5.8.4. Important limes

AIMS:

1) Students know definition of first important limit.
2) Students know how to apply first remarkable limit in solving tasks.
3) Students know different consequences of this limit.

Definition 1: First amazing limit is limit $\lim_{x\to 0}\frac{\sin x}{x}$.

Theorem 1: (Squeeze Theorem[1])

Suppose that for all x on [a, b] (except possibly at x = c) we have, f (x) ≤ h (x) ≤ g (x). Also suppose that, $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$ for some $a \le c \le b$. Then, $\lim_{x\to c} h(x) = L$.

Theorem 2: **First remarkable limit equals 1.**

Proove:

This proof of this limit uses the Squeeze Theorem. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we will try to take it slow. Let us start by assuming that $0 \le \theta \le \frac{\pi}{2}$. Since we are proving a limit that has $\Theta \to 0$ it is okay to assume that Θ is not too large (*i.e.*, $\theta \le \frac{\pi}{2}$). Also, by assuming that Θ is positive we are going to first prove that the above limit is true if it is the right-hand limit. As you will see if we can prove this then the proof of the limit will be easy.

So, now that we have got our assumption on Θ taken care of let us start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.,

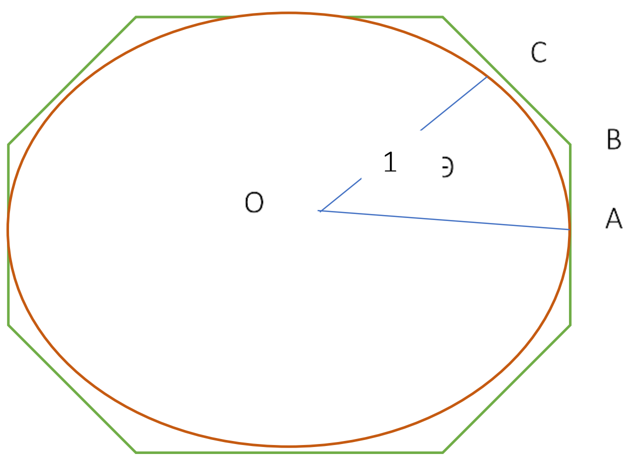

Figure 5.90

[1] Also called Two Soldiers Theorem, Sandwich Theorem

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Points A and C are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet B.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

Now denote the portion of the circle by arc *AC* and the lengths of the two portions of the octagon shown by |AB| and |BC|. Then by the observation about lengths we made above we must have,

$$\text{arc } AC < |AB| + |BC| \qquad (1)$$

Next, extend the lines *AB* and OC as shown below and call the point that they meet *D*. The triangle now formed by *AOD* is a right triangle. All this is shown in the figure below.

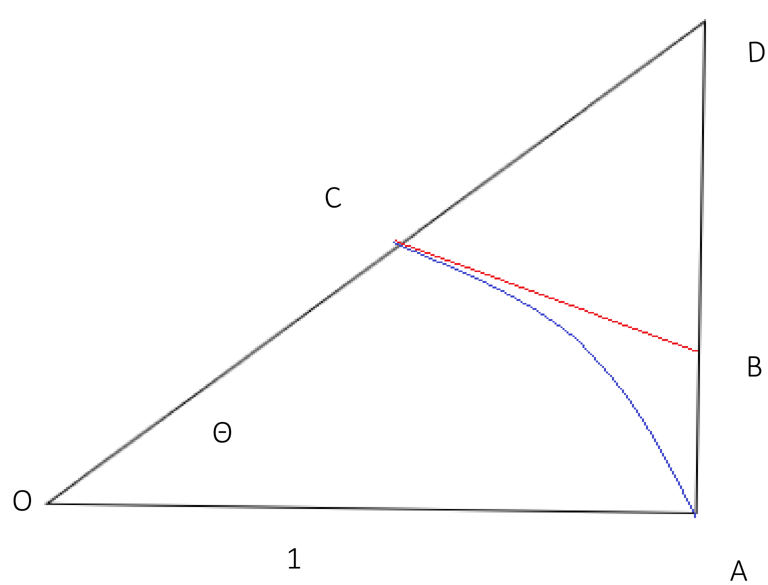

Figure 5.91

The triangle BCD is a right triangle with hypotenuse *BD* and so we know $|BC| < |BD|$. Also notice that $|AB| + |BD| = |AD|$. If we use these two facts in (1) we get,

$$\text{arc } AC < |AB| + |BC| < |AB| + |BD| = |AD| \qquad (2)$$

As noted already the triangle AOD is a right triangle and so we can use a little right triangle trigonometry to write $|AD| = |AO| \tan \Theta$. Also note that $|AO| = 1$ since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$\text{arc } AC < |AD| < |AO| \tan \Theta = \tan \Theta. \qquad (3)$$

Let's recall that the length of a portion of a circle is given by the radius of the circle times the angle (in radians) that traces out the portion of the circle we're trying to measure. For our portion this means that,

arc $AC = |AO|\Theta = \Theta$

So, putting this into (3) we see that

$$\theta = ark\ AC < \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ or } \cos\theta < \frac{\sin\theta}{\theta} \qquad (4).$$

Let's connect A and C with a line and drop a line straight down from C until it intersects AO at a right angle and let's call the intersection point E.

