

$$\sin\frac{5\pi}{4} = \sin\left(2\pi - \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \text{(the reference angle for } \frac{5\pi}{4} \text{ is } \frac{\pi}{4}\text{)}.$$

We evaluate $\arcsin\left(\sin\frac{5\pi}{4}\right)$ as follows:

$$\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.$$

To determine how to evaluate the composition of functions involving inverse trigonometric functions, first examine the value of sin(). You can use the inverse properties in the box only if x is in the specified interval.

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Example 5.57 Evaluating compositions of functions and their inverses	*Example 5.58 Evaluating compositions of functions and their inverses*
Find the exact value, if possible: $$\cos(\arccos 0.6)$$ **Solution:** The inverse property $\cos(\arccos x) = x$ applies for every $x \in [-1,1]$. To evaluate $\cos(\arccos 0.6)$, observe that $x = 0.6$. This value of x lies in $[-1,1]$, which is the domain of the inverse cosine function. This means that we can use the inverse property $\cos(\arccos x) = x$. Thus, $$\cos(\arccos 0.6) = 0.6.$$	Find the exact value, if possible: $$\arcsin\left(\sin\frac{3\pi}{2}\right)$$ **Solution:** The inverse property $\arcsin(\sin x) = x$ applies for every $x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. To evaluate $\arcsin\left(\sin\frac{3\pi}{2}\right)$, observe that $x = \frac{3\pi}{2}$. This value of x does not lie in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. To evaluate this expression, we first find $\sin\frac{3\pi}{2}$. $$\sin\frac{3\pi}{2} = \sin\left(2\pi - \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1.$$ The reference angle for $\frac{3\pi}{2}$ is $\frac{\pi}{2}$. $\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2}$. The angle in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whise sine is -1 is $-\frac{\pi}{2}$.

We can use points on terminal sides of angles in standard position to find exact values of expressions involving the composition of a function and a different inverse function.

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Example 5.59 Evaluating a composite trigonometric expression

Find the exact value, if possible:

$$\cos\left(\arctan\frac{5}{12}\right)$$

Solution:

The inner part of expression involves an angle.To evaluate such expression, we represent such angles by φ.

We let φ represent the angle in $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, whose tangent is $\frac{5}{12}$. Thus,

$$\varphi = \arctan\frac{5}{12}.$$

We are looking for the exact value of $\cos\left(\arctan\frac{5}{12}\right)$, with $\varphi = \arctan\frac{5}{12}$. Using the definition of the inverse tangent function, we can rewrite $\varphi = \arctan\frac{5}{12}$ as

$$\tan\varphi = \frac{5}{12}, where\ \varphi \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right).$$

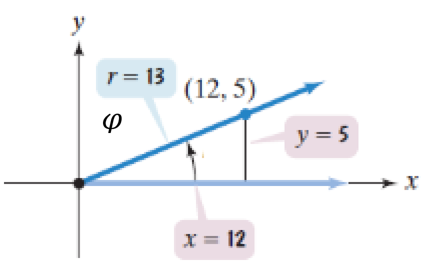

Figure 5.80 Representing $\tan\varphi = \frac{5}{12}$.

Because $\tan\varphi$ is positive, φ must be an angle in $\left(0;\frac{\pi}{2}\right)$. Thus, φ is a I quadrant angle. Figure 5.80 shows a right triangle in quadrant I w

$$\tan\varphi = \frac{5}{12}.$$

Side opposite φ, *or* y

Side adjacent to φ, *or* x

The hypotenuse of the triangle, r, or the distance from the origin to $(12{,}5)$, is found using $r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{x^2 + y^2} = \sqrt{12^2 + 5^2} = \sqrt{169} = 13.$$

We use the values for x and r to find the exact value of $\cos\left(\arctan\frac{5}{12}\right)$.

Example 5.60 Evaluating a composite trigonometric expression

Find the exact value, if possible:

$$\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$$

Solution:

The inner part of expression involves an angle.To evaluate such expression, we represent such angles by φ.