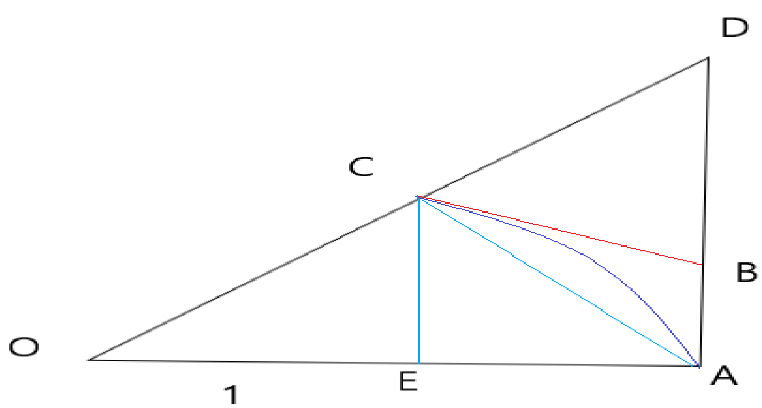

Figure 5.92

First thing to notice here is that,

$$|CE| < |AC| < \text{arc } AC. \qquad (5)$$

Triangle EOC is a right triangle with a hypotenuse of $|CO| = 1$. Using some right triangles trig we can see that

$$|CE| = |CO| \sin \Theta = \sin \Theta.$$

Putting this into (5) and recalling that arc $AC = \Theta$ we get,

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sin Θ =|CE|< arc AC = Θ

and with a little rewriting we get

$$\frac{\sin\theta}{\theta} < 1. \qquad (6)$$

Putting (4) and (6) together we see that $\cos\theta < \frac{\sin\theta}{\theta} < 1$ provided $0 \le \frac{\sin\theta}{\theta} \le \frac{\pi}{2}$. Let's also note that,

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad \lim_{\theta\to 0} 1 = 1.$$

We are now set up to use the Squeeze Theorem. The only issue that we need to worry about is that we are staying to the right of θ =0

Squeeze Theorem will tell us that, $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$

So, we know that the limit is true if we are only working with a right-hand limit. However, we know that sin Θ is an odd function and it means that,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

If we approach zero from the left (i.e., negative Θ 's) then we'll get the same values in the function as if we'd approached zero from the right (i.e., positive Θ's) and so,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1.$$

We have now shown that the two one-sided limits are the same and so we must also have,

$$\frac{\sin\theta}{\theta} = 1.$$

Example 5.68 Find the limit of the function $\lim_{x\to 0}\frac{\sin 7x}{5x}$.

Solution: As you can see, the function under the limit is close to the first remarkable limit, but the limit of the function itself is not equal to one. In such tasks for the limits, it is necessary to select the variable in the denominator with the same coefficient that is contained in the variable under the sine. In this case, divide and multiply by 7.

$$\lim_{x\to 0}\frac{\sin 7x}{5x} = \lim_{x\to 0}\frac{7\sin 7x}{5\cdot 7x} = \frac{7}{5}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{5}.$$

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Example 5.69 : Find the limit of the function $\lim_{x\to 0}\frac{\sin 6x}{\tan 11x}$.

Solution: To understand the result, we write the function as $\frac{\sin 6x}{\tan 11x}=\frac{\sin 6x}{\frac{\sin 11x}{\cos 11x}}=\frac{\cos 11x\cdot\sin 6x}{\sin 11x}$.

To apply the wonderful limit rules, we multiply and divide by factors. Further, we write the limit of the product of functions in terms of the product of the limits.

$$\lim_{x\to 0}\frac{\sin 6x}{\tan 11x}=\frac{6}{11}\lim_{x\to 0}\cos 11x\cdot\frac{11x}{\sin 11x}\cdot\frac{\sin 6x}{6x}=\frac{6}{11}\cdot\lim_{x\to 0}\cos 11x\cdot\lim_{x\to 0}\frac{\sin 6x}{6x}\cdot\lim_{x\to 0}\frac{11x}{\sin 11x}=\frac{6}{11}\cdot 1\cdot 1\cdot 1=\frac{6}{11}$$

5.8.5. Consequences

1) $\lim_{x\to 0}\frac{\tan x}{x}=1$
2) $\lim_{x\to 0}\frac{\arcsin x}{x}=1$
3) $\lim_{x\to 0}\frac{\arctan x}{x}=1$
4) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

5.8.6. Exercises

1) $\lim_{x\to 0}\frac{\sin 2x+\sin 3x}{2x}$
2) $\lim_{x\to 0}\frac{\sin 2x-\sin x}{3x}$
3) $\lim_{x\to 0}\frac{\cos x-1}{x^2}$
4) $\lim_{x\to 0}\frac{10x-3\sin x}{x}$
5) $\lim_{x\to 0}\frac{\sin 2x}{x}$

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