We let φ represent the angle in $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is $-\frac{1}{3}$. Thus,

$$\varphi = \arcsin\left(-\frac{1}{3}\right).$$

We are looking for the exact value of $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$, with $\varphi = \arcsin\left(-\frac{1}{3}\right)$. Using the definition of the inverse sine function, we can rewrite $\varphi = \arcsin\left(-\frac{1}{3}\right)$ as

$$\sin\varphi = -\frac{1}{3}, where\ \varphi \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right].$$

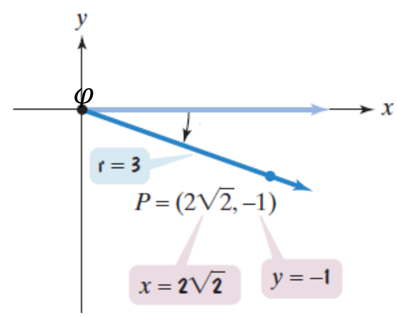

Figure 5.81 Representing $\sin\varphi = -\frac{1}{3}$.

Because $\sin\varphi$ is negative, φ must be an angle in $\left(-\frac{\pi}{2}, 0\right)$. Thus, φ is a IV quadrant angle. Figure 5.81 shows angle φ in quadrant IV with

$$\sin\varphi = -\frac{1}{3} = \frac{y}{r} = \frac{-1}{3}.$$

Thus, $y = -1$ and $r = 3$. The value of x can be found using $r = \sqrt{x^2 + y^2}$ or $r^2 = x^2 + y^2$.

Use $r^2 = x^2 + y^2$ with $y = -1$ and $r = 3$

$$3^2 = x^2 + (-1)^2$$
$$x^2 + (-1)^2 = 3^2$$
$$x^2 + 1 = 9$$
$$x^2 = 8$$
$$x = \sqrt{8} = 2\sqrt{2}.$$

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$\cos\left(\arctan\frac{5}{12}\right) = \cos\varphi = \frac{side\ adjacent\ to\ \varphi,\ or\ x}{hypotenuse,\ or\ r} = \frac{12}{13}.$	We use $x = 2\sqrt{2}$ and $y = -1$ to find the exact value of $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right)$. $\cot\left(\arcsin\left(-\frac{1}{3}\right)\right) = \cot\varphi = \frac{x}{y} = \frac{2\sqrt{2}}{-1} = -2\sqrt{2}.$

Some composite functions with inverse trigonometric functions can be simplified to algebraic expressions. To simplify such an expression, we represent the inverse trigonometric function in the expression by $\boldsymbol{\varphi}$. Then we use a right triangle.

Example 5.61 Simplifying an expression involving arcsin x

If $\boldsymbol{x} \in (\mathbf{0}, \mathbf{1}]$, write $\mathbf{cos}(\mathbf{arcsin}\,\boldsymbol{x})$ as an algebraic expression in $\boldsymbol{x}$.

Solution:

We let $\boldsymbol{\varphi}$ represent the angle in $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ whose sine is $\boldsymbol{x}$. Thus,

$\boldsymbol{\varphi} = \mathbf{arcsin}\,\boldsymbol{x}$ and $\mathbf{sin}\,\boldsymbol{\varphi} = \boldsymbol{x}$, where $\boldsymbol{x} \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

Because $\boldsymbol{x} \in (\mathbf{0}, \mathbf{1}]$, $\mathbf{sin}\,\boldsymbol{\varphi}$ is positive. Thus, $\boldsymbol{\varphi}$ is the I-quadrant angle and can be represented as an acute angle of a right triangle.

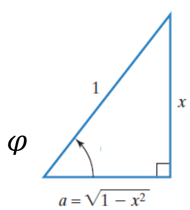

Figure 5.82 Representing $\boldsymbol{\sin\varphi = x}$.

Figure 5.82 shows a right triangle with

$$\mathbf{sin}\,\boldsymbol{\varphi} = \boldsymbol{x} = \frac{\boldsymbol{x}}{\mathbf{1}}.$$

Side opposite φ

hypotenuse

The third side $\boldsymbol{a}$, in Figure 14, can be found using the Pythagorean Theorem.

$$\mathbf{1}^2 = \boldsymbol{a}^2 + \boldsymbol{x}^2$$

$$\boldsymbol{a}^2 = \mathbf{1}^2 - \boldsymbol{x}^2$$

$$\boldsymbol{a} = \sqrt{\mathbf{1}^2 - \boldsymbol{x}^2}$$

We use the right triangle in ***Figure 5.82*** to write $\mathbf{cos}(\mathbf{arcsin}\,\boldsymbol{x})$ as an algebraic expression.

$$\cos(\arcsin x) = \cos\varphi = \frac{side\ adjacent\ \text{to}\ \varphi}{hypotenuse} = \frac{a}{hypotenuse} = \frac{\sqrt{1^2 - x^2}}{1} = \sqrt{1^2 - x^2}.$$

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